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THE FEKETE-SZEGO INEQUALITY FOR A CLASS OF $p-$VALENT FUNCTIONS

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ABSTRACT. In the present paper,the author considers a subclass of p-valent analytic functions of complex order which is denoted by $S^{\Omega}_{(\lambda,p)}(A,B,b)$ in the open unit disk U and gives the upper bounds for $\left|a_{p+2}-\mu a^2_{p+1}\right|$ when f belongs to $S^{\Omega}_{(\lambda,p)}(A,B,b)$.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots \tag{1}$$

which are analytic in the open unit disc $U=\{z : z\in \mathrm{C}, |z|<1\}$.
Let n, p be integers greater than zero; U is the open unit disc in the complex plane. Furthermore, let $A(p,n)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=n}^{\infty}a_{p+k}z^{p+k} \tag{2}$$

which are analytic and $p-$valent in the open unit disc U. Note that $A=A(1)$.
A function $f\in A(p,n)$ is said to be in the class $S(p,n,\alpha)$ of $p-$valently starlike functions of order α if it satisfies the condition

$$Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha,\quad (z\in U : 0\leq\alpha<p). \tag{3}$$

A function $f\in A(p,n)$ is in $K(p,n,\alpha)$, $p-$valently convex functions of order α, if it is satisfies the condition

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha,\quad (z\in U : 0\leq\alpha<p). \tag{4}$$

In 1991, S. Owa [7] studied the classes $S(p,n,\alpha)$ and $K(p,n,\alpha)$.

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The subclass of A consisting of univalent functions is denoted by S. In 1994, Ma and Minda [9] introduced and studied the class $S^*(\phi)$, consists of functions in $f \in S$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in U).$$

V. Ravichandran et al.[8] defined a class of functions which extends the class of starlike functions of complex order in the following.

Definition 1. [8]*Let $\phi(z)$ be an analytic functions with positive real part on U with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the open unit disc U onto a region starlike with respect to 1 and symmetric with respect to the real axis. Then the class $S_b^*(\phi)$ consists of all analytic functions $f \in A$ satisfying*

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z) \quad (z \in U, b \in \mathrm{C} - \{0\}).$$

The class $C_b(\phi)$ consists of all functions in $f \in A$ for which

$$1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \phi(z) \quad (z \in U, b \in \mathrm{C} - \{0\}).$$

In [8], V. Ravichandran et al. consider the classes $S_b^*(\phi)$ and $C_b(\phi)$ which are obtained by the suitable choices of A, B and b that are in the well-known classes of $S^*(A, B, b)$ and $C(A, B, b)$, where

$$\phi(z) = \frac{1 + Az}{1 + Bz}, \quad b \in \mathrm{C} - \{0\}, \quad (-1 \leq B < A \leq 1).$$

In [8], the following Fekete-Szegö inequality for functions in the class $S_b^*(\phi)$ is obtained.

Theorem 1. [8]*Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by Equation (1) belongs to $S_b^*(\phi)$, then*

$$\left|a_3 - \mu a_2^2\right| \leq 2\max\left\{1; \left|\frac{B_2}{B_1} + (1 - 2\mu)\, bB_1\right|\right\}. \tag{5}$$

This result is sharp.

A function $f(z) \in A$ is said to be starlike functions of complex order b, that is $f(z) \in S(b)$ if and only if

$$Re\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0, \quad (z \in U,\ b \in \mathrm{C} - \{0\}) \tag{6}$$

and $\frac{f(z)}{z} \neq 0$ in U. This class was introduced by Nasr and Aouf [5].
Let $S_p^\lambda(A, B, b)$ be the subclass that consists of functions $f(z) \in A(p, 1)$ that satisfy the condition

$$1 + \frac{1}{b}\left(\frac{1}{p}\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1\right) \prec \frac{1 + Az}{1 + Bz}, \quad b \in \{\mathrm{C} - \{0\}\} \tag{7}$$

where $\prec$ denotes subordination, $-1 \leq B < A \leq 1$ and $z \in U$. In 2004, Shenen et al. [3] introduced the operator $D^\lambda f(z)$ which is the entension of Salagean operator

[4] where

$$D^{\lambda} f(z) = D(D^{\lambda-1} f(z)) = z^p + \sum_{k=1}^{\infty} \left(\frac{p+k}{p}\right)^{\lambda} a_{p+k} z^{p+k}$$

with $\lambda \in \mathrm{N} \bigcup \{0\}$. Akbarally et al. [1] obtained the following Theorem 2 related to upper bounds of the Fekete-Szego functional for the class $S_p^{\lambda}(A, B, b)$.

Theorem 2. [1]*Let* $\frac{1+Az}{1+Bz} = 1 + F_1 z + F_2 z^2 + F_3 z^3 + \cdots$ *. If* $f(z) \in S_p^{\lambda}(A, B, b)$, *then for any complex number* μ,

$$\begin{array}{l} \left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{|b| F_1 p^{\lambda+1}}{(A-B)(p+1)^{\lambda}} \\ \max\left\{1, \left|\frac{1}{(A-B)}\left\{(A+B) + \frac{2F_2}{F_1} + 2pbF_1\left[\frac{(p+1)^{2\lambda} - 2\mu[p(p+2)]^{\lambda}}{(p+1)^{2\lambda}}\right]\right\}\right|\right\} \end{array} \tag{8}$$

This result is sharp.

In [6], the authors introduce the following equalities for the functions $f(z) \in A(p, n)$

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = z\left(D^0 f(z)\right)' = pz^p + \sum_{k=n}^{\infty} (p+k) a_{p+k} z^{p+k}$$

$$\vdots$$

$$D^{\Omega} f(z) = D(D^{\Omega-1} f(z)) = p^{\Omega} z^p + \sum_{k=n}^{\infty} (p+k)^{\Omega} a_{p+k} z^{p+k} \tag{9}$$

In 2011, Kamali and Sağsöz [6] define $\wp_{(\Omega,\lambda,p)} f(z) : A(p,n) \to A(p,n)$ such that

$$\wp_{(\Omega,\lambda,p)} f(z) = \left(\frac{1}{p^{\Omega}} - \lambda\right) D^{\Omega} f(z) + \frac{\lambda}{p} z \left(D^{\Omega} f(z)\right)' \tag{10}$$

where $0 \leq \lambda \leq \frac{1}{p^{\Omega}}$, $\Omega \in \mathrm{N} \bigcup \{0\}$. A function $f(z) \in A(p, n)$ is said to be in the class $\Im(\Omega, \lambda, p, \alpha)$ if ,

$$\begin{aligned} & Re\left\{\frac{z\left(\wp_{(\Omega,\lambda,p)} f(z)\right)'}{\wp_{(\Omega,\lambda,p)} f(z)}\right\} \\ = \quad & Re\left\{\frac{z\left\{\left(\frac{1}{p^{\Omega}} + (\frac{1}{p} - 1)\lambda\right)\left(D^{\Omega} f(z)\right)' + \frac{\lambda}{p} z \left(D^{\Omega} f(z)\right)''\right\}}{\left(\frac{1}{p^{\Omega}} - \lambda\right) D^{\Omega} f(z) + \frac{\lambda}{p} z \left(D^{\Omega} f(z)\right)'}\right\} > \alpha, \end{aligned} \tag{11}$$

for some α $(0 \leq \alpha < p)$, $0 \leq \lambda \leq 1/p^{\Omega}$, $\Omega \in \mathrm{N} \bigcup \{0\}$ and for all $z \in U$. This class was considered and studied earlier by Kamali and Sağsöz [6].
First, we introduce the subclass by denoted $S_{(\lambda,p)}^{\Omega}(A, B, b)$ that consists of functions $f(z) \in A(p, 1)$ that satisfy the condition

$$1 + \frac{1}{b}\left(\frac{1}{p} \cdot \frac{z\left(\wp_{(\Omega,\lambda,p)} f(z)\right)'}{\wp_{(\Omega,\lambda,p)} f(z)} - 1\right) \prec \frac{1+Az}{1+Bz} \tag{12}$$

where $\prec$ denotes subordination, $b \in \mathrm{C} - \{0\}$, A and B are the arbitrary fixed number, $-1 \leq B < A \leq 1$ and $z \in U$. We write $S_{(0,p)}^{\Omega}(A, B, b) = S_p^{\lambda}(A, B, b)$.
In the present paper, we obtain the upper bounds related to Fekete-Szegö inequality for functions in class $S_{(\lambda,p)}^{\Omega}(A, B, b)$.

2. The Fekete-Szego inequality for functions in the class $S^{\Omega}_{(\lambda,p)}(A,B,b)$

To prove our theorem, We need the following Lemma 3 by given Ma and Minda [9].

Lemma 3. [9]*If $p(z)=1+c_1z+c_2z^2+c_3z^3+\cdots$ is a function with positive real part, then for any complex number μ,*

$$\left|c_2-\mu c_1^2\right|\leq 2\max\{1,|2\mu-1|\} \tag{13}$$

and the result is sharp for functions given by $p(z)=\frac{1+z^2}{1-z^2}$ and $p(z)=\frac{1+z}{1-z}$.

Theorem 4. *Let $\zeta(z)=\frac{1+Az}{1+Bz}=1+\delta_1z+\delta_2z^2+\cdots$. If $f(z)\in S^{\Omega}_{(\lambda,p)}(A,B,b)$, then for some complex number μ,*

$$\left|a_{p+2}-\mu a_{p+1}^2\right|\leq \tfrac{|b|p}{(1+2\lambda p^{\Omega-1})}\left(\tfrac{p}{p+2}\right)^{\Omega}\left(\tfrac{\delta_1}{A-B}\right)$$
$$\max\left\{1,\left|\tfrac{1}{(A-B)}\left\{\tfrac{2\delta_2}{\delta_1}+2\delta_1pb\left[1-\tfrac{2p^{\Omega}(p+2)^{\Omega}\left(1+2\lambda p^{\Omega-1}\right)}{(1+\lambda p^{\Omega-1})^2(p+1)^{2\Omega}}\mu\right]+(A+B)\right\}\right|\right\}$$

The result is sharp.

Proof. If $f(z)\in S^{\Omega}_{(\lambda,p)}(A,B,b)$, then there exists a Schwarz function with $w(0)=0$ and $|w|<1$, analytic in the open unit disk such that

$$1+\frac{1}{b}\left(\frac{1}{p}.\frac{z\left(\wp_{(\Omega,\lambda,p)}f(z)\right)'}{\wp_{(\Omega,\lambda,p)}f(z)}-1\right)=\zeta(w(z)). \tag{14}$$

Let $\frac{1+Aw(z)}{1+Bw(z)}=1+c_1z+c_2z^2+c_3z^3+\cdots$, we obtain

$$1+Aw(z)=\{1+Bw(z)\}(1+c_1z+c_2z^2+c_3z^3+\cdots)\Rightarrow$$
$$w(z)=\frac{c_1z+c_2z^2+c_3z^3+\cdots}{(A-B)-B\{c_1z+c_2z^2+c_3z^3+\cdots\}}$$

and thus

$$w(z)=\left(\frac{c_1}{A-B}\right)z+\frac{1}{(A-B)}\left\{c_2+\frac{c_1^2B}{(A-B)}\right\}z^2+\cdots. \tag{15}$$

Since $\zeta(z)=1+\delta_1z+\delta_2z^2+\cdots$, therefore from (15)

$$\zeta(w(z))=1+\frac{\delta_1c_1}{(A-B)}z+\frac{1}{(A-B)}\left\{\frac{B\delta_1c_1^2}{(A-B)}+\delta_1c_2+\frac{\delta_2c_1^2}{(A-B)}\right\}z^2+\cdots. \tag{16}$$

Let

$$1+\frac{1}{b}\left(\frac{1}{p}.\frac{z\left(\wp_{(\Omega,\lambda,p)}f(z)\right)'}{\wp_{(\Omega,\lambda,p)}f(z)}-1\right)=1+h_1z+h_2z^2+\cdots \tag{17}$$

If following equality is used and compared for z and z^2, we obtain

$$1+\frac{\delta_1c_1}{(A-B)}z+\frac{1}{(A-B)}\left\{\frac{B\delta_1c_1^2}{(A-B)}+\delta_1c_2+\frac{\delta_2c_1^2}{(A-B)}\right\}z^2+\cdots$$
$$=\quad 1+h_1z+h_2z^2+\cdots\Rightarrow$$

$$h_1=\frac{\delta_1c_1}{A-B} \tag{18}$$

and

$$h_2=\frac{B\delta_1c_1^2}{(A-B)^2}+\frac{\delta_1c_2}{(A-B)}+\frac{\delta_2c_1^2}{(A-B)^2}. \tag{19}$$

From (17),

$$
\begin{aligned}
& 1+\frac{1}{b}\left(\frac{1}{p}\frac{pz^{p}+\sum\limits_{k=1}^{\infty}\frac{(k+p)^{\Omega+1}}{p^{\Omega}}\left(1+\lambda kp^{\Omega-1}\right)a_{k+p}z^{k+p}}{z^{p}+\sum\limits_{k=1}^{\infty}\frac{(k+p)^{\Omega}}{p^{\Omega}}\left(1+\lambda kp^{\Omega-1}\right)a_{k+p}z^{k+p}}-1\right) \\
= \; & 1+h_1 z+h_2 z^2+\cdots \Rightarrow
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\sum\limits_{k=1}^{\infty}\frac{(k+p)^{\Omega+1}}{p^{\Omega}}\left(1+\lambda kp^{\Omega-1}\right)a_{k+p}z^{k}-\sum\limits_{k=1}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}.p\left(1+\lambda kp^{\Omega-1}\right)a_{k+p}z^{k}}{1+\sum\limits_{k=1}^{\infty}\frac{(k+p)^{\Omega}}{p^{\Omega}}\left(1+\lambda kp^{\Omega-1}\right)a_{k+p}z^{k}} \\
= \; & pbh_1 z+pbh_2 z^2+\cdots \Rightarrow
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\left(\frac{1}{p^{\Omega}}\right)\left\{(p+1)^{\Omega}\left(1+\lambda p^{\Omega-1}\right)a_{p+1}z+(p+2)^{\Omega}\left(1+2\lambda p^{\Omega-1}\right)2a_{p+2}z^2+\cdots\right\}}{1+\frac{(p+1)^{\Omega}}{p^{\Omega}}\left(1+\lambda p^{\Omega-1}\right)a_{p+1}z+\frac{(p+2)^{\Omega}}{p^{\Omega}}\left(1+2\lambda p^{\Omega-1}\right)a_{p+2}z^2+\cdots} \\
= \; & pbh_1 z+pbh_2 z^2+\cdots
\end{aligned}
$$

which yields

$$
\begin{aligned}
& \left(\frac{1+p}{p}\right)^{\Omega}\left(1+\lambda p^{\Omega-1}\right)a_{p+1}z+\left(\frac{2+p}{p}\right)^{\Omega}\left(1+2\lambda p^{\Omega-1}\right)2a_{p+2}z^2+\cdots \\
= & \left\{\begin{array}{c} 1+\left(\frac{1+p}{p}\right)^{\Omega}\left(1+\lambda p^{\Omega-1}\right)a_{p+1}z \\ +\left(\frac{2+p}{p}\right)^{\Omega}\left(1+2\lambda p^{\Omega-1}\right)a_{p+2}z^2+\cdots \end{array}\right\}\left\{pbh_1 z+pbh_2 z^2+\cdots\right\}.
\end{aligned} \tag{20}
$$

Equalizing coefficients of terms z in the both side of equality (20), we have

$$
\frac{(1+p)^{\Omega}}{p^{\Omega}}\left(1+\lambda p^{\Omega-1}\right)a_{p+1}=pbh_1 \Rightarrow h_1=\frac{1}{pb}\left(\frac{1+p}{p}\right)^{\Omega}\left(1+\lambda p^{\Omega-1}\right)a_{p+1}. \tag{21}
$$

Furthermore, equalizing coefficients of terms z^2 in the both side of equality (20), we obtain

$$
\frac{2\,(p+2)^{\Omega}}{p^{\Omega}}\left(1+2\lambda p^{\Omega-1}\right)a_{p+2}=pbh_2+\frac{(p+1)^{\Omega}}{p^{\Omega}}\left(1+\lambda p^{\Omega-1}\right)a_{p+1}pbh_1.
$$

If this equation is withdrawn h_2 and used (21)

$$
\begin{aligned}
& 2\left(1+2\lambda p^{\Omega-1}\right)\frac{(p+2)^{\Omega}}{p^{\Omega}}a_{p+2}=pbh_2+\left(1+\lambda p^{\Omega-1}\right)^2\frac{(p+1)^{2\Omega}}{p^{2\Omega}}a_{p+1}^2 \Rightarrow \\
& h_2=\frac{1}{bp^{\Omega+1}}\left\{2\,(2+p)^{\Omega}\left(1+2\lambda p^{\Omega-1}\right)a_{p+2}-\frac{(1+p)^{2\Omega}}{p^{\Omega}}\left(1+\lambda p^{\Omega-1}\right)^2 a_{p+1}^2\right\}.
\end{aligned} \tag{22}
$$

Equating (18) and (21)

$$
\begin{aligned}
& \frac{\delta_1 c_1}{A-B}=\frac{1}{pb}\frac{(p+1)^{\Omega}}{p^{\Omega}}\left(1+\lambda p^{\Omega-1}\right)a_{p+1} \Rightarrow \\
& a_{p+1}=\frac{pb\delta_1 c_1}{(A-B)\left(1+\lambda p^{\Omega-1}\right)}\left(\frac{p}{1+p}\right)^{\Omega}.
\end{aligned} \tag{23}
$$

Equating (19) and (22), we obtain

$$\frac{1}{bp^{\Omega+1}}\left\{2\left(1+2\lambda p^{\Omega-1}\right)(2+p)^{\Omega}a_{p+2}-\frac{(1+p)^{2\Omega}}{p^{\Omega}}\left(1+\lambda p^{\Omega-1}\right)^{2}a_{p+1}^{2}\right\}$$

$$=\quad\left(\frac{1}{A-B}\right)\left\{\frac{B\delta_1c_1^2}{(A-B)}+\delta_1c_2+\frac{\delta_2c_1^2}{(A-B)}\right\}\Rightarrow$$

$$2(2+p)^{\Omega}\left(1+2\lambda p^{\Omega-1}\right)a_{p+2}=\left[\frac{p^{\Omega}bp\delta_1}{A-B}\right]\left\{\frac{pb\delta_1c_1^2}{(A-B)}+\left\{\begin{array}{c}\frac{Bc_1^2}{(A-B)}+c_2\\+\frac{\delta_2c_1^2}{\delta_1(A-B)}\end{array}\right\}\right\}\Rightarrow$$

$$a_{p+2}=\frac{b}{2}\frac{p^{\Omega+1}}{(p+2)^{\Omega}}\frac{1}{(1+2\lambda p^{\Omega-1})}\left(\frac{\delta_1}{A-B}\right)\left\{\begin{array}{c}\frac{Bc_1^2}{(A-B)}+c_2\\+\frac{\delta_2c_1^2}{\delta_1(A-B)}+\frac{\delta_1c_1^2}{A-B}pb\end{array}\right\}.\tag{24}$$

By using the obtained equalities (23) and (24), we can write

$$\left|a_{p+2}-\mu a_{p+1}^2\right|=\left|\begin{array}{l}\frac{b}{2}\frac{p^{\Omega+1}}{(p+2)^{\Omega}}\frac{1}{(1+2\lambda p^{\Omega-1})}\left(\frac{\delta_1}{A-B}\right)\left\{\begin{array}{c}\frac{Bc_1^2}{(A-B)}+c_2\\+\frac{\delta_2c_1^2}{\delta_1(A-B)}+\frac{\delta_1c_1^2}{A-B}pb\end{array}\right\}\\-\mu\left\{\frac{pb\delta_1c_1}{(A-B)(1+\lambda p^{\Omega-1})}\frac{p^{\Omega}}{(1+p)^{\Omega}}\right\}^2\end{array}\right|.\tag{25}$$

This equality numbered (25) is written as follows

$$\left|a_{p+2}-\mu a_{p+1}^2\right|=\left\{\frac{|b|}{2}\frac{p^{\Omega+1}}{(2+p)^{\Omega}}\frac{1}{(1+2\lambda p^{\Omega-1})}\left(\frac{\delta_1}{A-B}\right)\right\}$$
$$\left|\left\{c_2+\frac{1}{(A-B)}\left[Bc_1^2+\frac{\delta_2c_1^2}{\delta_1}+\delta_1c_1^2pb\right]\right\}-\mu\left\{\frac{2p^{\Omega}(p+2)^{\Omega}pb\delta_1c_1^2\left(1+2\lambda p^{\Omega-1}\right)}{(A-B)(1+\lambda p^{\Omega-1})^2(p+1)^{2\Omega}}\right\}\right|\Rightarrow$$

$$\left|a_{p+2}-\mu a_{p+1}^2\right|=\left\{\frac{|b|}{2(A-B)}\frac{p^{\Omega+1}}{(p+2)^{\Omega}}\frac{\delta_1}{(1+2\lambda p^{\Omega-1})}\right\}$$
$$\left|c_2-\frac{1}{(A-B)}\left\{\frac{2p^{\Omega}(p+2)^{\Omega}pb\delta_1\left(1+2\lambda p^{\Omega-1}\right)}{(1+\lambda p^{\Omega-1})^2(p+1)^{2\Omega}}\mu-\left(B+\frac{\delta_2}{\delta_1}+\delta_1pb\right)\right\}c_1^2\right|$$

and if it is taken as

$$\nu=\frac{1}{(A-B)}\left\{\frac{2p^{\Omega}(p+2)^{\Omega}pb\delta_1\left(1+2\lambda p^{\Omega-1}\right)}{\left(1+\lambda p^{\Omega-1}\right)^2(p+1)^{2\Omega}}\mu-\left(B+\frac{\delta_2}{\delta_1}+\delta_1pb\right)\right\}$$

equality is obtained

$$\left|a_{p+2}-\mu a_{p+1}^2\right|=\left\{\frac{|b|\,p}{2}\left(\frac{p}{p+2}\right)^{\Omega}\frac{1}{(1+2\lambda p^{\Omega-1})}\left(\frac{\delta_1}{A-B}\right)\right\}\left|c_2-\nu c_1^2\right|$$

By using Lemma 3, we obtain

$$\left|a_{p+2}-\mu a_{p+1}^2\right|\leq\left\{\frac{|b|}{2(A-B)}\frac{p^{\Omega+1}}{(p+2)^{\Omega}}\frac{\delta_1}{(1+2\lambda p^{\Omega-1})}\right\}\left[2\max\left\{1,|2\nu-1|\right\}\right]$$

and thus,

$$\left|a_{p+2}-\mu a_{p+1}^2\right|\leq\frac{|b|p}{(1+2\lambda p^{\Omega-1})}\left(\frac{p}{p+2}\right)^{\Omega}\left(\frac{\delta_1}{A-B}\right)$$
$$\max\left\{1,\left[\left|\frac{2}{(A-B)}\left\{\frac{2p^{\Omega}(p+2)^{\Omega}pb\delta_1\left(1+2\lambda p^{\Omega-1}\right)}{(1+\lambda p^{\Omega-1})^2(p+1)^{2\Omega}}\mu-\left(B+\frac{\delta_2}{\delta_1}+\delta_1pb\right)\right\}-\frac{A-B}{A-B}\right|\right]\right\}$$

or

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \frac{|b|p}{(1+2\lambda p^{\Omega-1})}\left(\frac{p}{p+2}\right)^{\Omega}\left(\frac{\delta_1}{A-B}\right)$$
$$\max\left\{1,\left|\frac{1}{(A-B)}\left\{\frac{2\delta_2}{\delta_1}+2\delta_1 pb\left[1-\frac{2p^{\Omega}(p+2)^{\Omega}\left(1+2\lambda p^{\Omega-1}\right)}{(1+\lambda p^{\Omega-1})^2(p+1)^{2\Omega}}\mu\right]+(A+B)\right\}\right|\right\}.$$

This result is sharp for the functions defined by

$$1+\frac{1}{b}\left(\frac{1}{p}\frac{z\left(\wp_{(\Omega,\lambda,p)}f(z)\right)'}{\wp_{(\Omega,\lambda,p)}f(z)}-1\right)=\frac{1+Az^2}{1+Bz^2}$$

and

$$1+\frac{1}{b}\left(\frac{1}{p}\frac{z\left(\wp_{(\Omega,\lambda,p)}f(z)\right)'}{\wp_{(\Omega,\lambda,p)}f(z)}-1\right)=\frac{1+Az}{1+Bz}.$$

□

Lemma 5. (i) *If we set* $\lambda=0$ *in Theorem 4, we obtain the upper bounds of the Fekete-Szego functional for the class* $S_p^{\lambda}(A,B,b)\equiv S_{(0,p)}^{\Omega}(A,B,b)$ *by Akbarally et al.* [1]

(ii) *If we take* $\lambda=\Omega=0$ *in Theorem 4, then we have the results for the class* $S_{(0,1)}^{0}(1,-1,b)\equiv S^*(b)$ *by V. Ravichandran et al.* [8]

Letting $\lambda=\Omega=0$, $p=b=1$, $A=1$ and $B=-1$ in Theorem 4, we have the following Corollary 6 given by Koegh and Merkes [2].

Corollary 6. *If* $f\in S_{(0,1)}^{0}(1,-1,1)\equiv S^*$, *then* $\left|a_3-\mu a_2^2\right|\leq\max\{1,|4\mu-3|\}$.

Also, if $\lambda=0$, $\Omega=p=b=A=1$ and $B=-1$ are taken in Theorem 4, the following Corollary 7 is obtained given by Koegh and Merkes [2].

Corollary 7. *If* $f\in S_{(0,1)}^{1}(1,-1,1)\equiv K$, *then* $\left|a_3-\mu a_2^2\right|\leq\max\left\{\frac{1}{3},|1-\mu|\right\}$

Note. When we get Corollary 6 and Corollary 7 given as a result of Theorem 4 we want to express that if $A=1$ and $B=-1$, it is obvious that $\delta_1=\delta_2=2$.

References

[1] A. Akbarally, S.C. Soh, M. Ismail, Coefficient bounds for a class multivalent function defined by Salagean operator, International Journal of Pure and Applied Mathematics, Volume 83, No. 3, 417-423, 2013.

[2] E.R. Keogh, E.P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc. Amer. Math. Soc., 20, 8-12, 1969.

[3] G.M. Shenen, T.Q. Salim, M.S. Marouf, A certain class of multivalent prestarlike functions involving the Srivastava-Saigo-Owa fractional integral operator, Kyungpook Math.J., 44, 353-362, 2004.

[4] G.S. Salagean, Subclasses of Univalent Functions, Lecture notes in Math., Springer-Verlage, 1013, 362-372, 1983.

[5] M.A. Nasr, M.K. Aouf, Starlike functions of complex order, J.Natural Sci.Math 25, 1-12, 1985.

[6] M. Kamali, F. Sağsöz, Some properties of subclasses of multivalent functions, Abstract and Applied Analysis, volume 2011, Article ID 361647, 15 Pages, 2011.

[7] S. Owa, "Some properties of certain multivalent functions", Applied Mathematics Letters, vol. 4, no.5, pp. 79-83, 1991.

[8] V. Ravichandran, Y. Polatoğlu, M. Bolcal, A. Şen, Certain subclasses of starlike and convex functions of complex order,Hacettepe Journal of Mathematics and Statistics, Volume 34, 9-15, 2005.

[9] W. Ma, D. Minda, A unified treatment of some special classes of univalent functions, In: Proceeding of the Internatinal Conferences on Complex Analysis , Z. Li, F. Ren, L. Yang, and S. Zhang (Eds.), Int. Press, 157-169, 1994.

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