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THE SHRINKING FIXED POINT MAP, CAPUTO AND INTEGRAL EQUATIONS: PROGRESSIVE CONTRACTIONS

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Abstract. We begin with a classical conjecture that every shrinking map of the closed unit ball in a Banach space into that ball has a fixed point. In our study we restrict the Banach space to $(\mathcal{B}, \|\cdot\|)$, the space of bounded continuous functions $\phi : [0, \infty) \to \Re$ with the supremum norm. Our mapping will be the natural mapping defined by an integral equation and we will have a Lipschitz map instead of a shrinking map. We take this as an opportunity to show the power and flexibility of a recent result on progressive contractions and show that there is, indeed, a unique fixed point residing in the closed ball. The problem is presented in the context of a Caputo fractional differential equation in which we point out that the standard inversion is incomplete. There is a continuation of that inversion which causes the equation to fit the conjecture.

1. Introduction: A Caputo Inversion

A brief sketch: In the first section we motivate a classical conjecture that a shrinking map of the closed unit ball in a Banach space has a fixed point by starting with a first order differential equation $x' = -f(t, x)$ with $xf(t, x) > 0$ if $x \neq 0$. Behavior of solutions is clear. We then show how it is related to a Caputo fractional differential equation which is inverted as an integral equation. Next, we show by a transformation that the integral equation can be transformed so that the natural mapping defined by the integral equation does, indeed, define a mapping of the closed unit ball into itself. If there is a fixed point it solves the integral equation and the Caputo fractional differential equation. The solution is very much like that of the beginning ordinary differential equation.

In the second section we introduce the concept of a progressive contraction which does not require the mapping to be shrinking. Rather than require a type of the classical contraction, it requires only a Lipschitz condition. We then show that a general integral equation which defines a mapping of the closed unit ball into itself has a fixed point under conditions weaker than shrinking. In fact, it more than solves the conjecture for that class of integral equations.

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One of the referees has pointed out that progressive contractions might be extended to Volterra integral equations in Banach spaces and offers two references ([10], [8]) which might start the investigation along that path. End of sketch.

Our first result, Theorem 1.3, concerns relations between Caputo fractional differential equations and ordinary differential equations. It tells us that a given equation may not seem to map the unit ball into itself, as is frequently required for a fixed point theorem and is the subject of this paper, but there is a transformation which may make it do exactly that. The transformation is especially effective for Caputo equations. This result also brings a classical conjecture into focus as being central in the study of integral equations. Stated briefly, the conjecture is that a shrinking map of the unit ball in a Banach space has a fixed point. Solving the conjecture for integral equations is the main thrust of this paper.

Recently we discovered a very simple fixed point technique which we call *progressive contractions*. This has evolved into a project of adapting it to a variety of integral equations which occur in applied mathematics ([2],[3]). We have tried to keep these short so that it is easy to grasp the idea without being led through complicated introductions or constructions.

One of those projects which has not yet appeared [4] involves showing that an integral equation of the form

$$x(t) = g(t, x(t)) + \int_0^t A(t-s)f(s, x(s))ds \tag{1.1}$$

with g a contraction, f Lipschitz in x, and $A : (0, \infty) \to \Re$ continuous and satisfying

$$t \downarrow 0 \implies \int_0^t |A(u)|du \downarrow 0$$

will have a unique global solution. There are several other applications but we mention this one because it offers a solution to an old conjecture which can be found on p. 39 of Smart [11]. The fact that on the previous page Smart studies a fixed point map for integral equations under far stronger conditions invites us to try to solve the conjecture for integral equations. Our earlier result solves it quickly.

Definition 1.1 A mapping L on a metric space into itself is said to be shrinking if $x \neq y \implies \rho(Lx, Ly) < \rho(x, y)$.

Conjecture. Every shrinking map of the closed unit ball in a Banach space into itself has a fixed point.

Notation We will substantially reduce the assumptions and prove the result for (1.1) when the Banach space is $(\mathcal{B}, \|\cdot\|)$, the space of bounded continuous functions $\phi : [0, \infty) \to \Re$ with the supremum norm. The unit ball in this space will be denoted by M. The mapping is defined by an integral equation.

Such problems are important. We often work a problem down toward Schauder's theorem, but are stopped by the $g(t, x)$ and unable to work it into Krasnoselskii's theorem on the sum of two operators or Darbo's theorem in the theory of measures of non-compactness. This conjecture offers a way out. Moreover, we have studied the problem of getting an equation like (1.1) into a form so that it does map the closed ball into itself. In [1] we developed a transformation which has been very effective in doing exactly that. We have used it in a number of places in the last six years, but an instructive example involving a fractional differential equation of Caputo type is given in [6] which displays the transformation in a clear way. Given

$$ {}^cD^q x(t) = -f(t,x(t)), \quad x(0) = x^0 \in \Re, \quad 0 < q < 1 \tag{1.2} $$

the inversion is given in Diethelm [7, p. 86] as

$$ x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))ds \tag{1.3} $$

and there is no chance that this could map the unit ball into itself. But the aforementioned transformation can, indeed, achieve exactly what we need provided that f satisfies classical stability properties that we see even in the most elementary considerations for ordinary differential equations as specified with (1.5) below. Such problems give real meaning to the conjecture.

The inversion (1.3) skews everything and we need to transform it back in line with the simple differential equation (1.2). We have posted an extensive annotated bibliography on Researchgate [5] listing papers and problems studied by means of this transformation. But to give concrete meaning to this note, we now show the result of that transformation for (1.3). The transformation is restricted to a wide class of kernels found in Miller [9, p. 209] including Caputo fractional differential equations.

Here is the point of the fourth paragraph of this introduction. If (1.2) were an ordinary differential equation with $xf(t,x) \geq 0$ then $x = 0$ would be stable: Solutions starting near zero would stay near. The mapping defined by solutions would map the unit ball into itself. But the same is true for Caputo equations when we make a complete inversion. Inversion (1.3) is the one found throughout the theory of fractional differential equations, but it is simply incomplete and hides the basic properties which are confronting us in (1.2). This is only one example given here to motivate the question of mapping the unit ball into itself. The aforementioned annotated bibliography gives many more such examples.

Example 1.2 The reference [6] gives complete details of the transformation so we will only give the effect here. Under that transformation (1.3) becomes

$$ x(t) = x(0)\left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s)\left[x(s) - \frac{f(s,x(s))}{J}\right]ds \tag{1.4} $$

where $0 < R(t)$, $\int_0^\infty R(t)dt = 1$, J is an arbitrary positive constant. Our stability considerations ask that $xf(t,x) \geq 0$, that there are positive constants $k < K$ so that $|x|, |y| \leq 1$ imply

$$ |f(t,x) - f(t,y)| \leq K|x-y| \quad \& \quad k|x| \leq |f(t,x)| \leq K|x| \tag{1.5} $$

and so for any $J > K$ we have

$$ \frac{k}{J} \leq \frac{f(t,x)}{Jx} \leq \frac{K}{J} < 1. $$

These are typical conditions required of f in order that the zero solution of $x' = -f(t,x)$ be asymptotically stable.

The natural mapping of the closed unit ball, M, is $\phi \in M$ and $|x(0)| \leq 1$ imply that

$$ (P\phi)(t) = x(0)\left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s)\left[\phi(s) - \frac{f(s,\phi(s))}{J}\right]ds. $$

We now show that the behavior of solutions of (1.2) mirror that of solutions of an ordinary differential equation when P has a fixed point, a property we later prove.

Theorem 1.3 If (1.5) holds then $P : M \to M$.

Proof Having in mind that $|x(0)| \leq 1$, we consider two cases. Assume first that

$$1 - \frac{k}{K} < |x(0)| \leq 1. \tag{1.6a}$$

As $1 - \frac{k}{K} < |x(0)|$, we may choose a $J > K$ such that $1 - \frac{k}{J} < |x(0)|$.

Then for $|\phi| \leq 1$ and using $k/J \leq f(t,x)/Jx < 1$ we have

$$\begin{aligned}
|(P\phi)(t)| &\leq |x(0)|\left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s)\left|\phi(s)\left[1 - \frac{f(s,\phi(s))}{J\phi(s)}\right]\right|ds \\
&\leq |x(0)|\left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s)|\phi(s)|\left[1 - \frac{k}{J}\right]ds \\
&\leq |x(0)| + \int_0^t R(s)ds\left[-|x(0)| + 1 - \frac{k}{J}\right] \\
&\leq |x(0)| \leq 1.
\end{aligned}$$

Now assume that

$$1 - \frac{k}{K} \geq |x(0)|, \tag{1.6b}$$

and choose any $J > K$. Then

$$1 \geq |x(0)| + \frac{k}{K} > |x(0)| + \frac{k}{J}$$

and for $|\phi| \leq 1$ we have

$$\begin{aligned}
|(P\phi)(t)| &\leq |x(0)|\left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s)|\phi(s)|\left[\left|1 - \frac{f(s,\phi(s))}{J\phi(s)}\right|\right]ds \\
&\leq |x(0)|\left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s)|\phi(s)|\left[1 - \frac{f(s,\phi(s))}{J\phi(s)}\right]ds \\
&\leq |x(0)|\left[1 - \int_0^t R(s)ds\right] + \int_0^t R(t-s)|\phi(s)|\left[1 - \frac{k}{J}\right]ds \\
&\leq |x(0)|\left[1 - \int_0^t R(s)ds\right] + \left[1 - \frac{k}{J}\right]\int_0^t R(s)ds \\
&= |x(0)| - |x(0)|\int_0^t R(s)ds + \left[1 - \frac{k}{J}\right]\int_0^t R(s)ds \\
&= |x(0)| + \left[1 - \frac{k}{J} - |x(0)|\right]\int_0^t R(s)ds \\
&\leq |x(0)| + \left[1 - \frac{k}{J} - |x(0)|\right] \\
&= 1 - \frac{k}{J} < 1.
\end{aligned}$$

Thus, in both case considered we see that for $\phi \in M$ we have $|(P\phi)(t)| \leq 1$ for any $t \geq 0$. Continuity of $(P\phi)(t)$ will follow from continuity of f and g so we will then have $P : M \to M$.

Our main result will show that since $P : M \to M$ then P has a fixed point.

2. The setting and solution

Consider a scalar integral equation

$$x(t) = g(t, x(t)) + \int_0^t A(t-s)f(s, x(s))ds \tag{2.1}$$

in which we suppose that for each $E > 0$ there is an $\alpha \in (0,1)$ so that $|x| \le 1$, $|y| \le 1$, and $0 \le t \le E$ imply that

$$|g(t,x) - g(t,y)| \le \alpha|x-y| \tag{2.2}$$

and for each $E > 0$ there is a $K > 0$ so that for $|x| \le 1, |y| \le 1$, and $0 \le t \le E$ we have

$$|f(t,x) - f(t,y)| \le K|x-y|. \tag{2.3}$$

Because K is unrestricted this cannot be a shrinking map.

We suppose that the kernel $A : (0,\infty) \to \Re$ is locally integrable so that when ϕ is continuous on $[0,\infty)$ then

$$\int_0^t A(t-s)\phi(s)ds \tag{2.4}$$

is continuous. Since $|A|$ is also locally integrable then $\int_0^t |A(s)|ds$ is continuous with

$$t \downarrow 0 \implies \int_0^t |A(s)|ds \downarrow 0. \tag{2.5}$$

Thus, for $T > 0$ and small enough we have

$$K\int_0^T |A(s)|ds < \frac{1-\alpha}{2}. \tag{2.6}$$

Moreover, $g : [0,\infty) \times [-1,1] \to \Re$ and $f : [0,\infty) \times [-1,1] \to \Re$ are continuous.

When $T_2 - T_1 \le T$ and $T_1 \le t \le T_2$ a change of variable yields

$$\int_{T_1}^t |A(t-s)|ds < \frac{1-\alpha}{2K}. \tag{2.7}$$

For our problem here we need to specify that for M the unit ball then

$$\phi \in M \implies |g(0, \phi(0))| \le 1. \tag{2.8}$$

Theorem 2.1 Let P be defined by $\phi \in M$ implies that

$$(P\phi)(t) = g(t, \phi(t)) + \int_0^t A(t-s)f(s, \phi(s))ds \tag{2.9}$$

and suppose that $\phi \in M$ implies $P\phi \in M$. Let (2.2)–(2.8) hold for (2.1). For every $E > 0$ there is a unique solution ξ of (2.1) on $[0,E]$ with $|\xi(t)| \le 1$.

Proof Let T satisfy (2.6). Divide $[0,E]$ into n equal parts, each of length $S < T$, and denote the end points by

$$0, T_1, T_2, ..., T_n = E.$$

Step 1. Let $(\mathcal{M}_1, |\cdot|_1)$ be the complete metric space of continuous functions $\phi : [0,T_1] \to \Re$ with $|\phi(t)| \le 1$.

As $P : M \to M$, if we define P_1 by $\phi \in M$ implies $(P_1\phi)(t) = (P\phi)(t)$ for $0 \le t \le T_1$ then it is clear that $|P_1\phi|_1 \le 1$. If $\phi, \psi \in \mathcal{M}_1$ then

$$\begin{aligned}
|(P_1\phi)(t) - (P_2\psi)(t)| &\le \alpha|\phi(t) - \psi(t)| \\
&+ \int_0^t |A(t-s)|K|\phi(s) - \psi(s)|ds \\
&\le |\phi - \psi|_1 \left[\alpha + K\int_0^T |A(s)|ds\right] \\
&\le |\phi - \psi|_1 \left[\alpha + \frac{1-\alpha}{2}\right] \\
&= |\phi - \psi|_1 \left[\frac{\alpha+1}{2}\right],
\end{aligned}$$

a contraction with unique fixed point ξ_1 on $[0, T_1]$ so

$$(P_1\xi_1)(t) = \xi_1(t) = g(t, \xi_1(t)) + \int_0^t A(t-s)f(s, \xi_1(s))ds. \tag{2.10}$$

The next step is the crucial one which shows us exactly how to take $n-2$ steps and arrive at E with the solution.

Step 2. Let $(\mathcal{M}_2, |\cdot|_2)$ be the complete metric space of continuous functions $\phi : [0, T_2] \to \Re$ with the supremum metric and

$$\phi(t) = \xi_1(t) \text{ on } [0, T_1], \quad |\phi(t)| \le 1.$$

Define $P_2 : \mathcal{M}_2 \to \mathcal{M}_2$ by $\phi \in \mathcal{M}_2$ implies

$$(P_2\phi)(t) = g(t, \phi(t)) + \int_0^t A(t-s)f(s, \phi(s))ds.$$

By (2.10) we see $P_2 : \mathcal{M}_2 \to \mathcal{M}_2$.

In our calculation below remember that if $\phi, \psi \in \mathcal{M}_2$ then their difference is zero on $[0, T_1]$ which accounts for the limits in the integral in the first line of the display below changing as seen in the second line of the display below. Thus,

$$\begin{aligned}
|(P_2\phi)(t) - (P_2\psi)(t)| &\le \alpha|\phi(t) - \psi(t)| \\
&+ \int_0^t |A(t-s)|K|\phi(s) - \psi(s)|ds \\
&\le \alpha|\phi - \psi|_2 \\
&+ K\int_{T_1}^t |A(t-s)||\phi(s) - \psi(s)|ds \\
&\le \alpha|\phi - \psi|_2 + |\phi - \psi|_2 \frac{1-\alpha}{2} \\
&\text{by (2.7)} \\
&= \frac{1+\alpha}{2}|\phi - \psi|_2.
\end{aligned}$$

Again we have a contraction and unique fixed point ξ_2 on $[0, T_2]$ agreeing with ξ_1 on $[0, T_1]$. In Step 3 we have a complete metric space on $[0, T_3]$ with all points being ξ_2 on $[0, T_2]$. In $n-2$ steps we have a unique solution on $[0, E]$.

This theorem gives a solution on any interval $[0, E]$. Here is a way to parlay that into a solution on $[0, \infty)$. Employ this theorem to get a solution on $[0, n]$, say x_n, for every positive integer n. Now, extend each of those solutions to a new function x_n^* by continuing with a horizontal line from $x_n^*(n)$ to infinity. Consider that sequence of functions and note that it converges uniformly on compact sets to a continuous function $x(t)$ on $[0, \infty)$ which is a solution on that interval because at every value of t it agrees with one of the x_n^* for $n > t$. This gives that global existence required in the conjecture and it is the fixed point in M.

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