OPTIMAL HOMOTOPY ASYMPTOTIC METHOD FOR FIRST-ORDER CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this article, optimal homotopy asymptotic method is presented to solve linear and nonlinear first-order conformable fractional differential equations that, is named conformable fractional optimal homotopy asymptotic method. So conformable fractional relaxation-oscillation and Riccati differential equations as a linear and nonlinear fractional differential equations, are solved by the proposed approach, respectively. The results obtained demonstrate the efficiency of the declared method for fractional equations.

1. INTRODUCTION

Many phenomena in our real world are described by fractional differential equations [3-14]. Fractional differential equations are often seeming perplexing to solve. Therefore, finding comprehensive methods for solving them sounds of high importance. Although having the exact solution of fractional equations in analyzing the phenomena is essential, there are many fractional differential equations, which cannot be solved exactly. Due to this fact, finding a desired approximate solutions of fractional differential equations is clearly vital. In recent years, many effective methods have been proposed for finding approximate solution to fractional differential equations, such as Adomian decomposition method [14, 15], homotopy perturbation method [16-19], homotopy analysis method [20], optimal homotopy asymptotic method [21-23], variational iteration method [24], generalized differential transform method [25], finite difference method [26], semi-disrete scheme and Chebyshev collocation method [27], wavelet operational [28, 29] and other methods [30-36]. In this paper, optimal homotopy asymptotic method is utilized to obtain an approximate solution of linear and nonlinear conformable fractional differential equations. The relaxation-oscillation and Riccati fractional differential equations, such as a linear and a nonlinear fractional equations are solved, respectively. The organization of this paper is as follows: In Section 2, the basic definitions like conformable fractional derivative and integral, is described. In Section 3, the conformable fractional

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homotopy asymptotic method for linear and nonlinear fractional differential equations, are presented. In Section 4, conformable fractional relaxation-oscillation and Riccati equations, as illustrative examples, by means of the proposed approach are solved. Finally, conclusions are given in Section 5.

2. Basic definitions

The purpose of this section is to recall some preliminaries of the proposed method.

2.1. **Conformable fractional derivative (CFD).** Given a function $f : [0, \infty) \longrightarrow \mathbb{R}$. The conformable fractional derivative of f, of order α, is defined by

$$\mathrm{T}_\alpha(f)(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}, \tag{2.1}$$

for all $x > 0$, $\alpha \in (0, 1]$. If f is $\alpha-$differentiable in some interval $(0, a)$, let's define $\mathrm{T}_\alpha(f)(0) = \lim_{x \to 0^+} \mathrm{T}_\alpha(f)(x)$, provided that $\lim_{x \to 0^+} \mathrm{T}_\alpha(f)(x)$ exists. If the conformable derivative of f of order α exists, then we simply say that f is $\alpha-$differentiable (see [1, 2]).

One can easily show that satisfies all the following properties (see [1]):

Let $a \in (0, 1]$ and f, g be a-differentiable at a point $x > 0$, then

    A. For $a, b \in \mathbb{R}$ $\mathrm{T}_\alpha(af + bg) = a\mathrm{T}_\alpha(f) + b\mathrm{T}_\alpha(g)$,
    B. For all $p \in \mathbb{R}$ $\mathrm{T}_\alpha(x^p) = px^{p-\alpha}$,
    C. For all constant functions $f(x) = \lambda$, $\mathrm{T}_\alpha(\lambda) = 0$,
    D. $\mathrm{T}_\alpha(f.g) = g.\mathrm{T}_\alpha(f) + f.\mathrm{T}_\alpha(g)$,
    E. $\mathrm{T}_\alpha\left(\frac{f}{g}\right) = \frac{g.\mathrm{T}_\alpha(f) - f.\mathrm{T}_\alpha(g)}{g^2}$,
    F. $\mathrm{T}_\alpha(f) = x^{1-\alpha}\frac{df}{dx}$.

If $\alpha \in (n, n+1]$ and is n-differentiable at $x > 0$, then the conformable fractional derivative of of order α is defined as follows

$$\mathrm{T}_\alpha(f)(x) = \lim_{\varepsilon \to 0} \frac{f^{([\alpha]-1)}\left(x + \varepsilon x^{([\alpha]-\alpha)}\right) - f^{([\alpha]-1)}(x)}{\varepsilon}, \tag{2.2}$$

where $[\alpha]$ is the smallest integer greater than or equal to α. When is $(n+1)$-differentiable at $x > 0$, as a consequence of (2.2), one can have (see [1])

$$\mathrm{T}_\alpha(f)(x) = x^{[\alpha]-\alpha}\frac{d^{[\alpha]}f}{dx^{[\alpha]}}.$$

2.2. **Conformable fractional integral.** Given a function $f : [a, \infty) \longrightarrow \mathbb{R}$, $a \geq 0$. The conformable fractional integral of f, is defined by

$$\mathrm{I}_\alpha^a(f)(x) = \int_a^x \frac{f(t)}{t^{1-\alpha}}dt, \tag{2.3}$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1)$ (see [1, 2]). For the sake of simplicity, lets consider $\mathrm{I}_\alpha^0(f)(x) = \mathrm{I}_\alpha(f)(x)$. One of the most useful results is the following statement (see [1]):

For all $x \geq a$ and any continuous function in the domain of I_α^a, we have
$\mathrm{T}_\alpha(\mathrm{I}_\alpha^a f(x)) = f(x)$.

3. Conformable fractional optimal homotopy asymptotic method

Consider conformable fractional functional equation with initial condition, as the following

$$\mathrm{T}_\alpha (u) = \mathcal{N}(u) + f(t), u(0) = \beta, \tag{3.1}$$

where $\mathcal{N}$ is a functional operator, $f(t)$ is a known function, and β is a constant. The Eq. (3.1) is changed into conformable fractional Riccati differential equation, when $\mathcal{N}(u) = g(t)u(t) + h(t)u^2(t)$, where g, h are given functions.
If $\mathcal{N}(u) = Bu(t)$, then Eq. (3.1) is converted into conformable fractional relaxation-oscillation differential equation, where B is a constant.
According to optimal homotopy asymptotic technique, a homotopy
$v(t,p) : \Omega \times [0,1] \to \mathbb{R}$ can be constructed as the following

$$(1-p)\left[\mathrm{T}_\alpha (v(t,p)) - f(t)\right] - H(p)\left[\mathrm{T}_\alpha (\nu(t,p)) - \mathcal{N}(\nu(t,p)) - f(t)\right] = 0. \tag{3.2}$$

Where $p \in [0,1]$ is an embedding parameter, $H(p)$ is a nonzero auxiliary function for $p \neq 0$ and $H(0) = 0$, $v(t,p)$ is an unknown function. By substituting $p = 0$ and 1 in equation (3.2), we have and respectively. Thus as p is changing from zero to unity, the solution $v(t,p)$ varies continuously from $u_0(t)$ to an exact solution $u(t)$. By substituting $p = 0$ in Eq. (3.2) the initial approximation $u_0(t) = v(t,0)$ is obtained as the solution of equation, the following

$$\mathrm{T}_\alpha (u_0(t)) - f(t) = 0, u(0) = \beta. \tag{3.3}$$

The auxiliary function $H(p)$ can be chose as the following

$$H(p) = c_1 p + c_2 p^2 + c_3 p^3 + \dots, \tag{3.4}$$

where $c_1, c_2, c_3, \dots$ are parameters, that will be determined later. Expanding $v(t, p, c_1, c_2, \dots)$ in a Taylor series of p is

$$\nu(t, p, c_1, c_2, \dots) = u_0(t) + \sum_{i=1}^{\infty} u_i(t, c_1, c_2, \dots, c_i) p^i. \tag{3.5}$$

Substituting Eqs. (3.4) and (3.5) into equation (3.2) and setting to zero the coefficient of like powers of p, then the zero order deformation equation is obtained as given in Eq. (3.3), and the other order deformation equations are given as the following

$$p^1; \mathrm{T}_\alpha (u_1(t, c_1)) - \mathrm{T}_\alpha (u_0(t)) + f(t) - c_1 \left[\mathrm{T}_\alpha (u_0(t)) - \mathcal{N}(u_0(t)) - f(t)\right] = 0, u_1(0) = 0,$$

$$p^2; \mathrm{T}_\alpha (u_2(t, c_1, c_2)) - \mathrm{T}_\alpha (u_1(t, c_1)) - c_2 \left[\mathrm{T}_\alpha (u_0(t)) - \mathcal{N}(u_0(t)) - f(t)\right]$$

$$-c_1 \left[\mathrm{T}_\alpha (u_1(t, c_1)) - u_1(t, c_1) \frac{\partial \mathcal{N}}{\partial u_0}(u_0(t))\right] = 0, u_2(0) = 0, \tag{3.6}$$

$$p^3; \mathrm{T}_\alpha (u_3(t, c_1, c_2, c_3)) - \mathrm{T}_\alpha (u_2(t, c_1, c_2)) - c_3 \left[\mathrm{T}_\alpha (u_0(t)) - \mathcal{N}(u_0(t)) - f(t)\right]$$

$$-c_2 \left[\mathrm{T}_\alpha (u_1(t, c_1)) - u_1(t, c_1) \frac{\partial \mathcal{N}}{\partial u_0}(u_0(t))\right]$$

$$-\frac{1}{2}c_1 \left[2\mathrm{T}_\alpha (u_2(t, c_1, c_2)) - u_1^2(t, c_1) \frac{\partial^2 \mathcal{N}}{\partial u_0^2}(u_0(t)) - 2u_2(t, c_1, c_2) \frac{\partial \mathcal{N}}{\partial u_0}(u_0(t))\right] = 0, u_3(0) = 0,$$

$$\vdots$$

It should be noted that $u_1, u_2, u_3, \dots$ are directed by linear equations (3.6), which can be easily solved. The convergence of the series given in Eq. (3.5 depends upon

the auxiliary parameters c_i for $i \geq 1$ If it converges at $p = 1$ we have,

$$u(t, c_1, c_2, \ldots) = u_0(t) + \sum_{i=1}^{\infty} u_i(t, c_1, c_2, \ldots, c_i). \qquad (3.7)$$

Generally, the mth order approximate solution of Eq. (3.1), can be denoted as the following

$$u^m(t, c_1, c_2, \ldots, c_m) = u_0(t) + \sum_{i=1}^{m} u_i(t, c_1, c_2, \ldots, c_i). \qquad (3.8)$$

By substitution of Eq. (3.8) into Eq. (3.1), the residual error can be expressed as follows

$$R(t, c_1, c_2, \ldots, c_m) = \mathrm{T}_\alpha(u^m(t, c_1, c_2, \ldots, c_m)) - \mathcal{N}(u^m(t, c_1, c_2, \ldots, c_m)) - f(t). \qquad (3.9)$$

When $R(t, c_1, c_2, \ldots, c_m) = 0$, results that $u^m(t, c_1, c_2, \ldots, c_m)$ is an exact solution. Such a case does not usually occur for nonlinear problems. In these cases we can apply least square approach:

$$J_m(c_1, c_2, \ldots, c_m) = \int_a^b R^2(t, c_1, c_2, \ldots, c_m)\, dt, \qquad (3.10)$$

where the values a, b depend on the given problem. The unknown convergence control parameters $c_1, c_2, c_3, \ldots, c_m$ can be optimally identified from the following conditions

$$\frac{\partial J_m}{\partial c_i} = 0, i = 1, 2, \ldots, m. \qquad (3.11)$$

It is interesting to point out that when these parameters are determined, the mth order approximate solution given by Eq. (3.8) can be constructed.

4. EXAMPLES

In this section, to illustrate the proposed approach, conformable fractional relaxation-oscillation and Riccati differential equations will be solved.

Example 4.1. *Consider the following fractional relaxation-oscillation differential equation with initial value*

$$\mathrm{T}_{\frac{3}{2}} u(t) = -u(t), 0 \leq t \leq 1, u(0) = 1, \qquad (4.1)$$

where $u(x) = \exp\left(-\frac{2}{3} t^{\frac{3}{2}}\right)$.
According to the proposed conformable fractional optimal homotopy asymptotic method, we have

$$(1-p)\left[\mathrm{T}_{\frac{3}{2}}(v(t,p))\right] - H(p)\left[\mathrm{T}_{\frac{3}{2}}(\nu(t,p)) + \nu(t,p)\right] = 0. \qquad (4.2)$$

Substituting Eqs. (3.4) and (3.5) into Eq. (4.2) and setting to zero the coefficient of the same powers of p, we derive

$$p^0; \mathrm{T}_{\frac{3}{2}}(u_0(t)) = 0, u_0(0) = 1,$$

$$p^1; \mathrm{T}_{\frac{3}{2}}(u_1(t, c_1)) - \mathrm{T}_{\frac{3}{2}}(u_0(t)) - c_1\left[\mathrm{T}_{\frac{3}{2}}(u_0(t)) + u_0(t)\right] = 0, u_1(0) = 0, \qquad (4.3)$$

$$p^2; \mathrm{T}_{\frac{3}{2}}\left(u_2\left(t, c_1, c_2\right)\right) - \mathrm{T}_{\frac{3}{2}}\left(u_1\left(t, c_1\right)\right) - c_2 \left[\mathrm{T}_{\frac{3}{2}}\left(u_0\left(t\right)\right) + u_0\left(t\right)\right]$$

$$-c_1 \left[\mathrm{T}_{\frac{3}{2}}\left(u_1\left(t, c_1\right)\right) + u_1\left(t, c_1\right)\right] = 0, \ u_2\left(0\right) = 0,$$

$$\vdots$$

Corresponding solution of this system of conformable differential equations (4.3), are

$$u_0\left(t\right) = 1,$$
$$u_1\left(t, c_1\right) = \tfrac{2}{3}c_1 t^{\frac{3}{2}},$$
$$u_2\left(t, c_1, c_2\right) = \tfrac{2}{3}c_1 t^{\frac{3}{2}} + \tfrac{2}{3}c_2 t^{\frac{3}{2}} + \tfrac{2}{3}c_1{}^2 t^{\frac{3}{2}} + \tfrac{2}{9}c_1{}^2 t^3,$$

$$\vdots$$

Therefore, two-terms approximation to the solution of Eq. (4.1), can be obtained as follows,

$$u^2\left(t, c_1, c_2\right) = 1 + 1.333333333 c_1 t^{\frac{3}{2}} + 0.6666666667 c_2 t^{\frac{3}{2}} \qquad (4.4)$$

$$+0.6666666667 c_1^2 t^{\frac{3}{2}} + 0.222222222 c_1^2 t^3.$$

For calculating unknown auxiliary constants c_1 and c_2 in $u^2(t, c_1, c_2)$ given in Eq. (4.4), we apply the procedure mentioned in (3.9) up to (3.11), we one obtains

$$c_1 = 0.8647018968, \ c_2 = -3.456662450.$$

By considering these values, in (4.4), two-order approximate solution of Eq. (4.1) reads to

$$u^2(t) = 1 - 0.653032857 t^{\frac{3}{2}} + 0.1661576378 t^3.$$

In Figure]4.1], the exact and approximate solutions of fractional relaxation-oscillation (4.1) equation are plotted.

Example 4.2. *Consider the following linear fractional relaxation-oscillation differential equation with initial value*

$$\mathrm{T}_\alpha u\left(t\right) = 1 + u\left(t\right), 0 \le t, 0 < \alpha \le 1, u\left(0\right) = 0. \qquad (4.5)$$

The exact solution of Eq. (4.4), is $u\left(t\right) = \exp\left(\frac{1}{\alpha}t^\alpha\right) - 1$.
According to the proposed conformable fractional optimal homotopy asymptotic method, we have

$$\left(1 - p\right)\left[\mathrm{T}_\alpha\left(v\left(t, p\right)\right) - 1\right] - H\left(p\right)\left[\mathrm{T}_\alpha\left(\nu\left(t, p\right)\right) - \nu\left(t, p\right) - 1\right] = 0. \qquad (4.6)$$

Substituting Eqs. (3.4) and (3.5) into Eq. (4.6) and setting to zero the coefficient of like powers of p, we obtain

$$p^0; \mathrm{T}_\alpha\left(u_0\left(t\right)\right) - 1 = 0, u_0\left(0\right) = 0,$$
$$p^1; \mathrm{T}_\alpha\left(u_1\left(t, c_1\right)\right) - \mathrm{T}_\alpha\left(u_0\left(t\right)\right) + 1 - c_1\left[\mathrm{T}_\alpha\left(u_0\left(t\right)\right) - u_0\left(t\right) - 1\right] = 0, u_1\left(0\right) = 0,$$
$$p^2; \mathrm{T}_\alpha\left(u_2\left(t, c_1, c_2\right)\right) - \mathrm{T}_\alpha\left(u_1\left(t, c_1\right)\right) - c_2\left[\mathrm{T}_\alpha\left(u_0\left(t\right)\right) - u_0\left(t\right) - 1\right]$$
$$-c_1\left[\mathrm{T}_\alpha\left(u_1\left(t, c_1\right)\right) - u_1\left(t, c_1\right)\right] = 0, u_2\left(0\right) = 0, \qquad (4.7)$$
$$p^3; \mathrm{T}_\alpha\left(u_3\left(t, c_1, c_2, c_3\right)\right) - \mathrm{T}_\alpha\left(u_2\left(t, c_1, c_2\right)\right) - c_3\left[\mathrm{T}_\alpha\left(u_0\left(t\right)\right) - u_0\left(t\right) - 1\right]$$
$$-c_2\left[\mathrm{T}_\alpha\left(u_1\left(t, c_1\right)\right) - u_1\left(t, c_1\right)\right] - c_1\left[\mathrm{T}_\alpha\left(u_2\left(t, c_1, c_2\right)\right) - u_2\left(t, c_1, c_2\right)\right] = 0, u_3\left(0\right) = 0,$$

$$\vdots$$

Matching solution of conformable equations (4.7), are

$$u_0(t) = \frac{1}{\alpha}t^\alpha,$$

$$u_1(t, c_1) = -\frac{1}{2}c_1\left(\frac{1}{\alpha}x^\alpha\right)^2,$$

$$u_2(t, c_1, c_2) = -\frac{1}{2}c_1^2\left(\frac{1}{\alpha}t^\alpha\right)^2 - \frac{1}{2}c_1\left(\frac{1}{\alpha}t^\alpha\right)^2 + \frac{1}{6}c_1^2\left(\frac{1}{\alpha}t^\alpha\right)^3 - \frac{1}{2}c_2\left(\frac{1}{\alpha}t^\alpha\right)^2,$$

$$u_3(t, c_1, c_2, c_3) = -\frac{1}{2}c_1^3\left(\frac{1}{\alpha}t^\alpha\right)^2 + \frac{1}{3}c_1^3\left(\frac{1}{\alpha}t^\alpha\right)^3 - c_1^2\left(\frac{1}{\alpha}t^\alpha\right)^2 - c_1 c_2\left(\frac{1}{\alpha}t^\alpha\right)^2$$

$$-\frac{1}{24}c_1^3\left(\frac{1}{\alpha}t^\alpha\right)^4 + \frac{1}{3}c_1^2\left(\frac{1}{\alpha}t^\alpha\right)^3 + \frac{1}{3}c_1 c_2\left(\frac{1}{\alpha}t^\alpha\right)^3 - \frac{1}{2}c_2\left(\frac{1}{\alpha}t^\alpha\right)^2 - \frac{1}{2}c_3\left(\frac{1}{\alpha}x^\alpha\right)^2,$$

$$\vdots$$

Thus, third-terms approximation to the solution of Eq. (4.5), will be obtained as the following

$$u^3(t, c_1, c_2, c_3) = \left(\frac{1}{\alpha}t^\alpha\right) - 1.5c_1\left(\frac{1}{\alpha}t^\alpha\right)^2 - 1.5c_1^2\left(\frac{1}{\alpha}t^\alpha\right)^2 + 0.5c_1^2\left(\frac{1}{\alpha}t^\alpha\right)^3 - c_2\left(\frac{1}{\alpha}t^\alpha\right)^2$$

$$-0.5000000001c_1^3\left(\frac{1}{\alpha}t^\alpha\right)^2 + 0.3333333334c_1^3\left(\frac{1}{\alpha}t^\alpha\right)^3 - c_1 c_2\left(\frac{1}{\alpha}t^\alpha\right)^2 \qquad (4.8)$$

$$-0.04166666668c_1^3\left(\frac{1}{\alpha}t^\alpha\right)^4 + 0.333333334c_1 c_2\left(\frac{1}{\alpha}t^\alpha\right)^3 - 0.5000000001c_3\left(\frac{1}{\alpha}t^\alpha\right)^2.$$

Table 4.1, shows the optimal values of the convergence control constants c_1, c_2 and c_3 in $u^3(t, c_1, c_2, c_3)$ given in Eq. (4.8) for different values of which can be obtained using the procedure mentioned in (3.9) up to (3.11). In Figures 4.2, the an exact and approximate solutions of fractional Relaxation-Oscillation equation for $\alpha = 0.4$, 0.6, 0.8, and 1.0 are plotted.

Example 4.3. *Consider the following conformable fractional Riccati differential equation with initial value*

$$T_\alpha u(t) = 1 - u^2(t), \quad 0 \le t \le 1, \quad 0 < \alpha \le 1, \ u(0) = 0, \qquad (4.9)$$

where $u(t) = \frac{\exp\left(\frac{2}{\alpha}t^\alpha\right) - 1}{\exp\left(\frac{2}{\alpha}t^\alpha\right) + 1}$.
Consistent with the conformable fractional homotopy asymptotic method, we obtain

$$(1 - p)\left[T_\alpha(v(t, p)) - 1\right] - H(p)\left[T_\alpha(\nu(t, p)) + \nu^2(t, p) - 1\right] = 0. \qquad (4.10)$$

Substituting Eqs. (3.4) and (3.5) into Eq. (4.10) and setting to zero the coefficient, we reads

$$p^0; T_\alpha(u_0(t)) - 1 = 0, \quad u_0(0) = 0,$$

$$p^1; T_\alpha(u_1(t, c_1)) - T_\alpha(u_0(t)) + 1 - c_1\left[T_\alpha(u_0(t)) + u_0^2(t) - 1\right] = 0, \quad u_1(0) = 0,$$

$$p^2; T_\alpha(u_2(t, c_1, c_2)) - T_\alpha(u_1(t, c_1)) - c_2\left[T_\alpha(u_0(t)) - u_0^2(t) - 1\right] \qquad (4.11)$$

$$-c_1\left[T_\alpha(u_1(t, c_1)) + 2u_0(t)u_1(t, c_1)\right] = 0, \quad u_2(0) = 0,$$

$$\vdots$$

Corresponding solution of system of equations (4.11), are

$$u_0\left(t\right) = \tfrac{1}{\alpha}t^{\alpha},$$

$$u_1\left(t, c_1\right) = \tfrac{1}{3}c_1\left(\tfrac{1}{\alpha}t^{\alpha}\right)^3,$$

$$u_2\left(t, c_1, c_2\right) = \tfrac{1}{3}\left(\tfrac{1}{\alpha}t^{\alpha}\right)^3\left(c_1^2 + c_1 + c_2\right) + \tfrac{2}{15}\left(\tfrac{1}{\alpha}t^{\alpha}\right)^5 c_1^2,$$

$$\vdots$$

Two-terms approximation to the solution of Eq. (4.9), will be obtained as follows

$$u^2\left(t, c_1, c_2\right) = \left(\tfrac{1}{\alpha}t^{\alpha} + \left(0.6666666667c_1 + 0.3333333333c_1^2\right)\right. \tag{4.12}$$

$$\left. + 0.3333333333c_2\right)\left(\tfrac{1}{\alpha}t^{\alpha}\right)^3 + 0.1333333333c_1^2\left(\tfrac{1}{\alpha}t^{\alpha}\right)^5.$$

Table 4.2, shows the optimal values of the convergence control constants c_1, c_2 and c_3 in $u^2(t, c_1, c_2)$ given in Eq. (4.12) for different values of α which can be obtained using the procedure mentioned in (3.9) up to (3.11). In Figure 4.3, the exact and approximate solutions of conformable fractional Riccati equation for $\alpha = 0.4, 0.6, 0.8$, and 1.0 are plotted.

Example 4.4. *Consider the following fractional Riccati differential equation with initial value*

$$\mathrm{T}_{\alpha}u\left(t\right) = 2u\left(t\right) - u^2\left(t\right) + 1 = 0, 0 \le t \le 1, 0 < \alpha \le 1, u\left(0\right) = 0. \tag{4.13}$$

The exact solution of Eq. (4.13), is $u\left(t\right) = \dfrac{\exp\left(\frac{2\sqrt{2}}{\alpha}t^{\alpha}\right) - 1}{\left(\sqrt{2}+1\right) + \left(\sqrt{2}-1\right)\exp\left(\frac{2\sqrt{2}}{\alpha}t^{\alpha}\right)}$.

By the proposed conformable fractional OHAM approach, we gives

$$\left(1 - p\right)\left[\mathrm{T}_{\alpha}\left(v\left(t, p\right)\right) - 1\right] - H\left(p\right)\left[\mathrm{T}_{\alpha}\left(\nu\left(t, p\right)\right) - 2\nu\left(t, p\right) + \nu^2\left(t, p\right) - 1\right] = 0. \tag{4.14}$$

Substituting Eqs. (3.4) and (3.5) into Eq. (4.14) and setting to zero the coefficient p, we reads

$$p^0; \mathrm{T}_{\alpha}\left(u_0\left(t\right)\right) - 1 = 0, \quad u_0\left(0\right) = 0,$$

$$p^1; \mathrm{T}_{\alpha}\left(u_1\left(t, c_1\right)\right) - \mathrm{T}_{\alpha}\left(u_0\left(t\right)\right) + 1 - c_1\left[\mathrm{T}_{\alpha}\left(u_0\left(t\right)\right) + u_0^2\left(t\right) - 2u_0\left(t\right) - 1\right] = 0, u_1\left(0\right) = 0,$$

$$p^2; \mathrm{T}_{\alpha}\left(u_2\left(t, c_1, c_2\right)\right) - \mathrm{T}_{\alpha}\left(u_1\left(t, c_1\right)\right) - c_2\left[\mathrm{T}_{\alpha}\left(u_0\left(t\right)\right) + u_0^2\left(t\right) - 2u_0\left(t\right) - 1\right] \tag{4.15}$$

$$- c_1\left[\mathrm{T}_{\alpha}\left(u_1\left(t, c_1\right)\right) - 2u_1\left(t, c_1\right) + 2u_0\left(t\right)u_1\left(t, c_1\right)\right] = 0, u_2\left(0\right) = 0,$$

$$\vdots$$

Matching solution of Eqs. (4.15) are

$$u_0\left(t\right) = \tfrac{1}{\alpha}t^{\alpha},$$

$$u_1\left(t, c_1\right) = c_1\left[\tfrac{1}{3}\left(\tfrac{1}{\alpha}t^{\alpha}\right)^3 - \left(\tfrac{1}{\alpha}x^{\alpha}\right)^2\right],$$

$$u_2\left(t, c_1, c_2\right) = -\left(c_1^2 + c_1 + c_2\right)\left(\tfrac{1}{\alpha}x^{\alpha}\right)^2 + \left(\tfrac{1}{3}c_2 - \tfrac{2}{3}c_1^2 + \tfrac{1}{3}c_1\right)\left(\tfrac{1}{\alpha}t^{\alpha}\right)^3 + \tfrac{2}{15}c_1^2\left(\tfrac{1}{\alpha}t^{\alpha}\right)^5,$$

$$\vdots$$

Consequently, two-terms approximation to the solution of Eq. (4.13), will be obtained as the following

$$u^2(t, c_1, c_2) = \left(\frac{1}{\alpha}t^\alpha\right) - \left[2c_1 + 0.9999999999c_1^2 + 0.9999999999c_2\right]\left(\frac{1}{\alpha}t^\alpha\right)^2$$

$$+ \left[0.6666666666c_1 + 0.9999999999c_1^2 + 0.3333333333c_2\right]\left(\frac{1}{\alpha}t^\alpha\right)^3$$

$$-0.6666666666c_1^2\left(\frac{1}{\alpha}t^\alpha\right)^4 + 0.1333333333c_1^2\left(\frac{1}{\alpha}t^\alpha\right)^5. \tag{4.16}$$

Table 4.3, shows the optimal values of the convergence control constants c_1, c_2 and c_3 in $u^2(t, c_1, c_2)$ given in Eq. (4.16) for different values of α which can be obtained using the procedure mentioned in (3.9) up to (3.11). In Figure 4.4, the exact solution and solution of conformable fractional OHAM of fractional Riccati equation, for $\alpha = 0.4$, 0.6, 0.8, and 1.0 are plotted.

5. Conclusion

In this paper, optimal homotopy asymptotic method is applied to obtain an approximate solution of fractional differential equations. Conformable fractional derivatives are used for fractional derivative in this study. In comparison, of the results of applying conformable fractional derivatives with the results reported in (see [21, 22]), one learns that, CFD is a simple tool to find an approximate solution to a linear and a nonlinear fractional differential equation. What can one learn from the plots: The closer values of to , the larger convergence interval. To show the effectiveness of the method, fractional relaxation-oscillation equation, and Riccati fractional differential equation have been solved by conformable fractional optimal homotopy asymptotic method.

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Figure 4.1. The exact and approximate solutions of fractional relaxation-oscillation (4.1) equation are plotted.
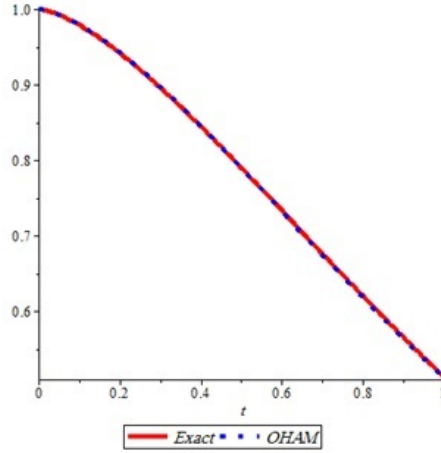

FIGURE 1. 2nd-order approximation of OHAM and Exact solution for Example 4.1.

Figures 4.2. The an exact and approximate solutions of fractional relaxation-oscillation equation for and are plotted.
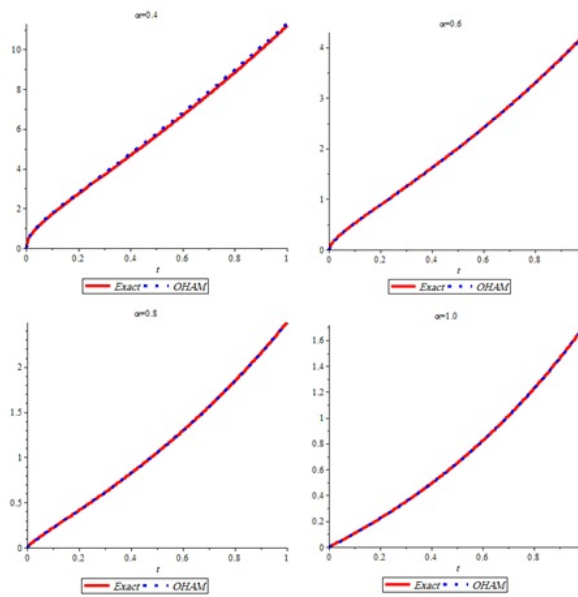

FIGURE 2. 3nd-order approximation of OHAM and exact solution for Example 4.2.

Figures 4.3. The exact and approximate solutions of conformable fractional Riccati equation for and are plotted.
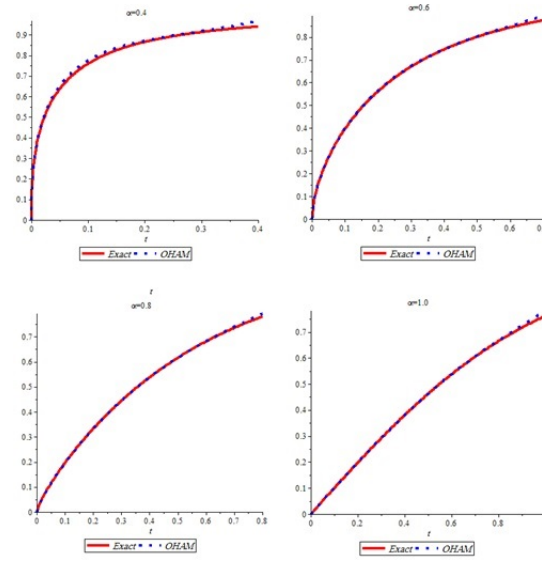
FIGURE 3. 2nd-order approximation of OHAM and exact solution
for Example 4.3.

Figures 4.4. The exact solution and solution of conformable fractional OHAM
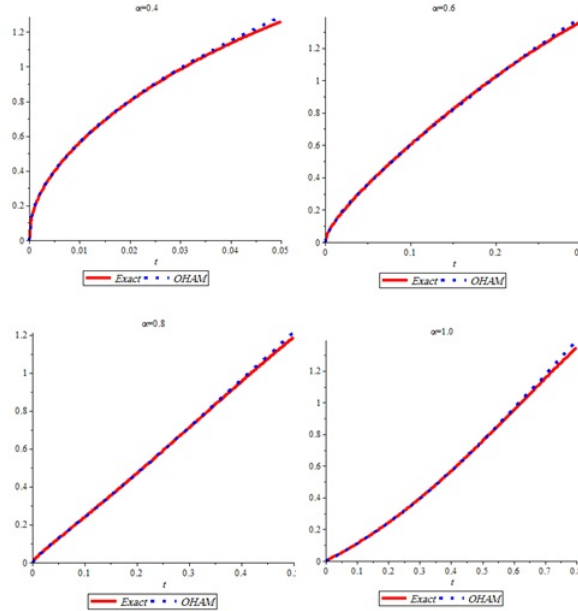of fractional Riccati equation, for and are plotted.

FIGURE 4. 2nd-order approximation of OHAM and exact solution
for Example 4.4.

TABLE 4.1. Values of auxiliary parameters for the third-order OHAM solution of Eq. (4.5) for different orders.

Auxiliary parameters	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
c_1	-1.457903741	-1.208867087	-1.208867087	-1.155095209
c_2	0.1110279352	0.01085942753	0.01085942753	0.003733184279
c_3	-0.1200484251	0.001321301525	0.001321301525	0.001571139761

TABLE 4.2. Values of auxiliary parameters for the third-order OHAM solution of Eq. (4.9) for different orders.

Auxiliary parameters	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
c_1	-0.5315048975	0.6936479614	0.8425060981	-0.9175405542
c_2	0.0002051308039	-2.773945592	-3.369657214	0.0001113528473

TABLE 4.3. Values of auxiliary parameters for the third-order OHAM solution of Eq. (4.9) for different orders.

Auxiliary parameters	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
c_1	0.7431673291	-0.6503249572	-1.046574182	-1.0621151875
c_2	-3.154368233	-0.2366912237	-0.001518446383	0.002147348534

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