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FRACTIONAL SYSTEM OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. We study a multi-dimensional coupled system of nonlinear fractional integro-differential equations. By using the contraction mapping principle and Schaefer fixed point theorem, we present new results on the existence and uniqueness of mild solutions. In addition, we investigate some types of Ulam stability for this fractional system. Finally, some examples are provided to demonstrate some applications of our results.

1. Introduction and Preliminaries

Fractional order calculus can represent systems with high-order dynamics and complex nonlinear phenomena using a few coeffcients. Indeed, the arbitrary order of the derivatives is often useful when the system has a specific behavior. It relevant to some applications in various scientific areas as mathematical modeling of systems and different physical systems: The diffusion equation, food engineering, robotics and control theory. For details, see [8, 12, 13, 15, 16, 17, 18]. Recent progress in the area of fractional derivatives and integrals implies a promising potential for future developments and application of the theory. It is important to notice that there are many researchs papers treated the existence and uniqueness of solutions for some fractional systems. We refer the reader to [1, 2, 3, 4, 5, 6, 11, 19, 24] and the references therein.

On the other hand, the Ulam stability of fractional differential equations can be considered as a new way for the researchers. Truthfully, we can inspect from it several topics in nonlinear analysis problems. Moreover, the analysis on stability of fractional order differential equations is more complex than that of classical differential equations, since fractional derivatives are nonlocal and have weakly singular kernels. Ulam's type stability problems has been attracted by many researchers, see [7, 9, 10, 20, 21, 22, 23].

Inspired by the above cited works, this paper is devoted to build the existence and uniqueness of mild solutions and some types of Ulam-Hyers stability for the

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following nonlinear coupled system:

$$\begin{cases} D^{\alpha_k} x_k(t) = f_k(t, x(t), hx(t)), \ k = 1, 2, ..., m, \ t \in J, \\ x_k^{(j)}(0) = a_j^k, \ j = 0, 1, ..., n-1, \ k = 1, 2, ..., m, \end{cases} \tag{1}$$

where $x = (x_k)_{k=1,2,...,m} \in \mathbb{R}^m$, and $hx(t) = \left(\int_0^t h_k(\tau, x_k(\tau)) \, d\tau \right)_{k=1,2,...,m} \in \mathbb{R}^m$, $n - 1 < \alpha_k < n$, $k = 1, 2, ..., m$, $n \in \mathbb{N}^* \setminus \{1\}$, $m \in \mathbb{N}^*$ and $J := [0, 1]$. The derivatives D^{α_k}, $k = 1, 2, ..., m$, are in the sense of Caputo. The operator J^α is the Riemann-Liouville fractional integral, defined by:

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds, \ \alpha > 0, \ t \geq 0. \tag{2}$$

where $\Gamma(\alpha) := \int_0^\infty e^{-x} x^{\alpha-1} dx$. The operator D^α is the derivative in the sense of Caputo, defined by:

$$D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s) \, ds = J^{n-\alpha} x^{(n)}(t), \ n - 1 < \alpha < n. \tag{3}$$

For each $k = 1, 2, ..., m$, the functions $f_k : [0, 1] \times \mathbb{R}^{2m} \to \mathbb{R}$ will be specified later. We need to recall some concepts and preparation results which are used throughout this paper, [13, 14, 15].

Lemma 1. *For $n \in \mathbb{N}^* \setminus \{1\}$, and $n - 1 < \alpha < n$, the general solution of the fractional differential equation $D^\alpha x(t) = 0$, is given by*

$$x(t) = \sum_{j=0}^{n-1} c_j \, t^j, \ (c_j)_{j=0,...,n-1} \in \mathbb{R}. \tag{4}$$

Lemma 2. *Let $n \in \mathbb{N}^* \setminus \{1\}$, and $n - 1 < \alpha < n$, then*

$$J^\alpha D^\alpha x(t) = x(t) + \sum_{j=0}^{n-1} c_j \, t^j, \ (c_j)_{j=0,...,n-1} \in \mathbb{R}. \tag{5}$$

Lemma 3. *Let $q > p > 0$, $f \in L^1([a, b])$. Then $D^p J^q f(t) = J^{q-p} f(t), t \in [a, b]$.*

Lemma 4. *Let E be Banach space. Assume that $T : E \to E$ is completely continuous operator and the set $V := \{x \in E : x = \mu T x, \ 0 < \mu < 1\}$ is bounded. Then T has a fixed point in E.*

In the following, we present the integral solution of (1).

Lemma 5. *Let $n - 1 < \alpha_k < n$, and assume that $\varphi_k(t) \in C([0, 1])$. Then, the following system:*

$$D^{\alpha_k} x_k(t) = \varphi_k(t), \ k = 1, 2, ..., m, \ t \in J, \tag{6}$$

associated with the conditions:

$$x_k^{(j)}(0) = a_j^k, \ j = 0, 1, ..., n-1, \ k = 1, 2, ..., m, \tag{7}$$

has a unique mild solution $(x_1, x_2, ..., x_m)$, where

$$x_k(t) = \int_0^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} \varphi_k(s) \, ds + \sum_{j=0}^{n-1} \frac{a_j^k}{j!} t^j, \ k = 1, 2, ..., m. \tag{8}$$

Proof. Applying Lemma 2, we get

$$x_k(t) = \int_0^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} \varphi_k(s)\, ds - \sum_{j=0}^{n-1} c_j^k t^j. \tag{9}$$

where

$$\begin{pmatrix} c_0^1 & c_1^1 & \dots & c_{n-1}^1 \\ c_0^2 & c_1^2 & \dots & c_{n-1}^2 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ c_0^m & c_1^m & \dots & c_{n-1}^m \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^n.$$

It is clear that,

$$x_k^{(j)}(0) = -j! c_j^k,\ j = 0, 1, \dots, n-1, \tag{10}$$

Combine (7) and (10), we obtain

$$c_j^k = -\frac{a_j^k}{j!},\ j = 0, 1, \dots, n-1. \tag{11}$$

Take (11) into (9) we have (8). This ends the proof of Lemma 5. □

Now, let us introduce the Banach space

$S := \left\{ x = (x_k)_{k=1,2,\dots,m} : x \in C(J, \mathbb{R}^m) \right\}$, endowed with the norm:

$$\|x\|_S = \max_{1 \le k \le m} \|x_k\|_\infty .$$

where

$$\|x_k\|_\infty = \sup_{t \in J} |x_k(t)|,\ J = [0, 1].$$

2. Existence and Uniqueness

In this section, we shall obtain sufficient conditions for the existence and uniqueness of mild solutions to (1).

Theorem 6. *Assume that:*

(H_1) : *There exist nonnegative constants* $\left(\mu_k^i\right)_{k=1,2,\dots,m,}^{i=1,2,\dots,m,}$ *and* $\left(\beta_k^i\right)_{k=1,2,\dots,m,}^{i=1,2,\dots,m,}$ *such that:*

$$\|f_k(t, x, u) - f_k(t, y, v)\|_\infty \le \sum_{i=1}^m \mu_k^i \|x_i - y_i\|_\infty + \sum_{i=1}^m \beta_k^i \|u_i - v_i\|_\infty ,$$

for all $t \in J$ *and all* $x = (x_k)_{k=1,2,\dots,m}$, $y = (y_k)_{k=1,2,\dots,m}$, $u = (u_k)_{k=1,2,\dots,m}$, $v = (v_k)_{k=1,2,\dots,m} \in S$.

(H_2) : $h_k \in C(J \times \mathbb{R}, \mathbb{R})$ *and there exist positive constants* $(\omega_i)_{i=1,2,\dots,m,}$;

$$\|h_i(t, x_i) - h_i(t, y_i)\|_\infty \le \omega_i \|x_i - y_i\|_\infty ,$$

for each $t \in J$, *and all* $x = (x_i)_{i=1,2,\dots,m}$, $y = (y_i)_{i=1,2,\dots,m} \in S$.

If

$$F := \max_{1 \le k \le m} \frac{1}{\Gamma(\alpha_k + 1)} \sum_{i=1}^m \left(\mu_k^i + \beta_k^i \omega_i\right) < 1, \tag{12}$$

then system (1) has a unique mild solution on J.

Proof. First, we define the nonlinear operator $R : S \to S$ as follows,

$$R(x)(t) := (R_k(x)(t))_{k=1,2,...,m}, \ t \in J,$$

$$R_k(x)(t) = \int_0^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} f_k(s, x(s), hx(s))\, ds + \sum_{j=0}^{n-1} \frac{a_j^k}{j!} t^j, \tag{13}$$

where $x = (x_k)_{k=1,2,...,m}$ and $hx(t) = \left(\int_0^t h_k(\tau, x_k(\tau))\, d\tau \right)_{k=1,2,...,m}$.

We show that the operator R is contractive on S:

Let $x, y \in S$ and $t \in J$, we have

$$|R_k(x)(t) - R_k(y)(t)|$$
$$\leq \frac{t^{\alpha_k}}{\Gamma(\alpha_k + 1)} \sup_{s \in J} |f_k(s, x(s), hx(s)) - f_k(s, y(s), hy(s))| \tag{14}$$

where $k = 1, 2, ..., m$.

Using (H_1), we get

$$\|R_k(x) - R_k(y)\|_\infty$$
$$\leq \frac{1}{\Gamma(\alpha_k + 1)} \left(\begin{array}{c} \mu_k^1 \sup_{s \in J} |x_1(s) - y_1(s)| + ... + \mu_k^m \sup_{s \in J} |x_m(s) - y_m(s)| \\ + \beta_k^1 \sup_{s \in J} \left| \int_0^s h_1(\tau, x_1(\tau))\, d\tau - \int_0^s h_1(\tau, y_1(\tau))\, d\tau \right| \\ + ... + \beta_k^m \sup_{s \in J} \left| \int_0^s h_m(\tau, x_m(\tau))\, d\tau - \int_0^s h_m(\tau, y_m(\tau))\, d\tau \right| \end{array} \right). \tag{15}$$

And by (H_2), we can write

$$\|R_k(x) - R_k(y)\|_\infty$$
$$\leq \frac{1}{\Gamma(\alpha_k + 1)} \left(\begin{array}{c} \mu_k^1 \sup_{s \in J} |x_1(s) - y_1(s)| + ... + \mu_k^m \sup_{s \in J} |x_m(s) - y_m(s)| \\ + \beta_k^1 \omega_1 \sup_{s \in J} \int_0^s d\tau \sup_{\tau \in J} |x_1(\tau) - y_1(\tau)| \\ + ... + \beta_k^m \omega_m \sup_{s \in J} \int_0^s d\tau \sup_{\tau \in J} |x_m(\tau) - y_m(\tau)| \end{array} \right). \tag{16}$$

Then,

$$\|R_k(x) - R_k(y)\|_\infty$$
$$\leq \frac{1}{\Gamma(\alpha_k + 1)} \left(\left(\mu_k^1 + \beta_k^1 \omega_1\right) \|x_1 - y_1\|_\infty + ... + \left(\mu_k^m + \beta_k^m \omega_m\right) \|x_m - y_m\|_\infty \right), \tag{17}$$

where $k = 1, 2, ..., m$.

Thus,

$$\|R(x) - R(y)\|_S \leq F \|(x - y)\|_S. \tag{18}$$

By (12), we have

$$F < 1.$$

Hence, the operator R is contractive. Then the system (1), has a unique mild solution. This completes the proof. $\square$

Example 7. *Consider the following system:*

$$\begin{cases} D^{\frac{10}{3}}x_1(t) = \frac{t}{9e^{2t+1}}\left(\begin{array}{c}\frac{|x_1(t)+x_2(t)+x_3(t)|}{(1+|x_1(t)+x_2(t)+x_3(t)|)} \\ +\int_0^t \frac{\cos x_1(\tau)}{2\pi}d\tau + \int_0^t \frac{\sin x_2(\tau)}{2\pi}d\tau + \int_0^t \frac{\sin x_3(\tau)}{e}d\tau\end{array}\right), \\ D^{\frac{7}{2}}x_2(t) = \frac{1}{16\pi^3 e^t}\left(\begin{array}{c}\sin(x_1(t)) + \sin(x_2(t)) - \sin(x_3(t)) \\ +\int_0^t \frac{\cos x_1(\tau)}{2\pi}d\tau + \int_0^t \frac{\sin x_2(\tau)}{2\pi}d\tau + \int_0^t \frac{\sin x_3(\tau)}{e}d\tau\end{array}\right), \\ D^{\frac{15}{4}}x_3(t) \\ = \frac{\cos x_1(t) - \cos x_2(t) + \cos x_3(t)}{12\pi} + \frac{\left|\int_0^t \frac{\cos x_1(\tau)}{2\pi}d\tau + \int_0^t \frac{\sin x_2(\tau)}{2\pi}d\tau + \int_0^t \frac{\sin x_3(\tau)}{e}d\tau\right|}{12\pi\left(1+\left|\int_0^t \frac{\cos x_1(\tau)}{2\pi}d\tau + \int_0^t \frac{\sin x_2(\tau)}{2\pi}d\tau + \int_0^t \frac{\sin x_3(\tau)}{e}d\tau\right|\right)}, \\ t \in [0,1], \\ x_1(0) = 0,\ x_1'(0) = \sqrt{2},\ x_1''(0) = \sqrt{5},\ x_1'''(0) = 3\sqrt{2}, \\ x_2(0) = 0,\ x_2'(0) = 1,\ x_2''(0) = -1,\ x_2'''(0) = \sqrt{7}, \\ x_3(0) = 0,\ x_3'(0) = 1,\ x_3''(0) = -1,\ x_3'''(0) = 0. \end{cases} \tag{19}$$

We have:

$$x = (x_k)_{k=1,2,3},\ hx(t) = \left(\int_0^t h_k(\tau, x_k(\tau))\,d\tau\right)_{k=1,2,3},$$

$$\begin{aligned} n &= 4, \alpha_1 = \frac{10}{3}, \alpha_2 = \frac{7}{2}, \alpha_3 = \frac{15}{4}, a_0^1 = 0, a_1^1 = \sqrt{2}, a_2^1 = \sqrt{5}, a_3^1 = 3\sqrt{2}, \\ a_0^2 &= 0, a_1^2 = 1, a_2^2 = -1, a_3^2 = \sqrt{7}, a_0^3 = 0, a_1^3 = 1, a_2^3 = -1, a_3^3 = 0, J = [0,1], \end{aligned}$$

$$f_1(t, x(t), hx(t)) =$$

$$\frac{t}{9e^{2t+1}}\left(\begin{array}{c}\frac{|x_1(t)+x_2(t)+x_3(t)|}{1+|x_1(t)+x_2(t)+x_3(t)|} \\ +\int_0^t \frac{\cos x_1(\tau)}{2\pi}d\tau + \int_0^t \frac{\sin x_2(\tau)}{2\pi}d\tau + \int_0^t \frac{\sin x_3(\tau)}{e}d\tau\end{array}\right),$$

$$f_2(t, x(t), hx(t)) =$$

$$\frac{1}{16\pi^3 e^t}\left(\begin{array}{c}\sin(x_1(t)) + \sin(x_2(t)) - \sin(x_3(t)) \\ +\int_0^t \frac{\cos x_1(\tau)}{2\pi}d\tau + \int_0^t \frac{\sin x_2(\tau)}{2\pi}d\tau + \int_0^t \frac{\sin x_3(\tau)}{e}d\tau\end{array}\right),$$

$$f_3(t, x(t), hx(t)) =$$

$$\frac{\cos x_1(t)-\cos x_2(t)+\cos x_3(t)}{12\pi} + \frac{\left|\int_0^t \frac{\cos x_1(\tau)}{2\pi}d\tau+\int_0^t \frac{\sin x_2(\tau)}{2\pi}d\tau+\int_0^t \frac{\sin x_3(\tau)}{e}d\tau\right|}{12\pi\left(1+\left|\int_0^t \frac{\cos x_1(\tau)}{2\pi}d\tau+\int_0^t \frac{\sin x_2(\tau)}{2\pi}d\tau+\int_0^t \frac{\sin x_3(\tau)}{e}d\tau\right|\right)}$$

and

$$h_1(\tau,x_1(\tau))=\frac{\cos x_1(\tau)}{2\pi}, h_2(\tau,x_2(\tau))=\frac{\sin x_2(\tau)}{2\pi}, h_3(\tau,x_3(\tau))=\frac{\sin x_3(\tau)}{e}.$$

So, for $t\in J$ *and*

$$\begin{aligned} x &= (x_k)_{k=1,2,3}, y=(y_k)_{k=1,2,3}\in\mathbb{R}^3,\\ u &= \left(\int_0^t h_k(\tau,x_k(\tau))\,d\tau\right)_{k=1,2,3}, v=\left(\int_0^t h_k(\tau,y_k(\tau))\,d\tau\right)_{k=1,2,3}\in\mathbb{R}^3, \end{aligned}$$

we have:

$$\begin{aligned} & |f_1(t,x,u)-f_1(t,y,v)| \\ \le\ & \frac{1}{9e}|x_1-y_1|+\frac{1}{9e}|x_2-y_2|+\frac{1}{9e}|x_3-y_3| \\ & +\frac{1}{18\pi e}|x_1-y_1|+\frac{1}{18\pi e}|x_2-y_2|+\frac{1}{9e^2}|x_3-y_3| \\ \le\ & \frac{2\pi+1}{18\pi e}|x_1-y_1|+\frac{2\pi+1}{18\pi e}|x_2-y_2|+\frac{e+1}{9e^2}|x_3-y_3|, \\ & |f_2(t,x,u)-f_2(t,y,v)| \\ \le\ & \frac{1}{16\pi^3}|x_1-y_1|+\frac{1}{16\pi^3}|x_2-y_2|+\frac{1}{16\pi^3}|x_3-y_3| \\ & +\frac{1}{32\pi^4}|x_1-y_1|+\frac{1}{32\pi^4}|x_2-y_2|+\frac{1}{16\pi^3 e}|x_3-y_3| \\ \le\ & \frac{2\pi+1}{32\pi^4}|x_1-y_1|+\frac{2\pi+1}{32\pi^4}|x_2-y_2|+\frac{e+1}{16\pi^3 e}|x_3-y_3|, \\ & |f_3(t,x,u)-f_3(t,y,v)| \\ \le\ & \frac{1}{12\pi}|x_1-y_1|+\frac{1}{12\pi}|x_2-y_2|+\frac{1}{12\pi}|x_3-y_3| \\ & +\frac{1}{24\pi^2}|x_1-y_1|+\frac{1}{24\pi^2}|x_2-y_2|+\frac{1}{12\pi e}|x_3-y_3| \\ \le\ & \frac{2\pi+1}{24\pi^2}|x_1-y_1|+\frac{2\pi+1}{24\pi^2}|x_2-y_2|+\frac{e+1}{12\pi e}|x_3-y_3|. \end{aligned}$$

Moreover, we get:

$$\omega_1=\omega_2=\frac{1}{2\pi},\ \omega_3=\frac{1}{e},$$

$$\mu_1^1=\mu_1^2=\mu_1^3=\beta_1^1=\beta_1^2=\beta_1^3=\frac{1}{9e},$$

$$\mu_2^1=\mu_2^2=\mu_2^3=\beta_2^1=\beta_2^2=\beta_2^3=\frac{1}{16\pi^3},$$

$$\mu_3^1=\mu_3^2=\mu_3^3=\beta_3^1=\beta_3^2=\beta_3^3=\frac{1}{12\pi},$$

$$\Gamma(\alpha_1+1)=9.256373,\ \Gamma(\alpha_2+1)=11.631728,\ \Gamma(\alpha_3+1)=16.586206,$$
$$F_1=0.016278,\ F_2=0.000639,\ F_3=0.005895.$$

Then it yields that:

$$\max(F_1,F_2,F_3)<1.$$

Thus, (19) has a unique mild solution on J.

Theorem 8. *Assume that the functions* $(f_k)_{k=1,2,\ldots,m}$ *,* $m\in\mathbb{N}^*$*, satisfy the following conditions:*
(H_3) : $f_k\in C\left(J\times\mathbb{R}^{2m},\mathbb{R}\right)$.
(H_4) : *There exists a nonnegative constant* λ *such that*

$$|f_k(t,x(t),hx(t))|\leq\lambda,$$

for each $t\in J$*, and all* $x=(x_k)_{k=1,2,\ldots,m}\in\mathbb{R}^m$ *and*

$$hx(t)=\left(\int_0^t h_k(\tau,x_k(\tau))\,d\tau\right)_{k=1,2,\ldots,m}\in\mathbb{R}^m.$$

Then the nonlinear fractional system (1) has at least one mild solution on J.

Proof. We will prove the theorem in two steps:
Step1: The operator R is completely continuous.
Let us consider the set $B_\rho:=\left\{x=(x_k)_{k=1,2,\ldots,m}\in S:\ \|x\|_S\leq\rho,\ \rho>0\right\}$. Then for each $x\in B_\rho$, we have

$$\|R_k(x)\|_\infty\leq\frac{1}{\Gamma(\alpha_k+1)}\sup_{s\in J}|f_k(s,x(s),hx(s))|+\sum_{j=0}^{n-1}\frac{|a_j^k|}{j!}.$$

Thanks to (H_4) yields,

$$\|R_k(x)\|_\infty\leq\left(\frac{\lambda}{\Gamma(\alpha_k+1)}+\sum_{j=0}^{n-1}\frac{|a_j^k|}{j!}\right).$$

Then, we get

$$\|R(x)\|_S=\max_{1\leq k\leq m}\left(\frac{\lambda}{\Gamma(\alpha_k+1)}+\sum_{j=0}^{n-1}\frac{|a_j^k|}{j!}\right)<\infty. \tag{20}$$

Which implies $R(B_\rho)$ is bounded.
By (H_3), the operator R is continuous in view of the continuity of f_k. On the other hand, for each $x=(x_k)_{k=1,2,\ldots,m}\in B_\rho$, and for all $t_1,t_2\in[0,1]$, $t_1<t_2$, we have:

$$\|R_k(x)(t_2)-R_k(x)(t_1)\|_\infty\leq$$
$$\left(\frac{\lambda\left(2(t_2-t_1)^{\alpha_k}+\left(t_2^{\alpha_k}-t_1^{\alpha k}\right)\right)}{\Gamma(\alpha_k+1)}+\sum_{j=1}^{n-1}\frac{|a_j^k|}{j!}\left(t_2^j-t_1^j\right)\right),$$

where $k=1,2,...,m$.

Thus,

$$\|R(x)(t_2)-R(x)(t_1)\|_S$$

$$\leq \max_{1\leq k\leq m}\left(\frac{\lambda\left(2\left(t_2-t_1\right)^{\alpha_k}+\left(t_2^{\alpha_k}-t_1^{\alpha_k}\right)\right)}{\Gamma\left(\alpha_k+1\right)}+\sum_{j=1}^{n-1}\frac{\left|a_j^k\right|}{j!}\left(t_2^j-t_1^j\right)\right). \quad (21)$$

Since the right-hand side of (21) is independent of $x=\left(x_k\right)_{k=1,2,\dots,m}$ and tend to zero as $t_2-t_1\to 0$, so R is equi-continuous. Hence, T is a completely continuous.

Step2: We show that the set:

$$\Sigma := \left\{x=\left(x_k\right)_{k=1,2,\dots,m}\in S:\ x=\delta R\left(x\right),\ 0<\delta<1\right\},$$

is bounded.

Let $x=\left(x_k\right)_{k=1,2,\dots,m}\in\Sigma$ and $t\in J$, then $x\left(t\right)=\delta R\left(x\right)\left(t\right)$. From (20), we get:

$$\left\|x\right\|_S\leq\delta\max_{1\leq k\leq m}\left(\frac{\lambda}{\Gamma\left(\alpha_k+1\right)}+\sum_{j=0}^{n-1}\frac{\left|a_j^k\right|}{j!}\right)<\infty.$$

Therefore, Σ is bounded.

It follows from the assumptions of lemma 4, that R has a fixed point in S which is a mild solution of system (1). Theorem 8 is thus proved. □

Example 9. *Consider the fractional coupled system*

$$\begin{cases} D^{\frac{5}{2}}x_1\left(t\right)=\begin{pmatrix} 2\pi e^t\cos\left(x_1\left(t\right)+x_2\left(t\right)\right) \\ \times\left(\int\limits_0^t\tau\sin x_1\left(\tau\right)d\tau-\int\limits_0^t\cos 2\left(\tau+1\right)x_2\left(\tau\right)d\tau\right)\end{pmatrix}, \\ D^{\frac{7}{3}}x_2\left(t\right)=\frac{\int_0^t\tau\sin x_1(\tau)d\tau-\int_0^t\cos 2(\tau+1)x_2(\tau)d\tau}{2\pi-\sin(x_1(t)+x_2(t))}, \\ t\in\left[0,1\right], \\ x_1\left(0\right)=0,\ x_1^{'}\left(0\right)=-1,\ x_1^{''}\left(0\right)=\sqrt{2}, \\ x_2\left(0\right)=0,\ x_2^{'}\left(0\right)=1,\ x_2^{''}\left(0\right)=\sqrt{3}.\end{cases} \quad (22)$$

For this second example, we have:

$$n=3, m=2, x=\left(x_k\right)_{k=1,2},$$

$$hx\left(t\right)=\left(\int\limits_0^t h_k\left(\tau,x_k\left(\tau\right)\right)d\tau\right)_{k=1,2},$$

$$\alpha_1=\frac{5}{2},\alpha_2=\frac{7}{3},a_0^1=0,a_1^1=-1,a_2^1=\sqrt{2},a_0^2=0,a_1^2=1,a_2^2=\sqrt{3},J=\left[0,1\right].$$

$$h_1\left(t,x_1\left(t\right)\right)=\int\limits_0^t\tau\sin x_1\left(\tau\right)d\tau, h_2\left(t,x_2\left(t\right)\right)=\int\limits_0^t\cos 2\left(\tau+1\right)x_2\left(\tau\right)d\tau.$$

Then for each $x,\ hx \in \mathbb{R}^2,$ *we get*

$$\begin{aligned} \left| f_1\left(t, x\left(t\right), hx\left(t\right)\right)\right| &= \left| \begin{array}{l} 2\pi e^t \cos\left(x_1\left(t\right)+x_2\left(t\right)\right) \\ \times\left(\int\limits_0^t \tau \sin x_1\left(\tau\right)d\tau - \int\limits_0^t \cos 2\left(\tau+1\right)x_2\left(\tau\right)d\tau\right) \end{array} \right| \\ &\leq 3\pi e, \end{aligned}$$

$$\begin{aligned} \left| f_2\left(t, x\left(t\right), hx\left(t\right)\right)\right| &= \left| \frac{\int\limits_0^t \tau \sin x_1\left(\tau\right)d\tau - \int\limits_0^t \cos 2\left(\tau+1\right)x_2\left(\tau\right)d\tau}{2\pi - \cos\left(x_1\left(t\right)+x_2\left(t\right)\right)} \right| \\ &\leq \frac{3}{2\left(2\pi-1\right)}. \end{aligned}$$

Since $f_k,\ k=1,2,$ *are continuous and bounded on* $J\times\mathbb{R}^4$*, we can take:*

$$\lambda = \max\left(3\pi e, \frac{3}{2\left(2\pi-1\right)}\right),$$

So, system (22) has at least one mild solution on J.

3. Ulam Stability

In this section, we impose some types of Ulam stability: Ulam-Hyers stability, generalized Ulam-Hyers stability and Ulam-Heyers-Rassias stability for system (1). We firstly give the definitions of each type of stability which are applicable to the system of equations appearing in this paper.

Definition 10. *The fractional system (1) is Ulam-Hyers stable if there exists a real number* $A>0$*, such that for each* $\left(\epsilon_k\right)_{k=1,2,...,m}>0$*, and for for each mild solution* $x=\left(x_k\right)_{k=1,2,...,m}\in S$ *of the following system*

$$\left|D^{\alpha_k}x_k\left(t\right)-f_k\left(t,x\left(t\right),hx\left(t\right)\right)\right|\leq\epsilon_k,\ t\in J, \tag{23}$$

there exists $y=\left(y_k\right)_{k=1,2,...,m}\in S$ *satisfying(1);* $y_k^{(j)}\left(0\right)=a_j^k,\ j=0,1,...,n-1,$ $k=1,2,...,m,$ *where*

$$\left\|x-y\right\|_S=\max_{1\leq k\leq m}\left\|x_k-y_k\right\|_\infty\leq A\epsilon,\ \epsilon=\max_{k\leq 1\leq n}\epsilon_k. \tag{24}$$

Definition 11. *The fractional system (1) has the generalized Ulam-Hyers stablity if there exist* $\Psi\in C\left(\mathbb{R}^+,\mathbb{R}^+\right),$ *such that for all* $\epsilon>0,$ *and for each mild solution* $x=\left(x_k\right)_{k=1,2,...,m}\in S$ *of (23), there exists* $y=\left(y_k\right)_{k=1,2,...,m}\in S$ *of (1);* $y_k^{(j)}\left(0\right)=a_j^k,\ j=0,1,...,n-1,\ k=1,2,...,m,$ *with*

$$\left\|x-y\right\|_S=\max_{1\leq k\leq m}\left\|x_k-y_k\right\|_\infty\leq\Psi\left(\epsilon\right),\ \epsilon>0,$$

Definition 12. *System (1) has the Ulam-Hyers-Rassias stablity if there exist functions* $T\in C\left(J,\mathbb{R}^+\right)$ *and* $\sigma>0$ *such that for each* $\left(\epsilon_k\right)_{k=1,2,...,m}>0$ *and for all mild solution* $x=\left(x_k\right)_{k=1,2,...,m}\in S$ *of*

$$\left|D^{\alpha_k}x_k\left(t\right)-f_k\left(t,x\left(t\right),hx\left(t\right)\right)\right|\leq\epsilon_kT\left(t\right),\ t\in J, \tag{25}$$

there exists $y=(y_k)_{k=1,2,...,m}\in S$ *of (1),* $y_k^{(j)}(0)=a_j^k,\ j=0,1,...,n-1,\ k=1,2,...,m,$ *with*

$$\|x-y\|_S=\max_{1\le k\le m}\|x_k-y_k\|_\infty\le\sigma\epsilon T(t),$$
$$\epsilon=\max_{k\le 1\le m}\epsilon_k,$$

Remark 13. $x=(x_k)_{k=1,2,...,m}\in S$ *is a mild solution of (1), if and only if there exist* $(q_k)_{k=1,2,...,m}\in C(J,\mathbb{R})$, *such that:*

$$|q_k(t)|\le\epsilon_k,\ t\in J,$$

and

$$D^{\alpha_k}x_k(t)=f_k(t,x(t),hx(t))+q_k(t),\ k=1,2,...,m,\ t\in J.$$

.

Theorem 14. *Suppose that the assumptions of Theorem 6 hold, and*

$$\sum_{i=1}^{m}\left(\mu_k^i+\beta_k^i\omega_i\right)<1,\ k=1,2,...,m. \tag{26}$$

Moreover, assume that the functions $(f_k)_{k=1,2,...,m}$, $m\in\mathbb{N}^*$, *satisfy the conditions* (H_3) *and* (H_4). *So, if the inequality*

$$\|D^{\alpha_k}x_k\|_\infty\ge\frac{\lambda}{\Gamma(\alpha_k+1)}+\sum_{j=0}^{n-1}\frac{|a_j^k|}{j!},\ k=1,2,...,m, \tag{27}$$

is valid, then (1) has the generalized Ulam-Hyers stability in S.

Proof. We begin by supposing that (27) is valid. According to Theorem 6, the problem (1) has a mild solution $y=(y_k)_{k=1,2,...,m}\in S$ satisfying:

$$\begin{cases} D^{\alpha_k}y_k(t)=f_k(t,y(t),hy(t)),\\ \\ y_k^{(j)}(0)=a_j^k,\ j=0,1,...,n-1,\ k=1,2,...,m.\end{cases} \tag{28}$$

Now, let

$$|D^{\alpha_k}x_k(t)-f_k(t,x(t),hx(t))|\le\epsilon_k,\ \epsilon_k>0,\ k=1,2,...,m. \tag{29}$$

According to (H_4), we have

$$\|x_k\|_\infty\le\frac{\lambda}{\Gamma(\alpha_k+1)}+\sum_{j=0}^{n-1}\frac{|a_j^k|}{j!},\ k=1,2,...,m. \tag{30}$$

Combining (27) and (30), yields

$$\|x_k\|_\infty\le\|D^{\alpha_k}x_k\|_\infty. \tag{31}$$

Replacing x_k by (x_k-y_k) in this inequality, we get

$$\|(x_k-y_k)\|_\infty \le \|D^{\alpha_k}(x_k-y_k)\|_\infty \tag{32}$$

$$\le \sup_{t\in J}\left|\begin{array}{l}(D^{\alpha_k}x_k(t)-f_k(t,x(t),hx(t)))\\ \\ -(D^{\alpha_k}y_k(t)-f_k(t,y(t),hy(t)))\\ \\ +(f_k(t,x(t),hx(t))-f_k(t,y(t),hy(t)))\end{array}\right| \tag{33}$$

And from (28) and (29), we obtain

$$\|(x_k - y_k)\|_\infty \le \epsilon_k + \sum_{i=1}^{m} \left(\mu_k^i + \beta_k^i \omega_i\right) \max_{1\le k\le m} \|x_k - y_k\|_\infty . \tag{34}$$

This implies that,

$$\max_{1\le k\le m} \|x_k - y_k\|_\infty \le \max_{1\le k\le m} \frac{\epsilon_k}{1 - \sum_{i=1}^{m} \left(\mu_k^i + \beta_k^i \omega_i\right)} := A\epsilon, \tag{35}$$

$$\epsilon = \max_{1\le k\le n} \epsilon_k, \;\; A = \max_{1\le k\le n} \frac{1}{1 - \sum_{i=1}^{m} \left(\mu_k^i + \beta_k^i \omega_i\right)}.$$

Or

$$\|x - y\|_S = \max_{1\le k\le m} \|x_k - y_k\|_\infty \le A\epsilon. \tag{36}$$

From (26) we get $A > 0$. Hence, system (1) has Ulam-Hyers stabily. Taking $\Psi(\epsilon) = A\epsilon$, we can state that the system (1) has the generalized Ulam-Hyers stability. This completes the proof. □

Theorem 15. *Let the assumptions of Theorem 6 hold. Then, system (1) has the Ulam-Hyers-Rassias stablity in S.*

Proof. Let $(x_k)_{k=1,2,...,m} \in S$ mild solution of (1). From remark 13, we get for all $k = 1, 2, ..., m$,

$$\begin{aligned} &\left| x_k(t) - \int_0^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} f_k(s, x(s), hx(s))\, ds - \sum_{j=0}^{n-1} \frac{a_j^k}{j!} t^j \right| \\ \le\;& J^{\alpha_k} \epsilon_k := \epsilon_k T(t), \end{aligned} \tag{37}$$

On the other hand, by Theorem 6, there exists $(y_k)_{k=1,2,...,m} \in S$ satisfying (28).

We have

$$|x_k(t) - y_k(t)| =$$

$$\left| \begin{array}{l} \left(x_k(t) - \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} f_k(s, x(s), hx(s))\, ds - \sum_{j=0}^{n-1} \frac{a_j^k}{j!} t^j \right) \\ - \left(y_k(t) - \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} f_k(s, y(s), hy(s))\, ds - \sum_{j=0}^{n-1} \frac{a_j^k}{j!} t^j \right) \\ + \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} f_k(s, x(s), hx(s))\, ds \\ - \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} f_k(s, y(s), hy(s))\, ds \end{array} \right|. \tag{38}$$

Then by (28) and (37), we get

$$|x_k(t) - y_k(t)| \leq \epsilon_k T(t) + \left| \begin{array}{c} \int\limits_0^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} f_k(s, x(s), hx(s))\, ds \\ \\ - \int\limits_0^t \frac{(t-s)^{\alpha_k - 1}}{\Gamma(\alpha_k)} f_k(s, y(s), hy(s))\, ds \end{array} \right|$$

which implies that:

$$\|x_k - y_k\|_\infty \leq \epsilon_k T(t) + \frac{1}{\Gamma(\alpha_k + 1)} \sum_{i=1}^{m} \left(\mu_k^i + \beta_k^i \omega_i\right) \max_{1 \leq k \leq n} \|x_k - y_k\|_\infty . \quad (39)$$

Then,

$$\max_{1 \leq k \leq n} \|x_k - y_k\|_\infty \leq \frac{\epsilon T(t)}{1 - F} := \sigma \epsilon T(t).$$

In addition,

$$\|(x - y)\|_S = \max_{1 \leq k \leq n} \|x_k - y_k\|_\infty \leq \sigma \epsilon T(t), \quad (40)$$

where

$$\sigma = \frac{1}{1 - F} > 0, \; \epsilon = \max_{1 \leq k \leq n} \epsilon_k.$$

Thus, System (1) has the Ulam-Hyers-Rassias stability. Theorem 15 is thus proved. □

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