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EXISTENCE OF SOLUTIONS FOR HADAMARD FRACTIONAL HYBRID DIFFERENTIAL EQUATIONS WITH IMPULSIVE AND NONLOCAL CONDITIONS

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Abstract. In this paper, we study the existence of solution for Hadamard Fractional Hybrid Differential equations with impulsive and nonlocal conditions. The main result is proved by means of a fixed point theorem. Finally, an example is also presented.

1. Introduction

In the last few decades, fractional differential equations (FDE) have gained considerably more attention and attracted by many researchers in fields of such as physics, mechanics, chemistry, aerodynamics and the electrodynamics of complex media. It comes from the fact that they have been proved to be valuable tools in the mathematical models for systems. For more details on fractional calculus and fractional differential equations theory, one can refer to the monographs of Kilbas, Srivastava and Trujillo [15], Lakshmikatham et al [19], Podlubny [9], Baleanu et al [5] and the references given therein.

The quadratic perturbation of nonlinear differential equations has considered more importance and served as special cases of dynamical systems. The details of different types of perturbations for a nonlinear differential and integral equations are given in Dhage [10].

The existence of solutions for an initial-value problem of nonlinear hybrid differential equations of Hadamard type has been discussed by B.Ahmad and S.K. Ntouyas in [4]. Very recently, JinRong Wang [12] has studied the existence results for nonlinear fractional order differential impulsive systems with Hadamard derivative. in Kilbas et al. [15], proved the existence and uniqueness of the solution of Cauchy problems for fractional differential equations involving the Hadamard derivatives involving in a nonsequential setting. Klimek [18] investigated existence and uniqueness of the solution of sequential fractional differential equations with Hadamard derivative by using the contraction principle and a new, equivalent norm. Wang et al. [23] discussed the existence of solutions and UlamHyers stability of

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fractional differential equations with Hadamard derivative by using some classical methods. Further, in [2] and [20], the authors studied two dimensional fractional differential systems with Hadamard derivative. [13, 14], the authors discussed Impulsive Fractional Integro-Differential Equations.

Inspired by above works, we study in this paper, a new existence results for Hadamard fractional hybrid differential equations with impulsive and nonlocal conditions

$$
\begin{gathered}
{}_HD^{\alpha}\Big(\frac{x(t)}{f(t,x(t))}\Big) = g(t,x(t)), \quad t\in(1,T],\ \alpha\in(0,1), \qquad (1)\\
x(t_k^+) = x(t_k^-) + y_k, \quad k=1,2,...,m, \quad y_k\in\mathbb{X}\\
{}_HJ^{1-\alpha}x(1) + \eta(x) = x_0,
\end{gathered}
$$

where ${}_HD^{\alpha}$ is the Hadamard fractional derivative, $f\in C([1,T]\times\mathbb{X},\mathbb{X}\setminus\{0\})$, $g : C([1,T]\times\mathbb{X},\mathbb{X})$ and $\eta : C(C,I)\to\mathbb{X}$, ${}_HJ^{(\cdot)}$ is the Hadamard fractional integral and $x_0\in\mathbb{X}$. t_k satisfy $1=t_0<t_1<...<t_m<t_{m+1}=T$, $x(t_k^+) = \lim\limits_{\epsilon\to 0^+} x(t_k+\epsilon)$ and $x(t_k^-) = \lim\limits_{\epsilon\to 0^-} x(t_k+\epsilon)$ represent the right and left limits of $x(t)$ at $t=t_k$.

In section 2 is devoted to preliminaries facts related to the existence of solution. The proof of main results of the paper discussed in section 3. Finally, an example is illustrated in the section 4.

2. Preliminaries

Let $C([1,T],\mathbb{X})$ denote the Banach space of all continuous real-valued functions defined on $[1,T]$ with the norm $\|x\| = \sup\{|x(t)| : t\in[1,T]\}$. For $t\in[1,T]$, we define $x_r(t) = (\log t)^r x(t), r\geq 0$.

Let $C_r([1,T],\mathbb{X})$ be the space of all continuous functions x such that $x_r\in C([1,T],\mathbb{X})$ which is indeed a Banach space endowed with the norm $\|x\|_C = \sup\{(\log t)^r|x(t)| : t\in[1,T]\}$.

Let $0\leq\gamma<1$ and $C_{\gamma,\log}[1,T]$ denote the weighted space of continuous functions defined by

$$C_{\gamma,\log}[1,T] = \{g(t) : (\log t)^{\gamma}g(t)\in C[1,T],\ \|y\|_{C_{\gamma,\log}} = \|(\log t)^{\gamma}g(t)\|_C\}.$$

In the following we denote $\|y\|_{C_{\gamma,\log}}$ by $\|y\|_C$.

Definition 1[[15]]

The Hadamard fractional integral of order q for a continuous function g is defined as

$$ {}_HJ^q g(t) = \frac{1}{\Gamma(q)}\int_1^t\Big(\log\frac{t}{s}\Big)^{q-1}\frac{g(s)}{s}ds,\ q>0.$$

Definition 2[[15]]

The Hadamard derivative of fractional order q for a continuous function $g : [1,\infty)\to\mathbb{X}$ is defined as

$$ {}_HD^q g(t) = \frac{1}{\Gamma(n-q)}\Big(t\frac{d}{dt}\Big)^n\int_1^t\Big(\log\frac{t}{s}\Big)^{n-q-1}\frac{g(s)}{s}ds,$$
$$n-1<q<n,\ n=[q]+1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Lemma 1 [[15, Theorem 3.28 p.213]]

Let $\alpha > 0$, $n = -[-\alpha]$ and $0 \leq \gamma < 1$. Let G be an open set in $\mathbb{R}$ and let $f : (a, b] \times G \to \mathbb{X}$ be a function such that: $f(x, y) \in C_{\gamma,\log}[a, b]$ for any $y \in G$, then the problem

$$ {}_HD^{\alpha}y(t) = f(t, y(t)), \quad \alpha > 0, \tag{2}$$

$$ {}_HJ^{\alpha-k}y(a+) = b_k,\ b_k \in \mathbb{R}, \quad (k = 1, \ldots, n,\ n = -[-\alpha]), \tag{3}$$

satisfies the Volterra integral equation

$$y(t) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)} \Big(\log\frac{t}{a}\Big)^{\alpha-j} + \frac{1}{\Gamma(\alpha)} \int_a^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} f(s, y(s))\frac{ds}{s}, \tag{4}$$

for $t > a > 0$; i.e., $y(t) \in C_{n-\alpha,\log}[a, b]$ satisfies the relations (2)-(3) if and only if it satisfies the Volterra integral equation (4).

In particular, if $0 < \alpha \leq 1$, problem (2)-(3) is equivalent to the equation

$$y(t) = \frac{b}{\Gamma(\alpha)}\Big(\log\frac{t}{a}\Big)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} f(s, y(s))\frac{ds}{s}, \quad s > a > 0. \tag{5}$$

Further details can be found in [15]. From Lemma 1 we have the following result.

Lemma 2 [15, Theorem 3.28 p.213]

Given $y \in C([1, T], \mathbb{R})$, the integral solution of initial-value problem

$$\begin{aligned} {}_HD^{\alpha}\Big(\frac{x(t)}{f(t, x(t))}\Big) &= y(t), \quad 0 < t < 1, \\ x(t_k^+) &= x(t_k^-) + y_k, \quad k = 1, 2, ..., m, \quad y_k \in \mathbb{X} \\ {}_HJ^{1-\alpha}x(1) + \eta(x) &= x_0, \end{aligned}$$

is given by

$$x(t) = \begin{cases} f(t, x(t))\Big(\frac{x_0-\eta(x)}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{y(s)}{s}ds\Big), \ for\ t \in (1, t_1] \\ f(t, x(t))\Big(\frac{x_0-\eta(x)+y_1}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{y(s)}{s}ds\Big), \ for\ t \in (t_1, t_2] \\ f(t, x(t))\Big(\frac{x_0-\eta(x)+y_1+y_2}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{y(s)}{s}ds\Big), \ for\ t \in (t_2, t_3] \\ \vdots \\ f(t, x(t))\Big(\frac{x_0-\eta(x)+\sum\limits_{i=0}^{m} y_i}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{y(s)}{s}ds\Big), \ for\ t \in (t_m, T] \end{cases} \tag{6}$$

Proof.

Assume that x satisfies equation (1). If $t \in (1, t_1]$, then ${}_HD^{\alpha}\Big(\frac{x(t)}{f(t,x(t))}\Big) = y(t), t \in (1, t_1]$ with ${}_HJ^{1-\alpha}x(1) + \eta(x) = x_0$, By virtue of lemma 1, one can obtain

$$x(t) = f(t, x(t))\Big(\frac{x_0 - \eta(x)}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{y(s)}{s}ds\Big), \ for\ t \in (1, t_1]$$

If $t \in (t_1, t_2]$ then ${}_HD^\alpha\Big(\frac{x(t)}{f(t,x(t))}\Big) = y(t), t \in (t_1, t_2]$ with $x(t_1^+) = x(t_1^-) + y_1$. Then we have

$$x(t_1^+) = f(t, x(t))\Big(x_0 - \eta(x) + \int_0^{t_1} y(s)\frac{ds}{s} + y_1\Big)$$

. By Lemma 1, we get

$$\begin{aligned} x(t) &= f(t,x(t))\Big(\frac{x(t_1^+) - \eta(x)}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{\int_1^{t_1} y(s)\frac{ds}{s}}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}\frac{y(s)}{s}ds\Big) \\ &= f(t,x(t))\Big(\frac{x_0 - \eta(x)}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{y_1}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}\frac{y(s)}{s}ds\Big), \\ &= f(t,x(t))\Big(\frac{x_0 - \eta(x) + y_1}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}\frac{y(s)}{s}ds\Big), t \in (t_1, t_2] \end{aligned}$$

without loss of generality, for $t \in (t_i, t_{i+1}]$, $i = 1, 2, ..., m$, we get

$$x(t) = f(t,x(t))\Big(\frac{x_0 - \eta(x) + \sum_{i=1}^m y_i}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}\frac{y(s)}{s}ds\Big), t \in (t_i, t_{i+1}]$$

On the otherhand, assume that x satisfies the fractional integral equation (6). If $t \in (1, t_1]$, then ${}_HJ^{1-\alpha}x(1) + \eta(x) = x_0$:, we get ${}_HD^\alpha\Big(\frac{x(t)}{f(t,x(t))}\Big) = y(t)$. Similarly, if $t \in (t_i, t_{i+1}]$, we obtain ${}_HD^\alpha\Big(\frac{x(t)}{f(t,x(t))}\Big) = y(t)$ and $x(t_k^+) = x(t_k^-) + y_k$, $k = 1, 2, ..., m,$. This completes the proof.

Theorem 1[10]
Let S be a non-empty, closed convex and bounded subset of the Banach algebra X let $A : X \to X$ and $B : S \to X$ be two operators such that:

(a) A is Lipschitzian with a Lipschitz constant k,
(b) B is completely continuous,
(c) $x = AxBy \Rightarrow x \in S$ for all $y \in S$, and
(d) $Mk < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$.

Then the operator equation $x = AxBx$ has a solution.

3. Existence Result

We introduce the following assumptions:

(H1) the function $f : [1,T] \times \mathbb{X} \to \mathbb{X} \setminus \{0\}$ is bounded continuous and there exists a positive bounded function ϕ with bound $\|\phi\|$ such that

$$||f(t, x(t)) - f(t, y(t))|| \leq \phi(t)||x(t) - y(t)||,$$

for $t \in [1, T]$ and for all $x, y \in \mathbb{X}$;

(H2) there exist a function $p \in C([1,T], \mathbb{X}^+)$ and a continuous nondecreasing function $\Omega : [0, \infty) \to (0, \infty)$ such that

$$||g(t, x(t))|| \leq p(t)\Omega(||x||), \quad (t, x) \in [1, T] \times \mathbb{X};$$

(H3) there exists a number $r > 0$ such that

$$r \geq K\big[\sum_{i=0}^m |y_i| + \frac{|x_0 + G|}{\Gamma(\alpha)} + \log T\frac{1}{\Gamma(\alpha+1)}\|p\|\Omega(r)\big]. \tag{7}$$

where $|f(t,x)| \leq K,\ \forall (t,x) \in [1,T] \times \mathbb{X}$ and

$$\|\phi\| \big[\sum_{i=0}^{m} |y_i| + \frac{|x_0| + G}{\Gamma(\alpha)} + \log T \frac{1}{\Gamma(\alpha+1)} \|p\| \Omega(r) \big] < 1.$$

Theorem 2

The assumptions assume that $[H_1], [H_2]$ and $[H_3]$ are satisfied. Then the problem (1) has at least one solution on $[1,T]$.

Proof

Set $X = C([1,T], \mathbb{X})$ and define a subset S of X as

$$S = \{x \in X : \|x\|_C \leq r\},$$

where r satisfies inequality (7).

Clearly S is closed, convex and bounded subset of the Banach space X and $G = \sup_{x \in \mathbb{X}} |\eta(x)|$. By Lemma 2, the initial-value problem (1) is equivalent to the integral equation

$$x(t) = \begin{cases} f(t,x(t))\Big(\frac{x_0-\eta(x)}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s,x(s))}{s}ds\Big), & for\ t \in (1,t_1] \\ f(t,x(t))\Big(\frac{x_0-\eta(x)+y_1}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s,x(s))}{s}ds\Big), & for\ t \in (t_1,t_2] \\ f(t,x(t))\Big(\frac{x_0-\eta(x)+y_1+y_2}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s,x(s))}{s}ds\Big), & for\ t \in (t_2,t_3] \\ \vdots \\ f(t,x(t))\Big(\frac{x_0-\eta(x)+\sum_{i=0}^{m} y_i}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_m}^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s,x(s))}{s}ds\Big), & for\ t \in (t_m,T] \end{cases} \tag{8}$$

Define the operators $\mathcal{A} : X \to X$ and $\mathcal{B} : S \to X$

$$\mathcal{A}x(t) = f(t,x(t)), \quad t \in (t_m,T], \tag{9}$$

$$\mathcal{B}x(t) = \frac{x_0 - \eta(x) + \sum_{i=0}^{m} y_i}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_{t_m}^{t}\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s,x(s))}{s}ds, \quad t \in (t_m,T]. \tag{10}$$

Then $x = \mathcal{A}x\mathcal{B}x$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 1 For the sake of clarity, we split the proof as follows.

Step: 1

We first show that $\mathcal{A}$ is a Lipschitz on X, i.e. (a) of Theorem 1 holds.

Let $x, y \in X$. Then by (H1) we have

$$\begin{aligned} |(\log t)^{1-\alpha}\mathcal{A}x(t) - (\log t)^{1-\alpha}\mathcal{A}y(t)| &= (\log t)^{1-\alpha}|f(t,x(t)) - f(t,y(t))| \\ &\leq \phi(t)(\log t)^{1-\alpha}|x(t) - y(t)| \\ &\leq \|\phi\|\|x-y\|_C, \quad \forall t \in (t_m,T] \end{aligned}$$

Taking the supremum over the interval $(t_m,T]$, we get

$$\|\mathcal{A}x - \mathcal{A}y\|_C \leq \|\phi\|\|x-y\|_C, \quad \forall x,y \in X$$

So $\mathcal{A}$ is a Lipschitz on X with Lipschitz constant $\|\phi\|$.

Step: 2

The operator $\mathcal{B}$ is completely continuous on S, i.e. (b) of Theorem 1 holds.

First we show that $\mathcal{B}$ is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then by Lebesque dominated convergence theorem,

$$
\begin{aligned}
&\lim_{n\to\infty}(\log t)^{1-\alpha}\mathcal{B}x_n(t)\\
&= \lim_{n\to\infty}\Big(\frac{x_0-\eta(x)+\sum_{i=0}^{m} y_i}{\Gamma(\alpha)} + (\log t)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s,x_n(s))}{s}ds\Big)\\
&= \Big(\frac{x_0-\eta(x)+\sum_{i=0}^{m} y_i}{\Gamma(\alpha)} + (\log t)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{\lim_{n\to\infty} g(s,x_n(s))}{s}ds\Big)\\
&= \Big(\frac{x_0-\eta(x)+\sum_{i=0}^{m} y_i}{\Gamma(\alpha)} + (\log t)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s,x(s))}{s}ds\Big)\\
&= (\log t)^{1-\alpha}\mathcal{B}x(t), \quad \forall t\in(t_m,T]
\end{aligned}
$$

This shows that $\mathcal{B}$ is continuous os S. It is sufficient to show that $\mathcal{B}(S)$ is a uniformly bounded and equicontinuous set in X.

$$
\begin{aligned}
(\log t)^{1-\alpha}|\mathcal{B}x(t)| &= \Big|\frac{x_0-\eta(x)+\sum_{i=0}^{m} y_i}{\Gamma(\alpha)} + (\log t)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s,x(s))}{s}ds\Big|\\
&\le \Big[\frac{|x_0|+G+\sum_{i=0}^{m}|y_i|}{\Gamma(\alpha)} + \|p\|\Omega(r)(\log T)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{1}{s}ds\Big]\\
&= \frac{|x_0|+G+\sum_{i=0}^{m}|y_i|}{\Gamma(\alpha)} + (\log T)\frac{1}{\Gamma(\alpha+1)}\|p\|\Omega(r), \quad \forall t\in(t_m,T]
\end{aligned}
$$

Taking supremum over the interval $(t_m, T]$, then we have,

$$
\|\mathcal{B}x\|_C \le \frac{|x_0|+G+\sum_{i=0}^{m}|y_i|}{\Gamma(\alpha)} + (\log T)\frac{1}{\Gamma(\alpha+1)}\|p\|\Omega(r), \quad \forall x\in S.
$$

This shows that $\mathcal{B}$ is uniformly bounded on S.

Next we show that $\mathcal{B}$ is an equicontinuous set in X. Let $t_1, t_2 \in (t_m, T]$ with $t_1 < t_2$ and $x \in S$. Then we have

$$\begin{aligned}
&\left|(\log t_2)^{1-\alpha}(\mathcal{B}x)(t_2) - (\log t_1)^{1-\alpha}(\mathcal{B}x)(t_1)\right| \\
&\leq \frac{\|p\|\Omega(r) + \sum_{i=0}^{m} y_i}{\Gamma(\alpha)} \left| \int_1^{t_2} (\log t_2)^{1-\alpha} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{1}{s} ds - \int_1^{t_1} (\log t_1)^{1-\alpha} \left(\log \frac{t_1}{s}\right)^{\alpha-1} \frac{1}{s} ds \right| \\
&\leq \frac{\|p\|\Omega(r) + \sum_{i=0}^{m} y_i}{\Gamma(\alpha)} \left| \int_1^{t_1} \left[(\log t_2)^{1-\alpha} \left(\log \frac{t_2}{s}\right)^{\alpha-1} - (\log t_1)^{1-\alpha} \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right] \frac{1}{s} ds \right| \\
&\quad + \frac{\|p\|\Omega(r)}{\Gamma(\alpha)} \left| \int_{t_1}^{t_2} (\log t_2)^{1-\alpha} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{1}{s} ds \right|.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in S$ as $t_2 - t_1 \to 0$.

Therefore, it follows from the Arzelá-Ascoli theorem that $\mathcal{B}$ is a completely continuous operator on S.

Step: 3

Next we show that hypothesis (c) of Theorem 1 is satisfied. Let $x \in X$ and $y \in S$ be arbitrary elements such that $x = \mathcal{A}x\mathcal{B}y$. Then we have

$$\begin{aligned}
(\log t)^{1-\alpha}|x(t)| &= (\log t)^{1-\alpha}|\mathcal{A}x(t)||\mathcal{B}y(t)| \\
&= |f(t, x(t))| \left(\left| \frac{x_0 - \eta(x) + \sum_{i=0}^{m} y_i}{\Gamma(\alpha)} + (\log t)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s, y(s))}{s} ds \right| \right) \\
&\leq K \left| \left(\frac{x_0 - \eta(x) + \sum_{i=0}^{m} y_i}{\Gamma(\alpha)} + (\log t)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s, y(s))}{s} ds \right) \right| \\
&\leq K \left[\frac{|x_0| + G + \sum_{i=0}^{m} |y_i|}{\Gamma(\alpha)} + (\log T)^{1-\alpha} \|p\|\Omega(r) \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{1}{s} ds \right] \\
&\leq K \left[\frac{|x_0| + G + \sum_{i=0}^{m} |y_i|}{\Gamma(\alpha)} + (\log T) \frac{1}{\Gamma(\alpha+1)} \|p\|\Omega(r) \right].
\end{aligned}$$

Taking supremum for $t \in (t_m, T]$, we obtain

$$\|x\|_C \leq K\Big[\frac{|x_0| + G + \sum_{i=0}^{m} |y_i|}{\Gamma(\alpha)} + (\log T)\frac{1}{\Gamma(\alpha+1)}\|p\|\Omega(r)\Big] \leq r,$$

that is, $x \in S$.

Step: 4

Now we show that $Mk < 1$, that is, (d) of Theorem 1 holds.

This is obvious by (H_4), since we have

$$M = \|B(S)\| = \sup\{\|\mathcal{B}x\| : x \in S\} \leq \frac{|x_0| + G + \sum_{i=0}^{m} |y_i|}{\Gamma(\alpha)} + (\log T)\frac{1}{\Gamma(\alpha+1)}\|p\|\Omega(r)$$

and $k = \|\phi\|$.

Thus all the conditions of Theorem 1 are satisfied and hence the operator equation $x = \mathcal{A}x\mathcal{B}x$ has a solution in S. In consequence, the problem (1) has a solution on $(t_m, T]$.

This completes the proof.

4. Example

Consider the problem

$$_HD^{\frac{1}{2}}\Big(\frac{x(t)}{f(t,x(t))}\Big) = g(t,x(t)), \quad t \in (1,T],\ \alpha \in (0,1), \tag{11}$$

$$x(t_k^+) = x(t_k^-) + \frac{1}{4}, \quad {}_HJ^{1-\alpha}x(1) + \sum_{i=1}^{m} c_i x(t_i) = 1,$$

where $f(t,x) = \frac{1}{5\sqrt{\pi}}(\sin t \tan^{-1} x + \frac{\pi}{2}), g(t,x) = \frac{1}{10}(\frac{1}{6}|x| + \frac{1}{8}\cos x + \frac{|x|}{4(1+|x|)} + \frac{1}{16})$. Obviously $|f(t,x)| \leq \frac{\sqrt{\pi}}{5} = K$, $||\phi|| = \frac{\sqrt{\pi}}{5}$ and $|g(t,x)| \leq \frac{1}{10}(\frac{1}{6}|x| + \frac{7}{16})$. We choose $||p|| = \frac{1}{10}, \Omega(r) = \frac{1}{6}r + \frac{7}{16}$. By the condition (H_3), it is found that $0.05473 \leq r < \frac{3}{8}(400\pi - 87)$. Clearly all the conditions of theorem 2 are satisfied. Hence by the conclusion of theorem 2, it follows that problem (11) has a solution.

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