Journal of Fractional Calculus and Applications
Vol. 9(1) Jan. 2018, pp. 195-203.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

COEFFICIENTS ESTIMATE FOR CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS ASSOCIATED WITH QUASI-SUBORDINATION

SHASHI KANT

Abstract. In this paper we introduce and investigate certain new subclasses of the function class Σ of bi-univalent function defined in the open unit disk, which are associated with the quasi-subordination. We find estimates on the Taylor-Maclaurin coefficient $|a_2|$ and $|a_3|$ for functions in these subclasses. Several known and new consequences of these results are also pointed out.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}), \tag{1.1}$$

and let $\mathcal{S}$ be the class of all functions from $\mathcal{A}$ which are univalent in $\mathbb{U}$. The Koebe one quarter theorem [5] states that the image of U under every function f from $\mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus such univalent function has an inverse f^{-1} which satisfies $f^{-1}(f(z)) = z$, $(z \in \mathbb{U})$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$. In fact the inverse function f^{-1} is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^2 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denotes the class of bi-univalent functions defined in the unit disc $\mathbb{U}$.

Ma - Minda [9] introduce the following classes by means of subordination :

$$\mathcal{S}^*(h) = \{f \in \mathcal{A} : \frac{zf^{'}(z)}{f(z)} \prec h(z)\},$$

where h is an analytic function with positive real part on $\mathbb{U}$ with $h(0) = 1, h(0)^{'} > 0$ which maps the unit disc $\mathbb{U}$ onto a region starlike with respect to 1 and which is

---

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. quasi - Subordination, bi - univalent function, coefficent estimates .

Submitted March 11, 2017.

symmetric with respect to real axis. A function $f \in \mathcal{S}^*(h)$ is called Ma - Minda starlike. $\mathcal{C}(h)$ is the class of convex function $f \in \mathcal{A}$ for which

$$1 + \frac{zf''(z)}{f'(z)} \prec h(z).$$

The classes $\mathcal{S}^*(h)$ and $\mathcal{C}(h)$ include several well-known subclasses of starlike and convex function as special case.The concept of subordination is generalized in 1970 by Robertson [18] through introducing a new concept of quasi-subordination.

For two analytic functions f and h, the function f is quasi subordination to h written as

$$f(z) \prec_q h(z) \qquad (z \in \mathbb{U}) \tag{1.3}$$

if there exist analytic functions ϕ and ω, with $|\phi(z)| \leq 1, \omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$\frac{f(z)}{\phi(z)} \prec h(z),$$

which is equivalent to

$$f(z) = \phi(z)h(\omega(z)) \qquad (z \in \mathbb{U}).$$

Observe that if $\phi(z) = 1$, then $f(z) = h(\omega(z))$, so that$f(z) \prec h(z)$ in $\mathbb{U}$, also if $\omega(z) = z$, then $f(z) = \phi(z)h(z)$ and it is said that $f(z)$ is majorized by $h(z)$ and written as $f(z) \ll h(z)$ in $\mathbb{U}$. Hence it is obvious that the quasi-subordination is a generalization of the usual subordination as well as majorization. The work on quasi - subordination is quite extensive which includes some recent investigations [2,7,8,10,12,17,18].

In 1967, Lewin [8] investigated the class Σ of bi-univalent functions and obtained the bound for the second coefficient a_2. Brannan and Taha [3] considered certain subclasses of bi-univalent functions similar to the familiar subclasses of univalent functions consisting of starlike, strongly starlike and convex functions. They introduced the bi-starlike function, bi-convex function classes and obtained non sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Recently Ali *et al.* [1], Deniz [4], Tang *et al.* [19] , Peng *et al.* [14] Ramchandran *et al.* [16], Murugusundaramoorthy *et al.* [11]etc. have introduced and investigated Ma-Minda type subclasses of bi-univalent functions class Σ. Further generalization of Ma - Minda type subclasses of class Σ have been made several authors including ([6], [13], [10], [20]) by means of quasi - subordination. Motivated by work in [7, 12] on quasi- subordination , we introduce and study here certain new subclasses of class Σ.

Throughout this paper it is assumed that $h(z)$ is analytic in $\mathbb{U}$ with $h(0) = 1$ and let

$$\phi(z) = A_0 + A_1 z + A_2 z^2 + \cdots \quad (|\phi(z)| \leq 1, z \in \mathbb{U}) \tag{1.3}$$

and

$$h(z) = 1 + B_1 z + B_2 z^2 + \cdots \qquad (B_1 \in \mathbb{R}^+). \tag{1.4}$$

Definition 1.1. *For $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$ is said to be in the class $\mathcal{S}^q_\Sigma(\lambda, \gamma, h)$, if the following two conditions are satisfied :*

$$\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\Big) \prec_q (h(z) - 1) \tag{1.5}$$

and

$$\frac{1}{\gamma}\Big(\frac{wg^{'}(w)}{(1-\lambda)w+\lambda g(w)}-1\Big) \prec_q (h(w)-1), \tag{1.6}$$

where $g = f^{-1}$ and h is given by (1.5) and $z, w \in \mathbb{U}$.

It follows that a function f is in the class $\mathcal{S}^q_\Sigma(\lambda, \gamma, h)$ if and only if there exists an analytic function ϕ with $|\phi(z)| \leq 1, (z \in \mathbb{U})$ such that

$$\frac{\frac{1}{\gamma}\Big(\frac{zf^{'}(z)}{(1-\lambda)z+\lambda f(z)}-1\Big)}{\phi(z)} \prec (h(z)-1) \tag{1.7}$$

and

$$\frac{\frac{1}{\gamma}\Big(\frac{wg^{'}(w)}{(1-\lambda)w+\lambda g(w)}-1\Big)}{\phi(w)} \prec (h(w)-1), \tag{1.8}$$

where $g = f^{-1}$ and h is given by (1.5) and $z, w \in \mathbb{U}$.

Definition 1.2. *For $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$ is said to be in the class $\mathcal{K}^q_\Sigma(\lambda, \gamma, h)$, if the following two conditions are satisfied :*

$$\frac{1}{\gamma}\Big(\frac{zf^{'}(z)+z^2f^{''}(z)}{(1-\lambda)z+\lambda zf^{'}(z)}-1\Big) \prec_q (h(z)-1), \tag{1.9}$$

and

$$\frac{1}{\gamma}\Big(\frac{wg^{'}(w)+w^2g^{''}(w)}{(1-\lambda)w+\lambda wg^{'}(w)}-1\Big) \prec_q (h(w)-1), \tag{1.10}$$

where $g = f^{-1}$ and h is given by (1.5) and $z, w \in \mathbb{U}$.

In the present paper, we find estimates on the Taylor- MacLaurin coefficients $|a_2|$ and $|a_3|$ for function f belonging in the classes $\mathcal{S}^q_\Sigma(\lambda, \gamma, h)$ and $\mathcal{K}^q_\Sigma(\lambda, \gamma, h)$. Several known and new consequences of these results are also pointed out.

In order to derive our main results , we have to recall here the following well-known Lemma:

Lemma 1.3.[15] *Let $p \in \mathcal{P}$ be family of all functions p analytic in $\mathbb{U}$ for which $\Re\{p(z)\} > 0$ and have the form $p(z) = 1 + p_1z + p_2z^2 + ...$ for $z \in \mathbb{U}$, then $|p_n| \leq 2$ for each n.*

2. Coefficient bounds for the function class $\mathcal{S}^q_\Sigma(\lambda, \gamma, h)$

Theorem 2.1. *Let $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$. If $f \in \mathcal{A}$ of the form (1.1) belonging to the class $\mathcal{S}^q_\Sigma(\lambda, \gamma, h)$, then*

$$|a_2| \leq min\Big\{\frac{B_1|\gamma||A_0|}{(2-\lambda)}, \sqrt{\frac{(B_1+|B_2-B_1|)|\gamma||A_0|}{\lambda^2-3\lambda+3}}\Big\} \tag{2.1}$$

and

$$\begin{aligned}|a_3| \leq min\Big\{&\frac{|\gamma|}{\lambda^2-3\lambda+3}(B_1+|B_2-B_1|)|A_0| + \frac{|\gamma|}{(3-\lambda)}|A_1|B_1,\\ &\frac{|\gamma|}{(3-\lambda)}\Big[\frac{|\gamma|\lambda B_1^2}{2-\lambda}|A_0|^2 + (B_1+|B_2-B_1|)|A_0| + B_1|A_1|\Big]\Big\}.\end{aligned} \tag{2.2}$$

Proof. *Let $f \in \mathcal{S}^q_\Sigma(\lambda,\gamma,h)$. In view of Definition1.1, there exist then Schwarz functions $r(z)$, $s(z)$ and an analytic function $\phi(z)$ such that*

$$\frac{1}{\gamma}\Big(\frac{zf^{'}(z)}{(1-\lambda)z+\lambda f(z)}-1\Big)=\phi(z)(h(r(z))-1) \tag{2.3}$$

and

$$\frac{1}{\gamma}\Big(\frac{wg^{'}(w)}{(1-\lambda)w+\lambda g(w)}-1\Big)=\phi(w)(h(s(w))-1). \tag{2.4}$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z)=\frac{1+r(z)}{1-r(z)}=1+c_1z+c_2z^2+\cdots \tag{2.5}$$

and

$$q(z)=\frac{1+s(z)}{1-s(z)}=1+d_1z+d_2z^2+\cdots, \tag{2.6}$$

which are equivalently

$$r(z)=\frac{p(z)-1}{p(z)+1}=\frac{1}{2}\big[c_1z+\big(c_2-\frac{c_1^2}{2}\big)z^2+\cdots\big] \tag{2.7}$$

and

$$s(z)=\frac{q(z)-1}{q(z)+1}=\frac{1}{2}\big[d_1z+\big(d_2-\frac{d_1^2}{2}\big)z^2+\cdots\big]. \tag{2.8}$$

It is clear that $p(z)$, $q(z)$ are analytic and have positive real parts in $\mathbb{U}$. In view of (2.3), (2.4), (2.7) and (2.8), clearly

$$\frac{1}{\gamma}\Big(\frac{zf^{'}(z)}{(1-\lambda)z+\lambda f(z)}-1\Big)=\phi(z)\Big[h\Big(\frac{p(z)-1}{p(z)+1}\Big)-1\Big] \tag{2.9}$$

and

$$\frac{1}{\gamma}\Big(\frac{wg^{'}(w)}{(1-\lambda)w+\lambda g(w)}-1\Big)=\phi(w)\Big[h\Big(\frac{q(w)-1}{q(w)+1}\Big)-1\Big]. \tag{2.10}$$

The series expansions for $f(z)$ and $g(w)$ as given in (1.1) and (1.2) respectively, provide us

$$\frac{1}{\gamma}\Big(\frac{zf^{'}(z)}{(1-\lambda)z+\lambda f(z)}-1\Big)=\frac{1}{\gamma}\big[(2-\lambda)a_2z+[(3-\lambda)a_3-\lambda(2-\lambda)a_2^2]z^2+\cdots\big] \tag{2.11}$$

and

$$\frac{1}{\gamma}\Big(\frac{wg^{'}(w)}{(1-\lambda)w+\lambda g(w)}-1\Big)=\frac{1}{\gamma}\big[(\lambda-2)a_2w+[(3-\lambda)(2a_2^2-a_3)-\lambda(2-\lambda)a_2^2]w^2+\cdots\big]. \tag{2.12}$$

Using (2.5) and (2.6) together with (1.4) and (1.5)

$$\phi(z)\Big[h\Big(\frac{p(z)-1}{p(z)+1}\Big)-1\Big]=\frac{1}{2}A_0B_1c_1z+\Big[\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\Big(c_2-\frac{c_1^2}{2}\Big)+\frac{A_0B_2c_1^2}{4}\Big]z^2+\cdots \tag{2.13}$$

and

$$\phi(w)\Big[h\Big(\frac{q(w)-1}{q(w)+1}\Big)-1\Big]=\frac{1}{2}A_0B_1d_1z+\Big[\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1\Big(d_2-\frac{d_1^2}{2}\Big)+\frac{A_0B_2d_1^2}{4}\Big]z^2+\cdots \tag{2.14}$$

Now equating (2.11) and (2.13) in view of (2.9) and comparing the coefficients of z and z^2, we obtain

$$\frac{2-\lambda}{\gamma}a_2 = \frac{1}{2}A_0B_1c_1 \tag{2.15}$$

and

$$\frac{(3-\lambda)a_3 - \lambda(2-\lambda)a_2^2}{\gamma} = \frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1\Big(c_2 - \frac{c_1^2}{2}\Big) + \frac{A_0B_2c_1^2}{4}. \tag{2.16}$$

Similarly (2.10) gives us

$$-\frac{2-\lambda}{\gamma}a_2 = \frac{1}{2}A_0B_1d_1 \tag{2.17}$$

and

$$\frac{(3-\lambda)(2a_2^2 - a_3) - \lambda(2-\lambda)a_2^2}{\gamma} = \frac{1}{2}A_1B_1d_1 + \frac{1}{2}A_0B_1\Big(d_2 - \frac{d_1^2}{2}\Big) + \frac{A_0B_2d_1^2}{4}. \tag{2.18}$$

From(2.15) and (2.17), we find that

$$a_2 = \frac{A_0B_1c_1\gamma}{2(2-\lambda)} = -\frac{A_0B_1d_1\gamma}{2(2-\lambda)} \tag{2.19}$$

which implies

$$|a_2| \leq \frac{|A_0\gamma|B_1}{2-\lambda}. \tag{2.20}$$

Adding (2.16) and (2.18) , we obtain

$$\frac{2(\lambda^2 - 3\lambda + 3)}{\gamma}{a_2}^2 = \frac{A_0B_1}{2}(c_2 + d_2) + \frac{A_0(B_2 - B_1)}{4}({c_1}^2 + {d_1}^2), \tag{2.21}$$

which implies

$$|a_2|^2 \leq \frac{|A_0\gamma|(B_1 + |B_2 - B_1|)}{\lambda^2 - 3\lambda + 3}, \tag{2.22}$$

hence, using (2.20) and (2.22) we get the bounds on $|a_2|$ as asserted in (2.1).

Next, in order to find the upper bound for $|a_3|$, by subtracting (2.18) from (2.16), we get

$$\frac{2(3-\lambda)}{\gamma}a_3 = \frac{2(3-\lambda)}{\gamma}{a_2}^2 + \frac{A_1B_1}{2}(c_1 - d_1) + \frac{A_0B_1}{2}(c_2 - d_2), \tag{2.23}$$

by using Lemma1.2 and (2.21) in (2.23), we obtain

$$|a_3| \leq \Big[\frac{|A_0|B_1}{\lambda^2 - 3\lambda + 3} + \frac{|A_0(B_2 - B_1)|}{\lambda^2 - 3\lambda + 3} + \frac{|A_1|B_1}{3-\lambda}\Big]|\gamma|. \tag{2.24}$$

Next, from (2.15) and (2.16), we have

$$\frac{(3-\lambda)a_3}{\gamma} = \frac{\lambda\gamma {A_0}^2{B_1}^2c_1^2}{4(2-\lambda)} + \frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1c_2 + \frac{1}{4}A_0(B_2 - B_1)c_1^2,$$

which implies

$$|a_3| \leq \frac{|\gamma|}{3-\lambda}\Big[B_1\big(\frac{\lambda}{2-\lambda}|A_0|^2|\gamma|B_1 + |A_1| + |A_0|\big) + |A_0(B_2 - B_1)|\Big]. \tag{2.25}$$

Further, from (2.15) and (2.18) , we deduce that

$$|a_3| \leq \frac{|\gamma|}{3-\lambda}\Big[B_1\big(\frac{\lambda^2 - 4\lambda + 6}{(2-\lambda)^2}|A_0|^2|\gamma|B_1 + |A_1| + |A_0|\big) + |A_0(B_2 - B_1)|\Big] \tag{2.26}$$

and thus we obtain the conclusion (2.2) of our theorem.

Remarks 2.2. *(i) For $\lambda = 1$, Theorem 2.1 provides improvement over the estimates obtained in [[10], Corollary 9, p 5].*

(ii) For $\lambda = \gamma = 1$, Theorem 2.1 reduces to a result in [[13], Theorem 3.2, p. 8].

(iii) For $\lambda = 0, \gamma = 1$, Theorem 2.1 reduces to a result in [[13], Corollary 2.4, p.8].

For $\phi(z) \equiv 1$, the above theorem reduces to following corollary:

Corollary 2.3. *For $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$, if $f \in \mathcal{A}$ of the form (1.1) satisfy the following subordination:*

$$1 + \frac{1}{\gamma}\Big(\frac{zf^{'}(z)}{(1-\lambda)z + \lambda f(z)} - 1\Big) \prec h(z) \tag{2.27}$$

and

$$1 + \frac{1}{\gamma}\Big(\frac{wg^{'}(w)}{(1-\lambda)w + \lambda g(w)} - 1\Big) \prec h(w), \tag{2.28}$$

where $g = f^{-1}$ and h is given by (1.5) and $z, w \in \mathbb{U}$, then

$$|a_2| \leq min\Big\{\frac{B_1|\gamma|}{(2-\lambda)}, \sqrt{\frac{(B_1 + |B_2 - B_1|)|\gamma|}{\lambda^2 - 3\lambda + 3}}\Big\} \tag{2.29}$$

and

$$|a_3| \leq min\Big\{\frac{|\gamma|}{\lambda^2 - 3\lambda + 3}\big(B_1 + |B_2 - B_1|\big), \quad \frac{|\gamma|}{(3-\lambda)}\big(\frac{|\gamma|\lambda}{2-\lambda}B_1^2 + B_1 + |B_2 - B_1|\big)\Big\}. \tag{2.30}$$

For$\lambda = \gamma = 1$, Corollary 2.4 gives the coefficient estimates for Ma - Minda bi-starlike functions. ***Remark 2.4.*** *For$\lambda = 0$ and $\gamma = 1$ Corollary 2.4 reduces to a result in [1, Theorem 2.1, p. 345].*

3. Coefficient bounds for the function class $\mathcal{K}_\Sigma^q(\lambda, \gamma, h)$

Theorem 3.1. *Let $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$. If $f \in \mathcal{A}$ of the form (1.1) belonging to the class $\mathcal{K}_\Sigma^q(\lambda, \gamma, h)$, then*

$$|a_2| \leq min\{\frac{B_1|\gamma||A_0|}{2(2-\lambda)}, \sqrt{\frac{(B_1 + |B_2 - B_1|)|\gamma||A_0|}{4\lambda^2 - 11\lambda + 9}}\} \tag{3.31}$$

and

$$\begin{aligned}|a_3| \leq min\Big\{&\frac{|\gamma|}{4\lambda^2 - 11\lambda + 9}(B_1 + |B_2 - B_1|)|A_0| + \frac{|\gamma|}{3(3-\lambda)}|A_1|B_1,\\ &\frac{|\gamma|}{3(3-\lambda)}\big[\frac{|\gamma|\lambda B_1^2}{2-\lambda}|A_0|^2 + (B_1 + |B_2 - B_1|)|A_0| + B_1|A_1|\big]\Big\}.\end{aligned} \tag{3.32}$$

Proof. Let $f \in \mathcal{K}_\Sigma^q(\lambda, \gamma, h)$. In view of Definition1.2, there exist then Schwarz functions $r(z)$, $s(z)$ and an analytic function $\phi(z)$ such that

$$\frac{1}{\gamma}\Big(\frac{zf^{'}(z) + z^2 f^{''}(z)}{(1-\lambda)z + \lambda zf^{'}(z)} - 1\Big) = \phi(z)(h(z) - 1) \tag{3.33}$$

and

$$\frac{1}{\gamma}\Big(\frac{wg^{'}(w) + w^2 g^{''}(w)}{(1-\lambda)w + \lambda wg^{'}(w)} - 1\Big) = \phi(z)(h(w) - 1), \tag{3.34}$$

where $r(z)$ and $s(z)$ are defined by (2.7) and (2.8) respectively. Under the same restrictions for $p(z), q(z), c_i$ and d_i as mentioned in Theorem2.1, obviously we have

$$\frac{1}{\gamma}\Big(\frac{zf^{'}(z)+z^2f^{''}(z)}{(1-\lambda)z+\lambda zf^{'}(z)}-1\Big)=\phi(z)\Big[h\Big(\frac{p(z)-1}{p(z)+1}\Big)-1\Big] \tag{3.35}$$

and

$$\frac{1}{\gamma}\Big(\frac{wg^{'}(w)+w^2g^{''}(w)}{(1-\lambda)w+\lambda wg^{'}(w)}-1\Big)=\phi(w)\Big[h\Big(\frac{q(w)-1}{q(w)+1}\Big)-1\Big]. \tag{3.36}$$

The series expansions for $f(z)$ and $g(w)$ as given in (1.1) and (1.2) respectively, provides us

$$\frac{1}{\gamma}\Big(\frac{zf^{'}(z)+z^2f^{''}(z)}{(1-\lambda)z+\lambda zf^{'}(z)}-1\Big)=\frac{1}{\gamma}\Big[2(2-\lambda)a_2z+\big((3-\lambda)a_3-4\lambda(2-\lambda)a_2^2\big)z^2+...\Big] \tag{3.37}$$

and

$$\frac{1}{\gamma}\Big(\frac{wg^{'}(w)+w^2g^{''}(w)}{(1-\lambda)w+\lambda wg^{'}(w)}-1\Big)=\frac{1}{\gamma}\Big[-2(2-\lambda)a_2w+\big(3(3-\lambda)(2a_2^2-a_3)-4\lambda(2-\lambda)a_2^2\big)w^2+...\Big]. \tag{3.38}$$

Now using (2.13) and (3.7) in (3.5) and comparing the coefficients of z and z^2, we get

$$\frac{2(2-\lambda)}{\gamma}a_2=\frac{1}{2}A_0B_1c_1 \tag{3.39}$$

and

$$\frac{1}{\gamma}\big(3(3-\lambda)a_3-4\lambda(2-\lambda)a_2^2\big)=\frac{1}{2}A_1B_1c_1+\frac{1}{2}A_0B_1\Big(c_2-\frac{c_1^2}{2}\Big)+\frac{A_0B_2c_1^2}{4}. \tag{3.40}$$

Similarly (2.14), (3.6) and (3.8) yields

$$-\frac{2(2-\lambda)}{\gamma}a_2=\frac{1}{2}A_0B_1d_1 \tag{3.41}$$

and

$$\frac{1}{\gamma}\big(3(3-\lambda)(2a_2^2-a_3)-4\lambda(2-\lambda)a_2^2\big)=\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1\Big(d_2-\frac{d_1^2}{2}\Big)+\frac{A_0B_2d_1^2}{4}. \tag{3.42}$$

From (3.9) and (3.11), we have

$$a_2=\frac{\gamma A_0B_1c_1}{4(2-\lambda)}=-\frac{\gamma A_0B_1d_1}{4(2-\lambda)}, \tag{3.43}$$

further by adding (3.10) and (3.12), we obtain

$$\frac{2(4\lambda^2-11\lambda+9)}{\gamma}{a_2}^2=\frac{A_0B_1}{2}(c_2+d_2)+\frac{A_0(B_2-B_1)}{4}({c_1}^2+{d_1}^2). \tag{3.44}$$

On using the Lemma1.3 in (3.13) and (3.14), we can get the desired bounds on $|a_2|$ as given in (3.1). Next, in order to find the upper bound for $|a_3|$, by subtracting (3.12) from (3.10) and using (3.14), we get

$$|a_3|\le\frac{|\gamma|}{4\lambda^2-11\lambda+9}[|A_0|B_1+|A_0(B_2-B_1)|]+\frac{|\gamma|}{3(3-\lambda)}|A_1|B_1. \tag{3.45}$$

For another bound on $|a_3|$*, we substitute the value of* a_2^2 *from (3.9) into (3.10) and use the Lemma1.3, which gives us*

$$|a_3| \leq \frac{|\gamma|}{3(3-\lambda)}\big[\frac{|\gamma|\lambda B_1^2}{2-\lambda}|A_0|^2 + (B_1 + |B_2 - B_1|)|A_0| + B_1|A_1|\big]. \tag{3.46}$$

With the help of (3.9) and (3.12) we obtain one more bound on $|a_3|$ *that is*

$$|a_3| \leq \frac{|\gamma|}{3(3-\lambda)}\big[\frac{|\gamma|B_1^2(2\lambda^2 - 7\lambda + 9)}{2(2-\lambda)^2}|A_0|^2 + (B_1 + |B_2 - B_1|)|A_0| + B_1|A_1|\big]. \tag{3.47}$$

Obviously the RHS of (3.17) is greater than the RHS of (3.16), so the desired bound on $|a_3|$ *is obtained from (3.15) and (3.16). For* $\phi(z) \equiv 1$*, the above theorem reduces to following corollary:* ***Corollary 3.2.*** *For* $0 \leq \lambda \leq 1$ *and* $\gamma \in \mathbb{C}\backslash\{0\}$*, if* $f \in \mathcal{A}$ *of the form (1.1) satisfy the following subordinations:*

$$1 + \frac{1}{\gamma}\Big(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\Big) \prec (h(z) \tag{3.48}$$

and

$$1 + \frac{1}{\gamma}\Big(\frac{wg'(w) + w^2 g''(w)}{(1-\lambda)w + \lambda w g'(w)} - 1\Big) \prec h(w), \tag{3.49}$$

where $g = f^{-1}$ *and* h *is given by (1.5) and* $z, w \in \mathbb{U}$*, then*

$$|a_2| \leq min\{\frac{B_1|\gamma|}{2(2-\lambda)}, \sqrt{\frac{(B_1 + |B_2 - B_1|)|\gamma|}{4\lambda^2 - 11\lambda + 9}}\} \tag{3.50}$$

and

$$\begin{aligned} |a_3| \leq min\Big\{&\frac{|\gamma|}{4\lambda^2 - 11\lambda + 9}(B_1 + |B_2 - B_1|), \\ &\frac{|\gamma|}{3(3-\lambda)}\big(\frac{|\gamma|\lambda}{2-\lambda}B_1^2 + B_1 + |B_2 - B_1|\big)\Big\}. \end{aligned} \tag{3.51}$$

Remarks 3.3. *(i) For* $\lambda = 1$*, Theorem 3.1 provides improvement over the estimates obtained in [*[10]*, Corollary 11, p 5].*
(ii) For $\lambda = \gamma = 1$*, Theorem 3.1 provides improvement over the estimates obtained in [*[13]*, Theorem 3.3, p. 9].*
Other interesting corollaries and consequences of Theorem 3.1 could be derived by specializing the parameters.

References

[1] R.M. Ali, S. K. Lee, V. Ravichandran and S. Subramaniam, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, Appl. Math. Lett. 25(3), 344-351, 2012.
[2] O. Altintas and S. Owa, Majorizations and quasi-subordinations for certain analytic functions, Proc. Jpn. Acad. Ser. A 68(7), 181 - 185, 1992.
[3] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, Studia Univ. Babes-Bolyai Math. 31(2), 70-77, 1986.
[4] E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, J. Classical Ana. 2(1), 49-60, 2013.
[5] P. L. Duren, Univalent functions, Springer-Verlag, New York, 1983.
[6] S. P. Goyal, O. Singh and R. Mukherjee, Certain results on a subclass of analytic and bi-univalent functions associated with coefficient estimates and quasi-subordination, Palestine Journal of Mathematics, 5(1), 79-85, 2016.
[7] S. Y. Lee, Quasi-subordinate functions and coefficient conjectures, J. Korean Math. Sco. 12(1), 43-50, 1975.
[8] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. Math. Soc. 18, 63-68, 1967.

[9] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, In: Proceedings of the conference on complex analysis,(Tianjin,1992), Conference Proceedings Lecturer Notes Analysis, International Press, Cambridge, Mass, USA, vol. 1, 157-169, 1994.

[10] N. Magesh, V. K. Balaji and J. Yamini, Certain subclasses of bistarlike and biconvex functions based on quasi - subordination, Abstract and analysis, 2016 Art. ID3102960, 6 pages ,2016.

[11] G. Murugusundaramoorthy, T. Janani and N. E. Cho, Bi-univalent functions of complex order based on subordinate conditions involving Hurwitz Lerch Zeta function, East Aasian Math. J. 32(1), 47-59, 2016.

[12] T. Panigrahi and R. K. Raina, Fekete-Szego coefficient functional for quasi-subordination class, Afro. Mat. DOI 10.1007/s13370-016-0477-1, 10 pages, 2017.

[13] A. B. Patil, and U. H. Naik, Estimates on initial coefficients of certain subclasses of bi-univalent functions associated with quasi-subordination, Global Journal of mathematical Analysis, 5(1), 6-10, 2017.

[14] Z. Peng, G. Murugusundaramoorthy and T. Janani, Coefficient estimate of bi-univalent functions of complex order associated with the Hohlov operator, Journal of Complex analysis, 2014 Art.ID 693908, 6 pages, 2014.

[15] Ch. Pommerenke, Univalent Functions. Vandenhoeck and Ruprecht, Gottingen, 1975.

[16] C. Ramachandran, R. Ambroseprabhu, and N. Magesh, Initial coefficient estimates for certain subclasses of bi-univalent functions of Ma-minda type, Applied Mathematical Sciences, 9(47), 2299-2308, 2015.

[17] F. Y. Ren, S. Owa and S. Fukui, Some inequalities on quasi-subodrdinate functions, Bull. Austral. Math. Soc. 43(2), 317-324,1991.

[18] M. S. Robertson, Quasi-subordination and coefficient conjecture, Bull. Amer. Math. Soc. 76, 1-9, 1970.

[19] H. Tang, G. Deng and S. Li, Coefficient estimates for new subclasses of Ma-Minda bi-univalent functions, J. Ineq. Appl. 2013 Art. 317, 10 pages, 2013 .

[20] P. P. Vyas and S. Kant, Certain Subclasses of bi-univalent functions associated with quasi-subordination, Journal of Rajasthan Academy of Physical Sciences, 15(4), 315 - 325, Dec. 2016.

Shashi Kant

Department of Mathematics, Government Dungar College, Bikaner-334001, INDIA

E-mail address: drskant.2007@yahoo.com