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FRACTIONAL q-CALCULUS OF A UNIFIED q-MITTAG-LEFFLER FUNCTION

B. V. NATHWANI, B. I. DAVE

Abstract. Motivated by the success of the applications of the Mittag-Leffler function in the problems of physics, biology, engineering and applied sciences, we propose here a q-extension of certain generalizations of Mittag-Leffler function. With the aid of q-Riemann-Liouville fractional integral operator, q-Kober fractional integral operator and fractional q-differential operator of arbitrary order, we study certain properties including the q-integro-differential equations of this proposed function.

1. Introduction

The Mittag-Leffler function

$$E_\alpha(z) = \sum_{n=0}^{z^n} \Gamma(\alpha n + 1)$$

was introduced in 1903 by Swedish mathematician Gosta Mittag-Leffler in connection with his method of study of some divergent series ([12], [13]). This function was later generalized by A. Wiman [20] and by T. R. Prabhakar [15] (Table-1 below). In 2007, Shukla and Prajapati [18] further generalized this function in the form :

$$E_{\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{1}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$; $\Re(\alpha, \beta, \gamma) > 0$ and $q \in (0,1) \cup \mathbb{N}$.
We define here a q-extension of (1) as follows.

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Definition 1.1. For $0 < q < 1,\ \alpha,\ \beta,\ \gamma,\ \lambda, \in \mathbb{C},\ \Re(\alpha,\beta,\gamma,\lambda) > 0,\ \delta,\ \mu > 0,\ r \in \{-1,0\} \cup \mathbb{N}, s \in \mathbb{N} \cup \{0\}$,

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q) = \sum_{n=0}^{\infty}(-1)^{pn}\ q^{pn(n-1)/2}\ \frac{(q^{\alpha n+\beta};q)_\infty}{(q;q)_n} \times \frac{[(q^{\lambda+\mu n};q)_\infty]^r}{[(q^{\gamma+\delta n};q)_\infty]^s} z^n, \tag{2}$$

where $p = \alpha^2 + r\mu^2 - s\delta^2 + 1$ with $\Re(p) > 0$.

The objective of constructing this function is (i) to include q-analogue of certain existing generalizations of Mittag-Leffler function, and (ii) to also include the q-analogue of the functions such as Bessel Maitland function, Saxena-Nishimoto function. In fact, it may be seen from Table-1 below that on specializing the parameters appropriately, the q-function (2) yields a q-analogue of the generalized Mittag-Leffler function (1) as well as the q-analogues of Bessel-Maitland function and Saxena-Nishimoto function (see [11], [17], [19]). It is noteworthy that if $p = 0$ then it also includes the q-analogues of Dotsenko function ($r = -1, s = 1, \alpha = \omega/\nu = \mu, \delta = 1$) and the Elliptic function ($r = -1, s = 1, \alpha = 1, \beta = 1, \gamma = 1/2, \delta = 1, \lambda = 1/2, \mu = 1$). substitutions.

Table-1

q-Function of	r	s	α	β	γ	δ	λ	μ
Mittag-Leffler	0	1	α	1	1	1	-	-
Wiman	0	1	α	β	1	1	-	-
Prabhakar	0	1	α	β	γ	1	-	-
Shukla and Prajapati	0	1	α	β	γ	q	-	-
Bessel-Maitland	0	0	μ	$\nu+1$	-	-	-	-
Saxena-Nishimoto	1	1	α_1	β_1	γ	K	β_2	α_2

By taking limit $q \to 1^-$, we get extended Mittag-Leffler function:

$$E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r) = \sum_{n=0}^{\infty} \frac{[(\gamma)_{\delta n}]^s}{\Gamma(\alpha n+\beta)\ [(\lambda)_{\mu n}]^r} \frac{z^n}{n!}, \tag{3}$$

defined in [14] in which the parameters $\alpha,\beta,\gamma,\lambda \in \mathbb{C}$ with $\Re(\alpha,\beta,\gamma,\lambda) > 0,\ \delta,\ \mu > 0,\ r \in \mathbb{N} \cup \{-1,0\}$ and $s \in \mathbb{N} \cup \{0\}$.
In [16], certain fractional calculus properties of such function were studied with the aid of the operator:

$$\left(\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p,\omega;a+}f\right)(x) = \int_a^x (x-t)^{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\rho,p}(\omega(x-t)^\alpha;s,r) \times f(t)\ dt. \tag{4}$$

Here, we consider its q-form and obtain certain results in subsequent sections.

2. Preliminaries

In what follows, the following definitions and formulas will be used.

Definition 2.1. For $a \in \mathbb{C}$, and $0 < |q| < 1$, the q-shifted factorial is defined by [5, Eq.(1.2.15), p.3 and Eq.(1.2.30), p.6]

$$(a;q)_n = \begin{cases} 1 & if\ n=0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & if\ n \in \mathbb{N} \\ \dfrac{(q;q)_\infty}{(aq^n;q)_\infty} & if\ n \in \mathbb{C} \end{cases} \tag{5}$$

where

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)\ , \quad |q|<1.$$

A further extension of thisl is given by [4]

$$[t-|a]_n = (t-a)(t-aq)(t-aq^2)\cdots(t-aq^{n-1}). \tag{6}$$

Definition 2.2. For $x \neq 0$, the q-derivative of a function $f(x)$ is defined by [5, Ex.1.12, p.22]

$$D_q f(x) = \frac{f(x)-f(xq)}{x-xq}. \tag{7}$$

Alternatively, [9]

$$\Delta_q f(x) = \frac{f(x)-f(xq^{-1})}{x-xq^{-1}}. \tag{8}$$

Definition 2.3. For $x \neq 0$, the q-derivative of product of two functions [3] is given by

$$D_q\left(f(x)g(x)\right) = g(qx)D_q f(x) + f(x)D_q(g(x)), \tag{9}$$

$$\Delta_q\left(f(x)g(x)\right) = g(q^{-1}x)\Delta_q f(x) + f(x)\Delta_q(g(x)). \tag{10}$$

Definition 2.4. The q-integrals are defined by [8]

$$\int_0^x f(t)\,\mathrm{d_q t} = x(1-q)\sum_{k=0}^{\infty} q^k\ f(xq^k), \tag{11}$$

and

$$\int_x^\infty f(t)\,\mathrm{d_q t} = x(1-q)\sum_{k=1}^{\infty} q^{-k}\ f(xq^{-k}). \tag{12}$$

Definition 2.5. The q-Beta integral due to W. Hahn [6] is

$$\int_0^1 t^{\lambda-1}E_q(tq)\,\mathrm{d_q t} = (1-q)\frac{(q;q)_\infty}{(q^\lambda;q)_\infty}, \quad \lambda>0. \tag{13}$$

Definition 2.6. A q-Beta function $\mathfrak{B}_q(x,y)$ is expressible in different ways [5].

$$\mathfrak{B}_q(x,y) = \int_0^1 t^{x-1}(tq)_{y-1}\,\mathrm{d_q t}, \tag{14}$$

$$\mathfrak{B}_q(x,y) = \frac{(1-q)\ (q)_\infty\ (q^{x+y})_\infty}{(q^x)_\infty\ (q^y)_\infty}, \tag{15}$$

and

$$\mathfrak{B}_q(x,y) = \int_0^1 t^{x-1} \frac{(tq;q)_\infty}{(tq^y;q)_\infty} \, \mathrm{d_q t} \tag{16}$$

in which $y \neq 0, -1, -2, ..., \Re(x) > 0$.

Definition 2.7. The q-Euler (Beta) transform is [5]:

$$\mathfrak{B}\{f(z) : a,\, b|q\} = \int_0^1 u^{\beta-1} \frac{(uq;q)_\infty}{(uq^\eta;q)_\infty} f(z) \, \mathrm{d_q u}. \tag{17}$$

A q-analogue of Stirling's asymptotic formula [10, Eq.(2.25), p.482] for the q-Gamma function is

$$\Gamma_q(x) \sim (1+q)^{\frac{1}{2}} \, \Gamma_{q^2}\left(\frac{1}{2}\right) \, (1-q)^{\frac{1}{2}-x} \, e^{\mu_q(x)}, \tag{18}$$

where $\mu_q(x) = \dfrac{\theta \, q^x}{1-q-q^x}$, $0 < \theta < 1$.
W. Hahn [6] defined the q-analogues of the Laplace transform:

$$F(S) \; = \; \phi(S) \; = \; \int_0^\infty e^{-St} \, f(t) \, \mathrm{dt},$$

by means of the following two integrals.

Definition 2.8. For $\Re(S) > 0$,

$$\mathcal{L}_q\{f(t)\} \; = \; \frac{1}{(1-q)} \int_0^{S^{-1}} E_q(qSt) \, f(t) \, \mathrm{d_q t}, \tag{19}$$

and

$$\mathcal{L}_q\{f(t)\} \; = \; \frac{1}{(1-q)} \int_0^{\infty} e_q(-St) \, f(t) \, \mathrm{d_q t}. \tag{20}$$

A q-Laplace transform of integration is given by [9]

$$\mathcal{L}_q\left[\int_0^x f(t) \, \mathrm{d_q t}\right] \; = \; \frac{1}{S} \, F_q(S), \tag{21}$$

whereas the formula for q-Laplace transform of differentiation [9] is

$$\mathcal{L}_q\left[D_q \; f(t)\right] \; = \; S \, F_q(S) - f(0). \tag{22}$$

$$\mathcal{L}_q\left[x f(x)\right] \; = \; -\frac{1}{q} \, \Delta_q F_q(S), \tag{23}$$

in which $F_q(S) = \mathcal{L}_q(f(x))(S)$.

Definition 2.9. A q-analogue of Laplace transform of convolution of two functions f_1, f_2 is [9]:

$$\mathcal{L}_q \left[\int_0^x f_1(t)\, f_2(x - tq)\, \mathrm{d_q t} \right] \;=\; F_{1_q}(S)\; F_{2_q}(S), \tag{24}$$

provided that the functions $F_{1_q}(S)$ and $F_{2_q}(S)$ exist; and moreover,

$$F_{1_q}(S) = \mathcal{L}_q(f_1(x))(S),\;\; F_{2_q}(S) = \mathcal{L}_q(f_2(x))(S).$$

Definition 2.10. A q-analogue of Riemann-Liouville fractional integral operator is given by [1]

$${}_qI^{\mu}_{a+}f(x) \;=\; \frac{1}{\Gamma_q(\mu)} \int_a^x (x - |yq)_{\mu-1}\; f(y)\, \mathrm{d_q y}, \tag{25}$$

where μ is an arbitrary order of integration with $Re(\mu) > 0$.
For instance, if $f(x) = x^{\nu-1}$, then (25) gives

$${}_qI^{\mu}_{0+}f(x)[x^{\nu-1}] \;=\; \frac{\Gamma_q(\nu)}{\Gamma_q(\nu+\mu)} x^{\nu+\mu-1}. \tag{26}$$

Definition 2.11. A basic analogue of the Kober fractional integral operator of type η, $\eta \in \mathbb{C}$, is given by [1]

$${}_qI^{\eta,\mu}_{0+}f(t) \;=\; \frac{t^{-\eta-\mu}}{\Gamma_q(\mu)} \int_0^t (t - |xq)_{\mu-1}\; x^{\eta}\; f(x)\, \mathrm{d_q x}, \tag{27}$$

where μ is an arbitrary order of integration with $\Re(\mu) > 0$.

Definition 2.12. A Riemann-Liouville fractional q-differential operator of arbitrary order α, is defined as [2] :

$$\left({}_qD^{\alpha}_{0+}f\right)(x) \;=\; \frac{1}{\Gamma_q(-\alpha)} \int_0^x (x - |yq)_{-\alpha-1}\; f(y)\, \mathrm{d_q y}, \tag{28}$$

in which $\Re(\alpha) < 0,\; 0 < |q| < 1$.

It is to be noted that $\left({}_qD^{\alpha}_{0+}f\right)(x) = D^{\alpha}_{x,q}f(x)$. In this context, we have

$$\left({}_qD^{\alpha}_{a+}f\right)(x) = \left(\frac{d_q}{d_q x}\right)^n \;\left({}_qI^{n-\alpha}_{a+}f\right)(x). \tag{29}$$

For instance, if $f(x) = x^{\mu-1}$, then (28) furnishes

$${}_qD^{\alpha}_{0+}[x^{\mu-1}] \;=\; \frac{\Gamma_q(\mu)}{\Gamma_q(\mu-\alpha)} x^{\mu-\alpha-1}. \tag{30}$$

Note 2.13. The q-analogue of Hilfer's ([7], [**?**]) generalized Riemann-Liouville fractional derivative operator $D^{\mu,\,\nu}_{a+}$ of order μ, $0 < \mu < 1$, and type ν, $0 \le \nu \le 1$, with respect to x may be written in the form:

$$\left({}_qD^{\mu,\,\nu}_{a+}f\right)(x) = \left({}_qI^{\nu(1-\mu)}_{a+}\, \frac{d}{dx}\, \left({}_qI^{(1-\nu)(1-\mu)}_{a+}f\right)\right)(x), \tag{31}$$

where ${}_qI_{a+}^{(1-\nu)(1-\mu)}$ denotes the q-analogue of the Kober fractional integral operator (27). The q-Laplace transform when applied to the equation (31) yields the formula

$$\begin{aligned}&\mathcal{L}_q[\,{}_qD_{0+}^{\mu,\,\nu} f(x)](S)\\ &= S^{\mu}\,\mathcal{L}_q[f(x)](S) - S^{\nu(1-\mu)}(\,{}_qI_{0+}^{(1-\nu)(1-\mu)} f)(0+),\end{aligned} \tag{32}$$

where $0 < \mu < 1$, and $(\,{}_qI_{0+}^{(1-\nu)(1-\mu)} f)(0+)$ is the Riemann-Liouville fractional integral operator of order $(1-\nu)(1-\mu)$ evaluated with the limit as $t \to 0+$. $t \to 0+$.

3. Main Results

3.1. **Convergence.**

Theorem 3.1. *Let $\Re(\alpha,\beta,\gamma,\lambda) > 0,\ \Re(\alpha^2) + r\mu^2 - s\delta^2 + 1 > 0,\ \delta,\mu > 0, r \in \{-1,0\} \cup \mathbb{N},\ s \in \mathbb{N} \cup \{0\}$ and $0 < q < 1$. Then $E_{\alpha,\,\beta,\,\lambda,\,\mu}^{\gamma,\delta}(z;s,r|q)$ is an entire function.*

Proof. Let us put

$$V_n = \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ \Gamma_q(n+1)} \tag{33}$$

to get

$$E_{\alpha,\,\beta,\,\lambda,\,\mu}^{\gamma,\delta}(z;s,r|q) = \sum_{n=0}^{\infty} V_n\ z^n.$$

Then in view of (18), we get after some simplification,

$$\begin{aligned} V_n \quad \sim \quad & \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (1+q)^{\frac{1}{2}(s-r-2)}\ (\Gamma_{q^2}(\frac{1}{2}))^{s-r-2}\ (1-q)^{n+\frac{1}{2}}}{(1-q)^{-s(\frac{1}{2}-\gamma-\delta n)}\ (1-q)^{\frac{1}{2}-\beta-\alpha n}\ (1-q)^{r(\frac{1}{2}-\lambda-\mu n)}} \\ & \times e^{\frac{\theta q^{\gamma+\delta n}}{1-q-q^{\gamma+\delta n}}}\ e^{-\frac{\theta q^{\beta+\alpha n}}{1-q-q^{\beta+\alpha n}}}\ e^{-\frac{\theta q^{\lambda+\mu n}}{1-q-q^{\lambda+\mu n}}}\ e^{-\frac{\theta q^{1+n}}{1-q-q^{1+n}}}. \end{aligned}$$

Hence,

$$\begin{aligned} \sqrt[n]{|V_n|} \quad \sim \quad & \left|\frac{(1+q)^{\frac{1}{2}(s-r-2)}\ (\Gamma_{q^2}(\frac{1}{2}))^{(s-r-2)}\ (1-q)^{s(\frac{1}{2}-\gamma-\delta n)}\ (1-q)^{n+\frac{1}{2}}}{(1-q)^{\frac{1}{2}-\beta-\alpha n}\ (1-q)^{r(\frac{1}{2}-\lambda-\mu n)}}\right|^{\frac{1}{n}} \\ & \times \left| e^{\frac{\theta q^{\gamma+\delta n}}{1-q-q^{\gamma+\delta n}}}\ e^{-\frac{\theta q^{\beta+\alpha n}}{1-q-q^{\beta+\alpha n}}}\ e^{-\frac{\theta q^{\lambda+\mu n}}{1-q-q^{\lambda+\mu n}}}\ e^{-\frac{\theta q^{1+n}}{1-q-q^{1+n}}}\right|^{\frac{1}{n}} \\ & \times \left|(-1)^p\ q^{p(n-1)/2}\right|. \end{aligned}$$

Now making limit $n \to \infty$, we get

$$\begin{aligned} \frac{1}{R} \quad &= \quad \lim_{n\to\infty} \sqrt[n]{|V_n|} \sim \left|(1-q)^{\alpha+r\mu-s\delta+1}\right|\ \lim_{n\to\infty}\left|q^{p(n-1)/2}\right| \\ &= \quad 0 \end{aligned}$$

when $\Re(\alpha^2) + r\mu^2 - s\delta^2 + 1 > 0$. Thus, the function (2) is an *entire* function. $\square$

3.2. **Fractional q-operators.** In this section, the following results are proved.

Theorem 3.2. *Let $a \in [0,\infty)$ and $\alpha, \beta, \gamma, \lambda, \delta, \mu \in \mathbb{N}$, $\eta > 0$ then for $x > a$*

$$\begin{aligned} & \left({}_qI^{\eta}_{a+}[t-|a]_{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[t-|aq^{\beta-1}]_{\alpha}; s, r|q)\right)(x) \\ = \ & [x-|a]_{\beta+\eta-1}\, E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu}(\omega\, [x-|aq^{\beta+\eta-1}]_{\alpha}; s, r|q), \end{aligned} \tag{34}$$

and

$$\begin{aligned} & \left({}_qD^{\eta}_{a+}[t-|a]_{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[t-|aq^{\beta-1}]_{\alpha}; s, r|q)\right)(x) \\ = \ & [x-|a]_{\beta-\eta-1}\, E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu}(\omega[x-|aq^{\beta-\eta-1}]_{\alpha}; s, r|q). \end{aligned} \tag{35}$$

Proof. To prove (34), we begin with

$$\begin{aligned} l.h.s. &= \left({}_qI^{\eta}_{a+}[t-|a]_{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[t-|aq^{\beta-1}]_{\alpha}; s, r|q)\right)(x) \\ &= {}_qI^{\eta}_{a+}[t-|a]_{\beta-1} \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s\ \omega^n}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n} [t-|aq^{\beta-1}]_{\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s\ \omega^n}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n}\ {}_qI^{\eta}_{a+}\big([t-|a]_{\alpha n+\beta-1}\big)(x) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s\ \omega^n}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n}\ \frac{\Gamma_q(\alpha n+\beta)}{\Gamma_q(\alpha n+\beta+\eta)} [x-|a]_{\alpha n+\beta+\eta-1} \\ &= [x-|a]_{\beta+\eta-1}\, E^{\gamma,\delta}_{\alpha,\beta+\eta,\lambda,\mu}(\omega[x-|aq^{\beta+\eta-1}]_{\alpha}; s, r|q) \\ &= r.h.s. \end{aligned}$$

□

The proof of (35) is similar hence omitted. Next, applying the fractional integral operator (25) with $a = 0$, we have

Theorem 3.3. *Let $\alpha, \beta, \gamma, \lambda, \delta, \mu \in \mathbb{N}$, $\eta > 0$ then*

$${}_qI^{\eta}_{0+}\ \big[E^{1,\delta}_{\alpha,1,\lambda,\mu}(t^{\alpha}; s, r|q)\big](x) = (x(1-q))^{\eta}\, E^{1,\delta}_{\alpha,\eta+1,\lambda,\mu}(x^{\alpha}; s, r|q). \tag{36}$$

Proof. Here the left hand member

$$\begin{aligned} & {}_qI^{\eta}_{0+}\ \big[E^{1,\delta}_{\alpha,1,\lambda,\mu}(t^{\alpha}; s, r|q)\big](x) \\ = \ & {}_qI^{\eta}_{0+}\left[\sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+1};q)_{\infty}\ [(q^{\lambda+\mu n};q)_{\infty}]^r\ (q^{n+1};q)_{\infty}\ t^{\alpha n}}{[(q^{1+\delta n};q)_{\infty}]^s}\right] \\ = \ & \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+1};q)_{\infty}\ (q^{n+1};q)_{\infty}}{[(q^{\lambda+\mu n};q)_{\infty}]^{-r}\ [(q^{1+\delta n};q)_{\infty}]^s}\ {}_qI^{\eta}_{0+}(t^{\alpha n})(x) \\ = \ & \sum_{n=0}^{\infty} \frac{(-1)^{pn} q^{pn(n-1)/2}\ (q^{\alpha n+1};q)_{\infty}\ (q^{n+1};q)_{\infty}}{[(q^{\lambda+\mu n};q)_{\infty}]^{-r}\ [(q^{1+\delta n};q)_{\infty}]^s} \frac{1}{\Gamma_q(\eta)} \int_0^x t^{\alpha n}(x-|tq)_{\eta-1}\, \mathrm{d_q}t. \end{aligned}$$

Now taking $t = xu$ and using (14), we get

$$\begin{aligned}
l.h.s. &= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+1};q)_\infty\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{n+1};q)_\infty}{[(q^{1+\delta n};q)_\infty]^s} \\
&\quad \times \frac{x^{\alpha n+\eta}}{\Gamma_q(\eta)} \int_0^1 u^{\alpha n}\ (uq;q)_{\eta-1}\ \mathrm{d_q}u \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+1};q)_\infty\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{n+1};q)_\infty}{[(q^{1+\delta n};q)_\infty]^s\ (q;q)_n} \\
&\quad \times \frac{x^{\eta+\alpha n}}{\Gamma_q(\eta)} \frac{(1-q)\ (q;q)_\infty\ (q^{\alpha n+\eta+1};q)_\infty}{(q^\eta;q)_\infty\ (q^{\alpha n+1};q)_\infty} \\
&= (x(1-q))^\eta \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{n+1};q)_\infty\ (q^{\alpha n+\eta+1};q)_\infty\ x^{\alpha n}}{[(q^{\lambda+\mu n};q)_\infty]^{-r}\ [(q^{1+\delta n};q)_\infty]^s\ (q;q)_n} \\
&= r.h.s.
\end{aligned}$$

□

The following theorem uses q-generalized differential operator (31).

Theorem 3.4. *Let $a \in [0,\infty),\ \alpha,\beta,\gamma,\lambda,\delta,\mu \in \mathbb{N},\ \eta,\nu > 0$ for $x > a$, then*

$$\begin{aligned}
&\left({}_qD^{\eta,\nu}_{a+}[t-|a]_{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[t-|aq^{\beta-1}]_\alpha;s,r|q)\right)(x) \\
&= [x-|a]_{\beta-\eta-1}\ E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu}(\omega[x-|aq^{\beta-\eta-1}]_\alpha;s,r|q). \qquad (37)
\end{aligned}$$

Proof. In view of (31), we have

$$\begin{aligned}
&\left({}_qD^{\eta,\nu}_{a+}[t-|a]_{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[t-|aq^{\beta-1}]_\alpha;s,r|q)\right)(x) \\
&= \left({}_qD^{\eta,\nu}_{a+}[t-|a]_{\beta-1}\sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s\ \omega^n}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n}[t-|aq^{\beta-1}]_{\alpha n}\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s\ \omega^n}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n}\ {}_qD^{\eta,\nu}_{a+}\big([t-|a]_{\alpha n+\beta-1}\big)(x) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s\ \omega^n}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ (q;q)_n}\ \frac{\Gamma_q(\alpha n+\beta)}{\Gamma_q(\alpha n+\beta-\eta)}[x-|a]_{\alpha n+\beta-\eta-1} \\
&= [x-|a]_{\beta-\eta-1}\ E^{\gamma,\delta}_{\alpha,\beta-\eta,\lambda,\mu}(\omega[x-|aq^{\beta-\eta-1}]_\alpha;s,r|q).
\end{aligned}$$

□

A q-analogue of the operator $\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+}$ stated in (4) may be defined with $p = \rho = 1$, as follows.

Definition 3.5. Let $\alpha,\beta,\gamma,\lambda,\delta,\mu \in \mathbb{N},\ r \in \mathbb{N}\cup\{-1,0\},\ s \in \mathbb{N}\cup\{0\},\ \omega \in \mathbb{C}$ and $x > a$, then

$$\left({}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+}f\right)(x) = \int_a^x (x-|tq)_{\beta-1}\ E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega(x-tq^\beta)_\alpha;s,r|q)\ f(t)\ \mathrm{d_q}t, \qquad (38)$$

wherein $\alpha^2 + r\mu^2 - s\delta^2 + 1 > 0$.

This q-operator turns out to be bounded. This is proved in

Theorem 3.6. *Let the function* ϕ *be in the space* $L(a,b) = \big\{ f : {}_q\|f\|_1 = \int_a^b |f(t)|\, \mathrm{d_q}t < \infty \big\}$ *of Lebesgue measurable functions on a finite interval* $[a,b]$*. Then the integral operator* ${}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+}$ *is bounded on* $L(a,b)$*.*

Proof. It suffice to show that

$$\begin{aligned} & {}_q\|{}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+}\phi\|_1 \\ = & \int_a^b \left| \int_a^x [x-tq]_{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[x-tq^\beta]_\alpha; s, r|q)\, \phi(t)\, \mathrm{d_q}t \right| \mathrm{d_q}x < \infty. \end{aligned}$$

By Fubini's theorem,

$$\begin{aligned} & {}_q\|{}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,a+}\phi\|_1 \\ = & \int_a^b \left| \int_a^x [x-tq]_{\beta-1}\, E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega[x-tq^\beta]_\alpha; s, r)\, \phi(t)\, \mathrm{d_q}t \right| \mathrm{d_q}x \\ \leq & \int_a^b \left[\int_t^b [x-tq]_{\beta-1} \left| E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\,\omega[x-tq^\beta]_\alpha; s, r) \right| \mathrm{d_q}x \right] |\phi(t)|\, \mathrm{d_q}t \\ = & \left| \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{\alpha n+\beta};q)_\infty\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s} \right| \\ & \times \int_a^b \int_t^b [x-tq]_{\alpha n+\beta-1}\, \mathrm{d_q}x\ |\phi(t)| \mathrm{d_q}t \\ = & \left| \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{\alpha n+\beta};q)_\infty\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s} \right| \mathcal{I}, \end{aligned} \tag{39}$$

where

$$\begin{aligned} \mathcal{I} &= \int_a^b \int_t^b [x-tq]_{\alpha n+\beta-1}\, \mathrm{d_q}x\ |\phi(t)|\, \mathrm{d}_qt \\ &\leq \int_a^b \left[\int_a^b [x-tq]_{\alpha n+\beta-1}\, \mathrm{d_q}x \right] |\phi(t)|\, \mathrm{d}_qt \\ &= \int_a^b \left[[x-tq]_{\alpha n+\beta} \left(\frac{1-q}{1-q^{\alpha n+\beta}} \right) \right]_a^b |\phi(t)|\, \mathrm{d_q}t \\ &= \left(\frac{1-q}{1-q^{\alpha n+\beta}} \right) \int_a^b \left([b-tq]_{\alpha n+\beta} - [a-tq]_{\alpha n+\beta} \right)\ |\phi(t)|\, \mathrm{d_q}t \\ &= \left(\frac{1-q}{1-q^{\alpha n+\beta}} \right) \left(\int_a^b [b-tq]_{\alpha n+\beta}\ |\phi(t)|\, \mathrm{d_q}t - \int_a^b [a-tq]_{\alpha n+\beta}\ |\phi(t)|\, \mathrm{d_q}t \right) \\ &= \left(\frac{1-q}{1-q^{\alpha n+\beta}} \right) (\mathcal{I}_1 - \mathcal{I}_2),\ \ \textit{say}. \end{aligned} \tag{40}$$

Now

$$\begin{aligned}
\mathcal{I}_1 &= \int_a^b [b-tq]_{\alpha n+\beta}\ |\phi(t)|\ \mathrm{d}_q t \\
&= \left(\int_0^b [b-tq]_{\alpha n+\beta}\ |\phi(t)|\ \mathrm{d_q} t - \int_0^a [b-tq]_{\alpha n+\beta}\ |\phi(t)|\ \mathrm{d_q} t\right) \\
&= \int_0^b [b-tq]_{\alpha n+\beta}\ D_q\left(\int_0^t |\phi(u)|\ \mathrm{d_q} u\right) \mathrm{d}_q t \\
&\quad - \int_0^a [b-tq]_{\alpha n+\beta}\ D_q\left(\int_0^t |\phi(u)|\ \mathrm{d_q} u\right) \mathrm{d}_q t \\
&= \mathcal{I}_{11} - \mathcal{I}_{12}. \qquad (41)
\end{aligned}$$

Here

$$\begin{aligned}
\mathcal{I}_{11} &= \int_0^b [b-tq]_{\alpha n+\beta}\ D_q\left(\int_0^t |\phi(u)|\ \mathrm{d_q} u\right)\ \mathrm{d_q} t \\
&= \left[[b-tq]_{\alpha n+\beta}\ \left(\int_0^t |\phi(u)|\ \mathrm{d_q} u\right)\right]_0^b - \int_0^b \left(\int_0^t |\phi(qu)|\ \mathrm{d_q} u\right) \\
&\quad \times \frac{(-q)(1-q^{\alpha n+\beta})}{1-q}[b-tq^2]_{\alpha n+\beta-1}\ \mathrm{d_q} t \\
&= [b-bq]_{\alpha n+\beta}\ \int_0^b |\phi(u)|\ \mathrm{d_q} u - \frac{(-q)(1-q^{\alpha n+\beta})}{1-q} \\
&\quad \times \int_0^b \left[\int_0^t |\phi(qu)|\ \mathrm{d_q} u\ [b-tq^2]_{\alpha n+\beta-1}\right]\ \mathrm{d_q} t \\
&\leq [b-bq]_{\alpha n+\beta}\ \int_0^b |\phi(u)|\ \mathrm{d_q} u - \frac{(-q)(1-q^{\alpha n+\beta})}{1-q} \\
&\quad \times \int_0^b \left[\int_0^b |\phi(qu)|\ \mathrm{d_q} u\ [b-tq^2]_{\alpha n+\beta-1}\right]\ \mathrm{d_q} t.
\end{aligned}$$

Since $\phi \in L(a,b)$,

$$\int_0^b |\phi(u)|\ \mathrm{d_q} u = \mathcal{M}_1 (= a\ finite\ value)$$

hence

$$\begin{aligned}
\mathcal{I}_{11} &\leq [b-bq]_{\alpha n+\beta}\ \mathcal{M}_1 - \frac{(-q)(1-q^{\alpha n+\beta})}{1-q}\int_0^b \mathcal{M}_1\ [b-tq^2]_{\alpha n+\beta-1}\,\mathrm{d_q}t \\
&= \mathcal{M}_1\Bigg([b-bq]_{\alpha n+\beta} - \frac{(-q)(1-q^{\alpha n+\beta})}{1-q}\frac{1-q}{(-q)(1-q^{\alpha n+\beta})} \\
&\quad \times [[b-tq]_{\alpha n+\beta}]_0^b\Bigg) \\
&= \mathcal{M}_1\left([b-bq]_{\alpha n+\beta} - ([b-bq]_{\alpha n+\beta} - [b-0q]_{\alpha n+\beta})\right) \\
&= \mathcal{M}_1\ b^{\alpha n+\beta}. \qquad (42)
\end{aligned}$$

Analogously, it can be shown that

$$\mathcal{I}_{12} \leq \mathcal{M}_2\ b^{\alpha n+\beta}. \qquad (43)$$

Using (42) and (43) in (41), one obtains

$$\mathcal{I}_1 \leq \mathcal{M}_1\ b^{\alpha n+\beta} - \mathcal{M}_2\ b^{\alpha n+\beta}$$

and likewise,

$$\mathcal{I}_2 \leq \mathcal{M}_1\ a^{\alpha n+\beta} - \mathcal{M}_2\ a^{\alpha n+\beta}.$$

Consequently, (40) leads us to

$$\begin{aligned}
\mathcal{I} &= \int_a^b \int_t^b [x-tq]_{\alpha n+\beta-1}\,\mathrm{d_q}x\ |\phi(t)|\,\mathrm{d_q}t \\
&\leq \left(\frac{1-q}{1-q^{\alpha n+\beta}}\right)\ \left(\left(\mathcal{M}_1\ b^{\alpha n+\beta} - \mathcal{M}_2\ b^{\alpha n+\beta}\right) - \left(\mathcal{M}_1\ a^{\alpha n+\beta} - \mathcal{M}_2\ a^{\alpha n+\beta}\right)\right) \\
&= \left(\frac{1-q}{1-q^{\alpha n+\beta}}\right)\ (\mathcal{M}_1-\mathcal{M}_2)\ b^{\alpha n+\beta} - (\mathcal{M}_1-\mathcal{M}_2)\ a^{\alpha n+\beta} \\
&= \left(\frac{1-q}{1-q^{\alpha n+\beta}}\right)\ (\mathcal{M}_1-\mathcal{M}_2)\ \left(b^{\alpha n+\beta} - a^{\alpha n+\beta}\right).
\end{aligned}$$

Using this in (39), we finally find

$$\begin{aligned}
&{}_q\|{}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\ a+}\phi\|_1 \\
&\leq \left|\sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{\alpha n+\beta};q)_\infty\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s}\right| \\
&\quad \times \left(\frac{1-q}{1-q^{\alpha n+\beta}}\right)\ (\mathcal{M}_1-\mathcal{M}_2)\ \left(b^{\alpha n+\beta} - a^{\alpha n+\beta}\right) \\
&\leq \sum_{n=0}^{\infty} \frac{q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{\alpha n+\beta};q)_\infty\ |\omega|^n}{[(q^{\gamma+\delta n};q)_\infty]^s} \\
&\quad \times \left(\frac{1-q}{1-q^{\alpha n+\beta}}\right)\ (\mathcal{M}_1-\mathcal{M}_2)\ \left(b^{\alpha n+\beta} - a^{\alpha n+\beta}\right).
\end{aligned}$$

The proof is completed in view of Theorem 3.1, as the series on right hand side represents the entire function. □

3.3. **Fractional q-differential equations involving the q-analogue of Hilfer derivative operator.** Here, the fractional q-differential equations corresponding to the q-analogue of Hilfer derivative operator are obtained. For that the following lemma is required.

Lemma 3.7. *In the notations of q-Laplace transform (19) and the operator defined by (38),*

$$\mathcal{L}_q\left({}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}\right)(x)(S) = \sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s} \times\frac{(q;q)_\infty\ q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n+\beta+1}}.$$

Proof. We have

$$\begin{aligned}
&\left({}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}\phi\right)(x)\\
=&\int_0^x (x-|tq)_{\beta-1}\,E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(\omega\ (x-tq^\beta)_\alpha;s,r|q)\ 1(t)\ \mathrm{d_q}t\\
=&\int_0^x (x-|tq)_{\beta-1}\sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{n+1};q)_\infty\ (q^{\alpha n+\beta};q)_\infty\ \omega^n}{[(q^{\lambda+\mu n};q)_\infty]^{-r}\ [(q^{\gamma+\delta n};q)_\infty]^s}\\
&\times(x-tq^\beta)_{\alpha n}\ \ 1(t)\ \mathrm{d_q}t\\
=&\sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r}{[(q^{\gamma+\delta n};q)_\infty]^s}\\
&\times(q^{\alpha n+\beta};q)_\infty\ \omega^n\int_0^x (x-|tq)_{\alpha n+\beta-1}\ 1(t)\ \mathrm{d_q}t.
\end{aligned}$$

By applying q-Laplace transform on both the sides, we get

$$\begin{aligned}
&\mathcal{L}_q\left({}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}\right)(x)(S)\\
=&\ \mathcal{L}_q\Bigg(\sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{\alpha n+\beta};q)_\infty\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s}\\
&\times\int_0^x (x-|tq)_{\alpha n+\beta-1}\ 1(t)\ \mathrm{d_q}t\Bigg)\\
=&\sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{\alpha n+\beta};q)_\infty\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s}\\
&\times\mathcal{L}_q\left(\int_0^x (x-|tq)_{\alpha n+\beta-1}\ 1(t)\ \mathrm{d_q}t\right).
\end{aligned}\tag{44}$$

From the definition of q-Laplace transforms (20),

$$\mathcal{L}_q\left(x^{\alpha n+\beta-1}\right)=\frac{1}{1-q}\int_0^\infty e_q(-Sx)\,x^{\alpha n+\beta-1}\,\mathrm{d_q}x. \tag{45}$$

Here letting $Sx=t$ and by making use of the q-integral formula:

$$\int_0^\infty t^{\alpha-1}\ e_q(-t)\,\mathrm{d_q}t=\frac{(1-q)\,(q;q)_\infty\,q^{-\alpha(\alpha-1)/2}}{(q^\alpha;q)_\infty\ S^{\alpha n+1}},$$

we get

$$\mathcal{L}_q(x^{\alpha n+\beta-1})=\frac{(q;q)_\infty\,q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n+\beta}\ (q^{\alpha n+\beta};q)_\infty}.$$

In (44), using the convolution theorem:

$$\mathcal{L}_q\left[\int_0^x f_1(t)\,f_2(x-tq)\,\mathrm{d_q}t\right]\ =\ F_{1_q}(S)\ F_{2_q}(S), \tag{46}$$

with $F_{1_q}(S)=\mathcal{L}_q(f_1(x))(S)$ and $F_{2_q}(S)=\mathcal{L}_q(f_2(x))(S)$, we find

$$\begin{aligned}\mathcal{L}_q\left({}_q\mathcal{E}^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}\right)(x)(S)\ &=\ \sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{\alpha n+\beta};q)_\infty\ \omega^n}{[(q^{\lambda+\mu n};q)_\infty]^{-r}\ [(q^{\gamma+\delta n};q)_\infty]^s}\\ &\quad\times\frac{(q;q)_\infty\ q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n+\beta}\ (q^{\alpha n+\beta};q)_\infty}\ \frac{1}{S}\\ &=\ \sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s}\\ &\quad\times\frac{(q;q)_\infty\ q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n+\beta+1}}.\end{aligned}$$

□

The fractional differential equation corresponding to the q-operator (38) is obtained in

Theorem 3.8. *If* $0<\eta<1,\ 0\le\nu\le 1,\ \omega,\xi\in\mathbb{C},\ \alpha>max\{0,\delta-1\}$ *then*

$$\left({}_qD^{\eta,\,\nu}_{0+}\ y\right)(x)\ =\ \xi\left({}_q\mathcal{E}^{\gamma,\,\delta}_{\alpha,\beta,\lambda,\mu,\omega;0+}\right)(x)\ +\ f(x) \tag{47}$$

with the initial condition

$$\left({}_qI^{(1-\nu)(1-\eta)}_{0+}\ y\right)(0+)\ =\ C, \tag{48}$$

has solution

$$\begin{aligned}y(x)\ &=\ C\ \frac{q^{(\eta-\nu(1-\eta))(\eta-\nu(1-\eta)-1)/2}}{\Gamma_q(\eta-\nu(1-\eta))}(1-q)^{1-\eta+\nu-\eta\nu}x^{\eta-\nu(1-\eta)-1}+\xi\,x^{\beta+\eta}\\ &\quad\times(1-q)^{-\eta-1}q^{\eta(\eta+1)/2+\beta(\eta+1)}\ E^{\gamma,\,\delta}_{\alpha,\ \beta+\eta+1,\ \lambda,\ \mu}\left(\omega\,(xq^{\eta+1})^\alpha;\ s,\ r|q\right)\\ &\quad+\frac{(1-q)1-\eta\ q^{\eta(\eta-1)/2}}{\Gamma_q(\eta)}\int_0^x f(t)\ (x-|tq)_{\eta-1}\,\mathrm{d_q}t, \end{aligned} \tag{49}$$

in the space $L(0,\infty)$ where C is arbitrary constant.

Proof. Applying q-Laplace transform on both the sides of (47), we have

$$\mathcal{L}_q\left({}_qD_{0+}^{\eta,\,\nu}\ y\right)(x)(S) \;=\; \xi\ \mathcal{L}_q\left({}_q\mathcal{E}_{\alpha,\,\beta,\,\lambda,\,\mu,\,\omega;0+}^{\gamma,\,\delta}\right)(x)(S)+\ \mathcal{L}_q\ f(x)(S).$$

In the light of Lemma 3.7 and the formula (32), this gives

$$\begin{aligned} S^{\eta}\,Y(S)-CS^{\nu(1-\eta)} \;&=\; \xi S^{-\beta-1}\sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{n+1};q)_\infty\ \omega^n}{[(q^{\lambda+\mu n};q)_\infty]^{-r}\ [(q^{\gamma+\delta n};q)_\infty]^{s}} \\ &\quad\times\frac{(q;q)_\infty\ q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n}}+F(S). \end{aligned}$$

That is,

$$\begin{aligned} Y(S) \;=\; CS^{\nu(1-\eta)-\eta}+\xi\, S^{-\beta-\eta-1}&\sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (q^{n+1};q)_\infty\ \omega^n}{[(q^{\lambda+\mu n};q)_\infty]^{-r}\ [(q^{\gamma+\delta n};q)_\infty]^{s}} \\ \times\frac{(q;q)_\infty\ q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n}}&+S^{-\eta}F(S). \end{aligned}$$

Here using the inverse q-Laplace transform given by

$$\mathcal{L}_q^{-1}\big(\frac{1}{S^{\alpha n+\beta}}\big)=\frac{(q^{\alpha n+\beta};q)_\infty\ x^{\alpha n+\beta-1}}{q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}\ (q;q)_\infty},$$

we further get

$$\begin{aligned} \mathcal{L}_q^{-1}(Y(S)) \;&=\; C\mathcal{L}_q^{-1}(S^{\nu(1-\eta)-\eta})(x) \\ &\quad+\xi\sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}\ (q^{n+1};q)_\infty\ \omega^n}{[(q^{\lambda+\mu n};q)_\infty]^{-r}\ [(q^{\gamma+\delta n};q)_\infty]^{s}\ (q;q)_n\ (q;q)_\infty} \\ &\quad\times\mathcal{L}_q^{-1}\big(\frac{1}{S^{\alpha n+\beta+\eta+1}}\big)+\mathcal{L}_q^{-1}\big(\frac{1}{S^{\eta}}F(S)\big). \end{aligned}$$

Thus,

$$\begin{aligned}
y(x) &= C\frac{q^{(\eta-\nu(1-\eta))(\eta-\nu(1-\eta)-1)/2}}{\Gamma_q(\eta-\nu(1-\eta))}(1-q)^{1-\eta+\nu-\eta\nu}x^{\eta-\nu(1-\eta)-1} \\
&\quad +\xi\sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}\ (q^{n+1};q)_\infty\ \omega^n\ (q;q)_\infty}{[(q^{\lambda+\mu n};q)_\infty]^{-r}\ [(q^{\gamma+\delta n};q)_\infty]^s} \\
&\quad \times\frac{(q^{\alpha n+\beta+\eta+1};q)_\infty\ x^{\alpha n+\beta+\eta}\ q^{(\alpha n+\beta)(\alpha n+\beta-1)/2}}{(q;q)_\infty} \\
&\quad +\frac{(1-q)^{1-\eta}\ q^{\eta(\eta-1)/2}}{\Gamma_q(\eta)}\int_0^x f(t)(x-|tq)_{\eta-1}\,\mathrm{d_q}t \\
&= C\frac{q^{(\eta-\nu(1-\eta))(\eta-\nu(1-\eta)-1)/2}}{\Gamma_q(\eta-\nu(1-\eta))}(1-q)^{1-\eta+\nu-\eta\nu}x^{\eta-\nu(1-\eta)-1} \\
&\quad +\xi x^{\beta+\eta}\sum_{n=0}^{\infty} q^{\eta(\eta+1)/2}q^{(1+\eta)(\alpha n+\beta)}(q^{n+1};q)_\infty \\
&\quad \times\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q^{\alpha n+\beta+\eta+1};q)_\infty\ (\omega\ x^\alpha)^n}{[(q^{\gamma+\delta n};q)_\infty]^s} \\
&\quad +\frac{(1-q)^{1-\eta}\ q^{\eta(\eta-1)/2}}{\Gamma_q(\eta)}\int_0^x f(t)(x-|tq)_{\eta-1}\,\mathrm{d_q}t \\
&= C\frac{q^{(\eta-\nu(1-\eta))(\eta-\nu(1-\eta)-1)/2}}{\Gamma_q(\eta-\nu(1-\eta))}(1-q)^{1-\eta+\nu-\eta\nu}x^{\eta-\nu(1-\eta)-1}+\xi\ x^{\beta+\eta} \\
&\quad \times q^{\eta(\eta+1)/2+\beta(\eta+1)}\ E^{\gamma,\,\delta}_{\alpha,\,\beta+\eta+1,\,\lambda,\,\mu}\,(\omega\,(xq^{\eta+1})^\alpha;s,r|q) \\
&\quad +\frac{(1-q)^{1-\eta}\ q^{\eta(\eta-1)/2}}{\Gamma_q(\eta)}\int_0^x f(t)(x-|tq)_{\eta-1}\,\mathrm{d_q}t.
\end{aligned}$$

□

The particular case $f(x)=x^\beta\ E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}\big((\omega(ax)^\alpha);s,r|q\big)$ of this theorem is

Theorem 3.9. *The q-differential equation*

$$\left({}_qD^{\eta,\,\nu}_{0+}\ y\right)(x)\ =\ \xi\ \left({}_q\mathcal{E}^{\gamma,\,\delta}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}\right)(x)\ +\ x^\beta\ E^{\gamma,\delta}_{\alpha,\beta+1,\lambda,\mu}\big((\omega(ax)^\alpha);s,r|q\big) \tag{50}$$

with the initial condition

$$\left({}_qI^{(1-\nu)(1-\eta)}_{0+}\ y\right)(0+)\ =\ C, \tag{51}$$

has solution in the space $L(0,\infty)$ which is given by

$$\begin{aligned}
y(x)=C\ &\frac{q^{(\eta-\nu(1-\eta))(\eta-\nu(1-\eta)-1)/2}}{\Gamma_q(\eta-\nu(1-\eta))}(1-q)^{1-\eta+\nu-\eta\nu}x^{\eta-\nu(1-\eta)-1}+q^{\eta(\eta+1)/2}\ x^{\beta+\eta} \\
&\times(\xi\ (1-q)^{-\eta-1}\ q^{\beta(\eta+1)}+1)\ E^{\gamma,\delta}_{\alpha,\beta+\eta+1,\lambda,\mu}(\omega(xq^{\eta+1})^\alpha;s,r|q)
\end{aligned} \tag{52}$$

in which C is arbitrary constant.

Another fractional q-differential equation with the operator ${}_qD_{0+}^{\eta,\,\nu}$ is replaced by $x\ {}_qD_{0+}^{\eta,\,\nu}$, is derived in

Theorem 3.10. *If* $0<\eta<\ 1,\ 0\leq\nu\leq 1,\ \omega,\xi\in\mathbb{C},\ \alpha>max\{0,\delta-1\}$ *then*

$$\left(x\ {}_qD_{0+}^{\eta,\,\nu}\ y\right)(x)\ =\ \xi\left({}_q\mathcal{E}_{\alpha,\beta,\lambda,\mu,\omega;0+}^{\gamma,\,\delta}\right)(x) \tag{53}$$

with the initial condition

$$\left({}_qI_0+^{(1-\nu)(1-\eta)}\ y\right)(0+)\ =\ C, \tag{54}$$

has solution in the space $L(0,\infty)$ *given by*

$$\begin{aligned}
y(x) &= (q^{\eta+1}-q)\sum_{j=0}^{\infty}q^j\ y(xq^j)\\
&\quad+C\frac{q^{\eta}(1-q^{-\nu(1-\eta)})}{1-q^{-1}}\left(\frac{x^{\eta-\nu(1-\eta)-1}\ (q^{\eta-\nu(1-\eta)+1};q)_{\infty}}{(q;q)_{\infty}\ q^{-(\eta-\nu(1-\eta)+1)(\eta-\nu(1-\eta))/2}}\right)\\
&\quad+\frac{x^{\beta+\eta-1}}{q^{-\eta(\eta-1)/2-\beta(\eta+1)}}E_{\alpha,\beta+\eta+1,\lambda,\mu}^{\gamma,\delta}(\omega\ (xq^{\eta+1})^{\alpha};s,r|q),
\end{aligned} \tag{55}$$

wherein C *is arbitrary constant.*

Proof. Applying q-Laplace transform on both the sides of (53), we get

$$\mathcal{L}_q\left(x\ {}_qD_{0+}^{\eta,\,\nu}\ y\right)(x)(S)\ =\ \xi\ \mathcal{L}_q\left({}_q\mathcal{E}_{\alpha,\beta,\lambda,\mu,\omega;\,0+}^{\gamma,\delta}\right)(x)(S).$$

In this, the left hand side simplification is given by

$$\begin{aligned}
\mathcal{L}_q\left(x\ {}_qD_{0+}^{\eta,\nu}\ y\right)(x)(S) &= \xi\sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_{\infty}]^r\ \omega^n}{[(q^{\gamma+\delta n};q)_{\infty}]^s}\\
&\quad\times\frac{(q;q)_{\infty}\ q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n+\beta+1}}.
\end{aligned} \tag{56}$$

Now from the formulas (23), (54), (10), and (8) in turn, it follows that

$$\begin{aligned}
&\mathcal{L}_q\left[x\left({}_qD_{0+}^{\eta,\,\nu}\ y\right)(x)\right]\\
&= -\frac{1}{q}\ \Delta_q\left(\mathcal{L}_q\left({}_qD_{0+}^{\eta,\,\nu}\ y\right)(x)(S)\right)\\
&= -\frac{1}{q}\ \Delta_q\left(S^{\eta}\ \mathcal{L}_q[y(x)](S)-S^{\nu(1-\eta)}(\,{}_qI_0^{(1-\nu)(1-\eta)}f)(0+)\right)\\
&= -\frac{1}{q}\ \Delta_q\left(S^{\eta}\ Y_q(S)-CS^{\nu(1-\eta)}\right)\\
&= -\frac{1}{q}\ \left((Sq^{-1})^{\eta}\ \Delta_q(Y_q(S))+Y_q(S)\ \Delta_q(S^{\eta})\right)-C\Delta_q\left(S^{\nu(1-\eta)}\right)\\
&= -\frac{1}{q}\left(S^{\eta}q^{-\eta}\ \Delta_q(Y_q(S))+Y_q(S)\ \frac{S^{\eta}-S^{\eta}q^{-\eta}}{S-Sq^{-1}}\right.\\
&\quad\left.-\ C\frac{S^{\nu(1-\eta)}-S^{\nu(1-\eta)}q^{-\nu(1-\eta)}}{S-Sq^{-1}}\right)\\
&= -\frac{1}{q}\left(q^{-\eta}\ \Delta_q(Y_q(S))+\frac{1-q^{-\eta}}{1-q^{-1}}Y_q(S)\frac{1}{S}-C\frac{1-q^{-\nu(1-\eta)}}{1-q^{-1}}S^{\nu(1-\eta)-\eta-1}\right).
\end{aligned}$$

Using this in (56), we further get

$$-\frac{1}{q}\ \left(q^{-\eta}\ \Delta_q(Y_q(S))+\frac{1-q^{-\eta}}{1-q^{-1}}Y_q(S)\frac{1}{S}-C\frac{1-q^{-\nu(1-\eta)}}{1-q^{-1}}S^{\nu(1-\eta)-\eta-1}\right)$$
$$=\ \sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s}\frac{(q;q)_\infty\, q^{-(\alpha n+\beta)(\alpha n+\beta-1)/2}}{S^{\alpha n+\beta+\eta+1}}.$$

Here applying the inverse q-Laplace transforms, we find

$$\frac{-q^{-\eta}}{q}\mathcal{L}_q^{-1}\left(\Delta_q(Y_q(S))\right)+\frac{1-q^{-\eta}}{1-q^{-1}}\mathcal{L}_q^{-1}\left(Y_q(S)\frac{1}{S}\right)-C\frac{1-q^{-\nu(1-\eta)}}{1-q^{-1}}$$
$$\times\mathcal{L}_q^{-1}\left(\frac{1}{S^{\eta-\nu(1-\eta)+1}}\right)$$
$$=\ \sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ (q;q)_\infty\ \omega^n}{q^{(\alpha n+\beta)(\alpha n+\beta-1)/2}\ [(q^{\gamma+\delta n};q)_\infty]^s}\ \mathcal{L}_q^{-1}\left(\frac{1}{S^{\alpha n+\beta+\eta+1}}\right).$$

Once again using (23) and convolution theorem (46), it gives

$$\frac{-q^{-\eta}}{q}\left(-q\ x\ y(x)\right)+\frac{1-q^{-\eta}}{1-q^{-1}}\left(\int_0^x y(t)\ \mathrm{d_q}t\right)\ -C\frac{1-q^{-\nu(1-\eta)}}{1-q^{-1}}$$
$$\times\frac{x^{\eta-\nu(1-\eta)}\ (q^{\eta-\nu(1-\eta)+1};q)_\infty}{(q;q)_\infty\ q^{-(\eta-\nu(1-\eta)+1)(\eta-\nu(1-\eta))/2}}$$
$$=\ \sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s}\frac{(q;q)_\infty}{q^{(\alpha n+\beta)(\alpha n+\beta-1)/2}}$$
$$\times\frac{x^{\alpha n+\beta+\eta}\ (q^{\alpha n+\beta+\eta+1};q)_\infty}{q^{-(\alpha n+\beta+\eta+1)(\alpha n+\beta+\eta)/2}\ (q;q)_\infty}.$$

Finally, the q-integral formula (11):

$$\int_0^x f(t)\ \mathrm{d_q}t\ =\ x(1-q)\sum_{k=0}^{\infty}q^k\ f(xq^k),$$

simplifies this to

$$x\ y(x)+(q-q^{\eta+1})\ x\ \sum_{j=0}^{\infty}q^j\ y(xq^j)$$
$$-C\frac{q^\eta(1-q^{-\nu(1-\eta)})}{1-q^{-1}}\left(\frac{x^{\eta-\nu(1-\eta)}\ (q^{\eta-\nu(1-\eta)+1};q)_\infty}{(q;q)_\infty\ q^{-(\eta-\nu(1-\eta)+1)(\eta-\nu(1-\eta))/2}}\right)$$
$$=\ \sum_{n=0}^{\infty}\frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\lambda+\mu n};q)_\infty]^r\ x^{\alpha n+\beta+\eta}\ (q^{\alpha n+\beta+\eta+1};q)_\infty\ \omega^n}{[(q^{\gamma+\delta n};q)_\infty]^s\ q^{-\alpha n\eta+1}\ q^{-\eta(\eta-1)/2-\beta(\eta+1)}}$$
$$=\ q^{\eta(\eta-1)/2-\beta(\eta+1)}x^{\beta+\eta}E_{\alpha,\beta+\eta+1,\lambda,\mu}^{\gamma,\delta}\left(\omega\ (xq^{\eta+1})^\alpha;s,r|q\right).$$

This is (55). □

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References

[1] R. P. Agarwa, Certain fractional q-integrals and q-derivatives, Proc. Camb. Phil. Soc. **66** (1969), No. 3, 365-370.

[2] W. A. Al-Salam, Some fractional q-integrals and q-derivatives, Proc. Edinburgh Math. Soc. **15** (1966), 135-140.

[3] G. Bangerezako, An introduction to q-difference equations, University of Burundi, Faculty of Science, Department of Mathematis (2007) https://www.uclouvain.be/cps/ucl/doc/math/documents/RAPSEM354.pdf

[4] B. I. Dave, Extensions of certain inverse reseries relations and associated polynomials, Ph. D. Thesis, The M. S. University of Baroda, Vadodara (1994).

[5] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, (1990).

[6] W. Hahn, Über Orthogonal polynome, die q-Differenzengleichungen, Mat. Nachr. **2** (1949), 4-34.

[7] R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong (2000), 87-130.

[8] F. H. Jakson, Basic integration, Quart. J. Math. (Oxford) **2** (1951), 1-16.

[9] N. Kobachi, On q-Laplace transformation, Research Repotrs of Kumamoto-NCT **3** (2011), 69-76.

[10] M. Mansour, An asymptotic expansion of the q-Gamma function $\Gamma_q(x)$, Journal of nonlinear Mathematical Physics **13** (2006), No. 4, 479-483.

[11] A. M. Mathai, R. K. saxena, and H. J. haubold, The H-function: Theory and applications, Springer, New York (2010).

[12] G. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$, C. R. Acad. Sci. Paris **137** (1903), 554-558.

[13] G. Mittag-Leffler, Une generalisation de l'integrale de Laplace-Abel, Comptes Rendus de l'Academie des Sciences Serie **137** (1903), 537-539.

[14] B. V. Nathwani and B. I. Dave, Generalized Mittag-Leffler function and its properties, To appear in Math. Student.

[15] T. R. Prabhakar, A singular equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J. **19** (1971), No. 4, 7-15.

[16] J. C. Prajapati and B. V. Nathwani, Fractional calculus of a unified Mittag-Leffler function, Ukrainian Mathematical Journal **66** (2015), No. 8, 1267-1280.

[17] R. K. Saxena, S. Kalla and R. Saxena, Multivariate analogue of generalized Mittag-Leffler function, Integral Transform. spec. Funct.

[18] A. K. Shukla and J. C. Prajapati, On a generalization of Mittag-Leffler function and its properties, J. Math. Anal. Appl. **336** (2007), No. 2, 797-811.

[19] A. K. Shukla and J. C. Prajapati, some properties of a class of polynomials suggested by Mittal, Proyecciones J. Math. **26** (2007), No. 2, 145-156.

[20] A. Wiman, Über de fundamental satz in der theoric der funktionen $E_\alpha(x)$, Acta Math. **29** (1905), 191-201.

B. V. Nathwani

Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda, Vadodara-390 002, INDIA.

E-mail address: bharti.nathwani@yahoo.com

B. I. Dave

Department of Mathematics, Faculty of Science, The Maharaja Sayajirao University of Baroda, Vadodara-390 002, INDIA.

E-mail address: bidavemsu@yahoo.co.in