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# EXISTENCE AND UNIQUENESS OF SOLUTION FOR SOME TWO-POINT BOUNDARY VALUE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we will prove the existence and uniqueness of solution for some two-point boundary value fractional differential equations, using the Banach mapping principle. Equally, we will exhibit the stability property of our problem in the sense of Hyers-Ulam-Rassias.

## 1. Introduction

In the recent years, the study of fractional differential equations has been in the limelight by many researchers in the areas of applied sciences, such as engineering, physics, biology and economics. This is basically because, it finds applications in several real world problems. For details on the theory and some applications of fractional differential equations, see the monographs of [[5],[13], [7], [3]]. In the qualitative theory of (classical and fractional) differential equations, various theorems have been extensively deployed by researchers in establishing the existence and uniqueness of solutions to both the initial and boundary value problems. For a detailed study on the existence and uniqueness of solutions of fractional equations , ([see [6],[10],[2],[9],[4]]) and the references therein.

On the one hand, Hyers-Ulam-Rassias stability properties of all kinds of equations have continued to hold sway in the literature. According to Jung[[11]], what is today known as Hyers-Ulam-Rassias stability originated in the fall of 1940, when Ulam proposed a number of problems, of which one of them was on the stability of homomorphisms. This was brilliantly answered by Hyers using functional equations. This attracted a number of research works in the area of stability of functional equations such as classical and fractional differential equations. Some of these works can be seen in the papers of Miura et al[[14]], Abbas[[8]], Wang et al[[12]], and a host of other research papers too numerous to mention here.

In their papers titled, "New concepts and results in stability of fractional differential equations, Wang et al[[15]] introduced some new concepts in stability of

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fractional differential equations from different perspectives. Relying on some fixed point theorems in a generalized complete metric space, they were able to prove that a given nonlinear Caputo fractional derivative of order $\alpha \in (0,1)$ has Hyers-Ulam Rassis stability as well as Hyers Ulam stability. Equally, Wang and Xu [[16]] studied the Hyers-Ulam stability of two types of linear fractional differential equations with the Caputo fractional derivatives. By applying Laplace transform method, they were able to show that the two types of equations has Hyers-Ulam stability.

To the best of our knowledge, there has been very few works on the Hyers-Ulam-Rassis stability as well as Hyers-Ulam stability of nonlinear two-point boundary value fractional differential equations with the Riemann-Liouville fractional derivatives. This will be the focus of our paper. We will employ the Banach contaction mapping principle on a given metric space to prove that our type of equation has a unique solution and also show that our problem is stable in the sense of Hyers-Ulam Rassis.

Our work is organized as follows. Some tools of fractional calculus, definitions of terms and other preliminary facts will be introduced in section 2. In section three, we present an existence and uniqueness result for our problem using a fixed point approach. Lastly, in section 4, we are going to prove that our problem has a Hyers-Ulam Rassis stability.

## 2. Preliminaries

In this section, we introduce our problem, some tools of fractional calculus, some notations, definitions, and preliminary facts used in the entire work. First, let us consider the following fractional boundary value problem:

$$D^{\alpha}_{a_+}x(t) + k\, D^{\beta}_{a_+}x(t) + g(t, x(t)) = h(t),\ t \in [a,b], \tag{1}$$

$$D^{\alpha-1}_{a_+}x(a_+) = D^{\alpha-1}_{a_+}x(b_-), \tag{2}$$

$$I^{2-\alpha}_{a_+}x(a_+) = I^{2-\alpha}_{a_+}x(b_-), \tag{3}$$

$$I^{1-\beta}_{a_+}x(a_+) = I^{1-\beta}_{a_+}x(b_-), \tag{4}$$

in the space of $W^{\alpha,\beta}(a,b)$, where $0 < \beta < 1 < \alpha < 2$, $k$ is a positive constant; $D^{\alpha}_{a_+}x(t)$ is the Riemann-Liouville fractional derivative of $x$, of order $\alpha$, $D^{\alpha-1}_{a_+}x(a_+) := \lim_{t\longrightarrow a^+} D^{\alpha}_{a_+}x(t)$, $h \in L^{\frac{1}{\beta}}(a,b)$, $g$ is an $L_\infty-$ Carathéodory function.

$$W^{\alpha,\beta}(a,b) = \Big\{x \in C^{2-\alpha}[a,b] : D^{\alpha}_{a_+}x(t) \in L^{\frac{1}{\beta}}(a,b), \Big\}$$

and $C^{2-\alpha}[a,b] = \big\{x : x(t-a)^{2-\alpha} \in C^0[a,b]\big\}$.

**Definition 2.1.** [[5]]The Riemann-Liouville fractional integral of a function $x$, of order $\gamma > 0$, with lower limit $a$, is defined as

$$I^{\gamma}_{a_+}x(t) = \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}x(s)ds, \tag{5}$$

while the Riemann-Liouvile fractional derivative of a function $x$, of order $\gamma$, with lower limit as a real number $a$, is defined as

$$D^{\gamma}_{a_+}x(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\gamma-1}x(s)ds,\ n = [\gamma]+1. \tag{6}$$

It is worthy of note from (5) and (6), that

$$D^{\gamma}_{a_+}x(t) = \frac{d^n}{dt^n} I^{n-\gamma}_{a_+}x(t). \tag{7}$$

**Definition 2.2.**[[13]] A two-parameter Mittag-Leffler function of $z \in \mathbb{C}$, denoted by $E_{\alpha,\beta}(z)$ is defined as

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)},$$

where $\alpha > 0,\ \beta > 0$.

**Definition 2.3.**[[**?**]] The boundary value problem (1)-(4) is said to have Hyers-Ulam stability if there exists a real number $\delta(\epsilon) > 0$, such that for each $\epsilon > 0$ and a function $x \in W^{\alpha,\beta}(a,b)$, with

$$|D^{\alpha}_{a_+}x(t) + k\ D^{\beta}_{a_+}x(t) + g(t,x(t)) - h(t)| \leq \epsilon, \tag{8}$$

there exists a solution $y \in W^{\alpha,\beta}(a,b)$ of the differential equation (1) such that

$$|x(t) - y(t)| < \delta(\epsilon)$$

and $\lim_{\epsilon \longrightarrow 0} \delta(\epsilon) = 0$. If this statement is also true when we replace $\epsilon$ and $\delta(\epsilon)$ respectively by $F, C : [a,b] \longrightarrow [0,\infty)$, where $F, C$ are appropriate functions not depending on $x$ and $y$ explicitly, then we say that the boundary value problem has the Hyers-Ulam-Rassias stability.

**Definition 2.4.** A function $f : [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a Carathéodory function if it satisfies the following conditions

- $f(t,x)$ is Lebesgue measurable with respect to $t$ in $[a,b]$,
- $f(t,x)$ is continuous with respect to $x$ on $\mathbb{R}$

A function $f(t,x)$ defined on $[a,b] \times \mathbb{R}$ is said to be an $L^p-$ Carathéodory function, $p \geq 1$, if it is a Carathéodory function and $\forall r > 0$, there exists $h_r \in L^p(a,b)$, such that $\forall x \in [-r,r]$ and $\forall t \in [a,b]$, then $f(t,x) \leq h_r(t)$.

**Lemma 2.5.**[[5]] The space $AC^n[a,b]$ consists of those and only function $f$, which can be represented in the form

$$f(x) = I^n_{a_+}\varphi(x) + \sum_{k=0}^{n-1} c_k(x-a)^k, \tag{9}$$

where $\varphi \in L_1(a,b),\ c_k(k = 0,1,2,\cdots,n-1)$ are arbitrary constants.

**Lemma 2.6.**[[5]] If $f \in L_1(a,b)$ and $I^{n-\alpha}_{a_+}f(t) \in AC^n[a,b]$, then the equality

$$I^{\alpha}_{a_+}(D^{\alpha}_{a_+}) = f(x) - \sum_{j=1}^{n} \frac{D^{\alpha-j}_{a_+}f(a_+)(x-a)^{\alpha-j}}{\Gamma(\alpha - j + 1)}. \tag{10}$$

A particular case, where $0 < \beta < 1 < \alpha < 2$, we have that

$$\begin{aligned} I^{\alpha}_{a_+}D^{\beta}_{a_+}x(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} D^{\beta}_{a_+}x(s)ds \\ &= \frac{1}{\Gamma(\alpha+1)} \frac{d}{dt} \int_a^t (t-s)^{\alpha} D^{\beta}_{a_+}x(s)ds \end{aligned}$$

But,

$$\frac{1}{\Gamma(\alpha+1)}\int_a^t (t-s)^\alpha D_{a_+}^{\beta} x(s)ds = \frac{1}{\Gamma(\alpha+1)}\int_a^t (t-s)^\alpha \frac{d}{ds} I_{a_+}^{1-\beta} x(s)ds$$

Using integration by parts on the right-hand side, one obtains

$$\begin{aligned}\frac{1}{\Gamma(\alpha+1)}\int_a^t (t-s)^\alpha \frac{d}{ds} I_{a_+}^{1-\beta} x(s)ds &= \frac{1}{\Gamma(\alpha+1)}(t-s)^\alpha I_{a_+}^{1-\beta} x(s)\mid_a^t + \frac{\alpha}{\Gamma(\alpha+1)}\int_a^t (t-s)^{\alpha-1} I_{a_+}^{1-\beta} x(s)ds \\ &= -\frac{1}{\Gamma(\alpha+1)}(t-a)^\alpha I_{a_+}^{1-\beta} x(a_+) + I_{a_+}^{\alpha} I_{a_+}^{1-\beta} x(t).\end{aligned}$$

Therefore,

$$\begin{aligned}I_{a_+}^{\alpha} D_{a_+}^{\beta} x(t) &= \frac{d}{dt}\{-\frac{1}{\Gamma(\alpha+1)}(t-a)^\alpha I_{a_+}^{1-\beta} x(a_+) + I_{a_+}^{\alpha} I_{a_+}^{1-\beta} x(t)\} \\ &= -\frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1} I_{a_+}^{1-\beta} x(a_+) + I_{a_+}^{\alpha-1} I_{a_+}^{1-\beta} x(t) \\ &= I_{a_+}^{\alpha-\beta} x(t) - \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1} I_{a_+}^{1-\beta} x(a_+)\end{aligned}$$

This implies that

$$I_{a_+}^{\alpha} D_{a_+}^{\beta} x(t) = I_{a_+}^{\alpha-\beta} x(t) - \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1} I_{a_+}^{1-\beta} x(a_+). \tag{11}$$

**Lemma 2.7.** If $x \in W^{\alpha,\beta}(a,b)$, then $x$ satisfies the equations (1)- (4) if, and only if $x$ satisfies the integral equation

$$\begin{aligned}x(t) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} D_{a_+}^{\alpha-1} x(b_-) - \frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)} I_{a_+}^{2-\alpha} x(a_+) + k I_{a_+}^{\alpha-\beta} x(t) - k\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} I_{a_+}^{1-\beta} x(b_-) \\ = I_{a_+}^{\alpha}[h(t) - g(t,x(t))].\end{aligned}$$

We note that the above equation is the Volterra-integral equation associated to equations (1)- (4).

**Proof:** We first prove the necessity. Let $x \in W^{\alpha,\beta}(a,b)$, satisfies equations (1)-(4), we show that $x$ satisfies the above volterra-integral equation. Then by definition of $W^{\alpha,\beta}(a,b), D_{a_+}^{\alpha} x(t) \in L^{\frac{1}{\beta}}(a,b)$. Equally, $D_{a_+}^{\beta} x(t), g(t,x(t)) \in L^{\frac{1}{\beta}}(a,b) \subset L_1(a,b)$. From (7), we have that

$$D_{a_+}^{\alpha} x(t) = \frac{d^2}{dt^2} I_{a_+}^{2-\alpha} x(t). \tag{12}$$

By Lemma 2.5, $I_{a_+}^{2-\alpha} x(t) \in AC^2[a,b]$. Thus, we can apply Lemma 2.6 . With this, we operate $I_{a_+}^{\alpha}$ to both sides of (1), i.e.,

$$I_{a_+}^{\alpha}\left(D_{a_+}^{\alpha} x(t) + k\, D_{a_+}^{\beta} x(t) + g(t,x(t)) = h(t)\right).$$

Making use of equations (10) and (11),we have

$$x(t) - \sum_{j=1}^{2} \frac{D_{a_+}^{\alpha-j} x(a_+)(x-a)^{\alpha-j}}{\Gamma(\alpha-j+1)} + k\left(k I_{a_+}^{\alpha-\beta} x(t) - \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1} I_{a_+}^{1-\beta} x(a_+)\right) +$$

$$I_{a_+}^{\alpha}\left[h(t) - g(t,x(t))\right]$$

Considering our boundary conditions (2)-(4), then

$$x(t)-\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}D_{a_+}^{\alpha-1}x(b_-)-\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)}I_{a_+}^{2-\alpha}x(a_+)+kI_{a_+}^{\alpha-\beta}x(t)-k\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}I_{a_+}^{1-\beta}x(b_-)$$
$$=I_{a_+}^{\alpha}[h(t)-g(t,x(t))].$$

This proves the necessity.

To prove the sufficiency, supposing that $x\in W^{\alpha,\beta}(a,b)$, satisfies a.e., the Volterra-integral equation above, then applying the operator $D_{a_+}^{\alpha}$ to it, we have

$$D_{a_+}^{\alpha}(x(t)-\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}D_{a_+}^{\alpha-1}x(b_-)-\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)}I_{a_+}^{2-\alpha}x(a_+)+kI_{a_+}^{\alpha-\beta}x(t)-k\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}I_{a_+}^{1-\beta}x(b_-)$$
$$=I_{a_+}^{\alpha}[h(t)-g(t,x(t))]).$$

This establishes (1). Equally, applying $D_{a_+}^{\alpha-1}$ to the volterra equation, we have

$$D_{a_+}^{\alpha-1}x(t)-D_{a_+}^{\alpha-1}x(b_-)+kI_{a_+}^{1-\beta}x(t)-kI_{a_+}^{1-\beta}x(b_-)=I[h(t)-g(t,x(t))].$$

Taking the limit as $t\longrightarrow a_+$, we obtain that

$$D_{a_+}^{\alpha-1}x(a_+)=D_{a_+}^{\alpha-1}x(b_-).$$

Similarly, applying $I_{a_+}^{2-\alpha}$, $I_{a_+}^{1-\beta}$ respectively to the Volterra-integral equation, we obtain equations (3) and (4) respectively. Thus, our lemma is proved.

Lastly, we introduce a fundamental results of Banach fixed point that will be used in our main result.

**Theorem 2.8.**[[16]]((Banach's contraction principle) Let (X,d) be a complete metric space, and consider a mapping $J:X\longrightarrow X$, which is strictly contractive, i.e.,

$$d(Jx,Jy)\le Ld(x,y),\ \forall x,y\in X,$$

for some (Lipschitz constant) $L<1$. Then,

(1) The mapping $J$ has one, and only one, fixed point $x^\star=J(x^\star)$;
(2) the fixed point $x^\star$ is globally attractive, i.e.,

$$\lim_{n\longrightarrow\infty}J^nx=x^\star;$$

(3)

$$d(J^nx,x^\star)\le L^nd(x,x^\star),\ \forall n\ge 0,\ \forall x\in X;$$

(4)

$$d(J^nx^n,x^\star)\le\frac{1}{1-L}d(J^nx,J^{n+1}x),\ \forall n\ge 0,\ \forall x\in X;$$

(5)

$$d(x,x^\star)\le\frac{1}{1-L}d(x,Tx),\ \forall x\in X$$

## 3. Existence Results

In this section, using Banach contraction principle, we will prove that the fractional boundary value problem (1)- (4) has a unique solution.

**Theorem 3.1.** Assume that $\partial_s g$ with respect to the second variable is an $L^\infty-$ Carathéodory function, and suppose also that there are positive constants $C, k^\star M, L$ such that

$$\frac{1}{\Gamma(\alpha-\beta)}\int_a^t (t-s)^{2\alpha-3}R(s)ds \le k^\star R(t), \tag{13}$$

and $(k+M)(b-a)^{2-\alpha} < \frac{1}{k^\star}$, with $R:[a,b]\longrightarrow(0,\infty)$. Then equations (1) - (4) has a unique solution in $W^{\alpha,\beta}(a,b)$.

**Proof:** Define a metric $d$ on $W^{\alpha,\beta}(a,b)$ as follows:

$$d(x,y) = \inf\left\{C\in[0,\infty) : (t-a)^{2-\alpha}|x(t)-y(t)| \le CR(t).\right\} \tag{14}$$

We claim that $(W^{\alpha,\beta}(a,b), d)$ is complete. To verify our claim, let $x_n$ be a Cauchy sequence in $(W^{\alpha,\beta}(a,b), d)$. Then, by definition, $x_n \in C^{2-\alpha}[a,b]$ such that $x_n(t-a)^{2-\alpha} \in C^0[a,b]$. This implies that $x_n \in C^0[a,b]$. Since $C^0[a,b]$ is complete, it follows that $x_n$ converges to a point $x\in C^0[a,b]$, which implies that $x(t-a)^{2-\alpha}$ is continuous. Therefore, $x\in C^{2-\alpha}[a,b]$, which in turn implies that $x\in W^{\alpha,\beta}(a,b)$. Hence, $W^{\alpha,\beta}(a,b)$ is complete.

Next, we define an operator $T : W^{\alpha,\beta}(a,b)\longrightarrow W^{\alpha,\beta}(a,b)$ by

$$Tx(t) = \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}D^{\alpha-1}_{a_+}x(b_-) + \frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)}I^{2-\alpha}_{a_+}x(a_+) - kI^{\alpha-\beta}_{a_+}x(t) + k\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}I^{1-\beta}_{a_+}x(b_-)$$
$$+I^{\alpha}_{a_+}[h(t)-g(t,x(t))],$$

It is easy to see that $T$ is well-defined. We go ahead to show that $T$ is strictly contractive. For arbitrary $x,y\in W^{\alpha,\beta}(a,b)$, we pick any constant $C_{xy}\in[0,\infty)$ such that $d(x,y)\le C_{xy}$. That is to say from equation (14) that,

$$(t-a)^{2-\alpha}|x(t)-y(t)| \le CR(t).$$

Now,

$$(t-a)^{2-\alpha}|Tx(t)-Ty(t)| \le \frac{k(t-a)^{2-\alpha}}{\Gamma(\alpha-\beta)}\int_a^t (t-s)^{\alpha-\beta-1}|x(s)-y(s)|ds$$
$$+\frac{(t-a)^{2-\alpha}}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|g(s,x(s))-g(s,y(s))|ds$$

We observe that since $g$ and $\partial_s g$, with respect to its second variable are $L^\infty-$ Caratheódory functions, there exists a positive constant $M\in L^\infty$ such that

$$|g(t,x(t))-g(t,y(t))| \le M|x(t)-y(t)|.$$

Therefore,

$$(t-a)^{2-\alpha}|Tx(t)-Ty(t)| \le \frac{k(t-a)^{2-\alpha}}{\Gamma(\alpha-\beta)}\int_a^t (t-s)^{\alpha-1}|x(s)-y(s)|ds$$
$$+\frac{M(t-a)^{2-\alpha}}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}|x(s)-y(s)|ds$$

$$\leq (k+M)(t-a)^{2-\alpha}\frac{1}{\Gamma(\alpha-\beta)}\int_a^t (t-s)^{2-\alpha}(t-s)^{2\alpha-3}|x(s)-y(s)|ds.$$

From the definition of our metric in (14) and equation (13), we have that

$$\begin{aligned}(t-a)^{2-\alpha}|Tx(t)-Ty(t)| &\leq (k+M)(t-a)^{2-\alpha}\frac{1}{\Gamma(\alpha-\beta)}\int_a^t (t-s)^{2\alpha-3}CR(s)ds\\ &\leq (k+M)Ck^\star(t-a)^{2-\alpha}R(t)\\ &\leq (k+M)Ck^\star(b-a)^{2-\alpha}R(t)\end{aligned}$$

This implies that

$$\begin{aligned}d(Tx,Ty) &\leq (k+M)Ck^\star(b-a)^{2-\alpha}R(t),\\ \Longrightarrow d(Tx,Ty) &\leq (k+M)k^\star(b-a)^{2-\alpha}d(x,y)\end{aligned}$$

Therefore,

$$d(Tx,Ty) \leq Ld(x,y),$$

with $L=(k+M)k^\star(b-a)^{2-\alpha}<1$. This shows that $T$ is strictly contractive. Hence, by Theorem 2.8.(1), $T$ has a unique fixed point $x_0 \in W^{\alpha,\beta}(a,b)$, defined by

$$x_0(t) = \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}D_{a_+}^{\alpha-1}x(b_-)+\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)}I_{a_+}^{2-\alpha}x(a_+)-kI_{a_+}^{\alpha-\beta}x_0(t)+k\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}I_{a_+}^{1-\beta}x(b_-)$$
$$+I_{a_+}^{\alpha}[h(t)-g(t,x_0(t))].$$

## 4. Hyers-Ulam-Rassias stability

In this section, we will show that our problem has a Hyers-Ulam-Rassias stability using the conclusions of Theorem 2.8.

**Theorem 4.1.** Suppose that the conditions of Theorem 3.1 hold. Let

$$|D_{a_+}^{\alpha}x(t)+k\,D_{a_+}^{\beta}x(t)+g(t,x(t))-h(t)| \leq R(t)(t-a)^{\alpha-2},$$

then there exists a unique solution $x_0 \in W^{\alpha,\beta}(a,b)$ of equations (1)-(4)such that

$$(t-a)^{2-\alpha}|x(t)-x_0(t)| \leq \frac{k^\star(b-a)^{4-2\alpha}}{1-(k+M)(b-a)^{2-\alpha}} \tag{15}$$

**Proof:** From our assumption that

$$|D_{a_+}^{\alpha}x(t)+k\,D_{a_+}^{\beta}x(t)+g(t,x(t))-h(t)| \leq R(t)(t-a)^{\alpha-2},$$

it follows that

$$-R(t)(t-a)^{\alpha-2} \leq D_{a_+}^{\alpha}x(t)+k\,D_{a_+}^{\beta}x(t)+g(t,x(t))-h(t) \leq R(t)(t-a)^{\alpha-2},\ \forall t\in[a,b].$$

Thus,

$$D_{a_+}^{\alpha}x(t)+k\,D_{a_+}^{\beta}x(t)+g(t,x(t))-h(t) \leq R(t)(t-a)^{\alpha-2}. \tag{16}$$

Operating $I_{a_+}^{\alpha}$ to both sides of the inequality (16) and making use of equations (10) and (11), we have

$$x(t)-\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}D_{a_+}^{\alpha-1}x(b_-)-\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)}I_{a_+}^{2-\alpha}x(a_+)+kI_{a_+}^{\alpha-\beta}x(t)-k\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}I_{a_+}^{1-\beta}x(b_-)$$
$$+I_{a_+}^{\alpha}[g(t,x(t))-h(t)] \leq \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}(t-a)^{2-\alpha}R(s)ds.$$

From the definition of our map $T$, the preceding inequality becomes

$$|x(t) - T(x(t))| \leq |\frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}(t-a)^{2-\alpha} R(s)ds|.$$

$$\Longrightarrow |x(t) - T(x(t))| \leq \frac{(t-a)^{2-\alpha}}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} R(s)ds$$

$$\leq \frac{(t-a)^{2-\alpha}}{\Gamma(\alpha-\beta)} \int_a^t (t-s)^{2\alpha-3} R(s)ds.$$

Thus,

$$(t-a)^{2-\alpha}|x(t) - T(x(t))| \leq \frac{(b-a)^{4-2\alpha}}{\Gamma(\alpha-\beta)} \int_a^t (t-s)^{2\alpha-3} R(s)ds.$$

From (13), we have that,

$$(t-a)^{2-\alpha}|x(t) - T(x(t))| \leq (b-a)^{4-2\alpha} k^\star R(t)$$

$$\Longrightarrow d(x, Tx) \leq (b-a)^{4-2\alpha} k^\star.$$

Now, from Theorem 3.1, we are guaranteed of an existence of a unique solution $x_0 \in W^{\alpha,\beta}(a,b)$ of equations (1)-(4). Finally, from Theorem 2.11(5), we obtain that

$$d(x, x_0) \leq \frac{1}{1-(k+M)(b-a)^{2-\alpha}} d(x, Tx)$$

$$\leq \frac{k^\star (b-a)^{4-2\alpha}}{1-(k+M)(b-a)^{2-\alpha}}.$$

Thus (15) is satisfied and our theorem is proved.

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