*Journal of Fractional Calculus and Applications*
Vol. 9(1) Jan. 2018, pp. 15-34.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

# STRUCTURE OF SOLUTION SETS FOR IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study the structure of solution sets for impulsive fractional differential equations, first we have proved that the solution sets is contractible to a point, using the Leray-Schauder alternative, we extend the classical Kneser?s theorem and Aronszajn type result for this class of equations by showing that the set of all solutions is compact and $R_\delta$-set.

## 1. Introduction

In this paper, we are concerned with the solution set for the initial value problems for impulsive differential equation with fractional order of the form

$$^{RL}D^{\alpha}y(t) = f(t, y(t)) \quad \text{a.e} \quad t \in J = (0, T], \; t \neq t_k, \tag{1}$$

$$\Delta^* y|_{t_k} = I_k(y(t_k^-)), \tag{2}$$

$$\lim_{t \longrightarrow 0^+} t^{1-\alpha} y(t) = c_0, \tag{3}$$

where $k = 1, \ldots, m$, $0 < \alpha \leq 1$, $^{RL}D^{\alpha}$ is the standard Riemman-Liouville fractional derivative, $f : J \times R \longrightarrow R$ is a given function , $c_0 \in R$, $I_k : R \to R$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$ and $\Delta^* y|_{t_k} = y^*(t_k^+) - y(t_k^-)$, where $y^*(t_k^+) = \lim_{t \longrightarrow t_k^+}(t - t_k)^{1-\alpha} y(t)$ and $y(t_k^-) = \lim_{t \longrightarrow t_k^-} y(t)$.
Impulsive functional differential equations arise in a variety of areas of biological, physical and engineering applications; see for example the books of Hale [20, 21]. The dynamics of many processes in physics, population dynamics, biology and medicine may be subject to abrupt changes such as shocks or perturbations ; see for instance [1, 2], and the references therein. These perturbations may be seen as model of impulsive differential equations. For instance, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment or a missing product. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. Their models may be described by impulsive differential equations. The mathematical study of initial and boundary value problems for differential equations with impulses was first considered in 1960

2010 *Mathematics Subject Classification.* 34A37, 26A33, 26A42, 34A60.
*Key words and phrases.* Differential equations, topological structure, solution set, fixed point theorem, impulsive, Riemman-Liouville fractional derivative, Aronszajn type theorem, contractible space, acyclic space, Homotopy, $R_\delta$-set.
Submitted March 30, 2017. Revised April 20, 2017.

by Milman and Myshkis [27], and then followed by a period of active research which culminated in 1968 with the monograph by Halanay and Wexler [22], for more details we also refer to the monographs of Bainov and Simeonov [5], Samoilenko and Perestyuk [28].
Fractional differential equations have been of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various scientific fields such as physics, mechanics, chemistry, engineering, etc. For details ; see [6, 7, 10, 24, 26], and the references [23, 24].
In 1923, Kneser was the first whom extend Peano result concerning the existence of solutions to study the topological properties, He prove that the solution sets is continuum. After, in 1942, Aronszajn extend Kneser?s theorem by showing that the set of all solutions is even an $R_\delta$-set, the monograph [13], Chapter 4 is an excellent reference to study Aronszajn-type results.
Topological structure of the solution set for ordinary differential equations and inclusions is developed recently; see for example [3, 4, 8, 9, 19, 17, 29], and the monograph [11, 12]. The aim is to investigate the topological structure of the solution set of problem $(1)-(3)$.
The paper is organized as follows, in Section 2 we give some general results and preliminaries and in Section 3 we have proved that the solution sets is contractible to a point, using the Leray-Schauder alternative, we extend the classical Kneser?s theorem and Aronszajn type result for this class of equations by showing that the set of all solutions is compact and an $R_\delta$-set and in the last section we give an example which illustrate our result.
To our best knowledge, there are very few results for solution set for fractional differential equations with Riemman-Liouville derivative. Also, as far as we know, no papers exist in the literature devoted to such problems with solution set for impulsive differential equations with Riemman Liouville fractional order.

## 2. Existence and uniqueness

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C([0,1],R)$ we denote the Banach space of all continuous functions from $[0,1]$ into $R$ with the norm

$$\|y\|_\infty = \sup_{t\in[0,1]} |y(t)|$$

To consider the impulsive condition 3, it is convenient to introduce some additional concepts and notations.
First, we recall some elementary notions and notations from geometric topology. For details, we recommend [3, 11, 25]. In what follows $(X,d)$ and $(Y,d^{'})$ stand for two metric spaces. Denote by $P(X)=\{Y\in P(E): Y\neq\emptyset.\}$. Let E be a Banach space $P_{cv,cl}(E)=\{Y\in P(E): Y convex, closed\}$.

**Definition 2.1.** *Let $A\in P(X)$. The set $A$ is called a contractible space provided there exists a continuous homotopy $H: A\times[0,1]\longrightarrow A$ and $x_0\in A$ such that*

(a) $H(x,0)=x,$ *for every* $x\in A$,
(b) $H(x,1)=x_0,$ *for every* $x\in A$,

*namely if the identity map is homotopic to a constant map, $A$ is homotopically equivalent to a point.*

Note that if $A \in P_{cv,cl}(X)$, then $A$ is contractible, but the class of contractible sets is much larger than the class of closed convex sets.

**Definition 2.2.** *$A \in P(X)$ is a retract of $X$ if there exists a continuous map $r : X \longrightarrow A$ such that $r(a) = a$ for every $a \in A$.*

**Definition 2.3.** *A compact nonempty space $X$ is called an $R_\delta -$set provided there exists a decreasing sequence of compact nonempty contractible spaces $\{X_n\}_{n\in N}$ such that $X = \bigcap_{n=1}^{\infty} X_n$.*

**Definition 2.4.** *A space $X$ is called an absolute retract (in short $X \in AR$) provided that for every space $Y$, every closed subset $B \subseteq Y$ and any continuous map $\varphi : B \longrightarrow X$, there exists a continuous extension $g : Y \longrightarrow X$ of $\varphi$ over $Y$ that is*

$$g(x) = \varphi(x) \text{ for every } x \in B.$$

*In other words, for every space $Y$ and for any embedding $\varphi : X \longrightarrow Y$ , the set $\varphi(X)$ is a retract of $Y$.*

Let us recall the well-known Lasota-Yorke approximation lemma, [12, 14].

**Lemma 2.1.** *Let $E$ be a normed space, $X$ a metric space and $F : X \longrightarrow E$ be a continuous map. Then for each $\varepsilon > 0$ there is a locally Lipschitz map $F_\varepsilon : X \longrightarrow E$ such that*

$$\|F(x) - F_\varepsilon(x)\| < \varepsilon, \text{ for every } x \in X.$$

Next, we present a result about the topological structure of the solution set of some nonlinear functional equations due to N. Aronszajn and developed by Browder and Gupta [3, 8].

**Theorem 2.1.** *Let $(X, d)$ be a metric space, $(E, \|.\|)$ a Banach space and $F : X \longrightarrow E$ a proper map, i.e., $F$ is continuous and for every compact $K \subset E$, the set $F^{-1}(K)$ is compact. Assume further that for each $\varepsilon > 0$, a proper map $F_\varepsilon : X \longrightarrow E$ is given, and the following two conditions are satisfied*

(a) $\|F_\varepsilon(x) - F(x)\| < \varepsilon$, *for every* $x \in X$,
(b) *for every $\varepsilon > 0$ and $u \in E$ in a neighborhood of the origin such that $\|u\| \leq \varepsilon$, the equation $F_\varepsilon(x) = u$ has exactly one solution $x_\varepsilon$,*

*then the set $S = F^{-1}(0)$ is an $R_\delta -$ set.*

**Lemma 2.2.** *Let $E$ be a Banach space, $C \subset E$ be a nonempty closed bounded subset of $E$ and $F : C \longrightarrow E$ is an completely continuous map, then $G = Id - F$ is a proper.*

**Lemma 2.3.** *Let $v : [0, b] \longrightarrow [0, +\infty)$ be a real function and $w(.)$ is a nonnegative, locally integrable function on $[0, b]$. Assume that there are constants $a > 0$ and $0 < \beta < 1$ such that*

$$v(t) \leq w(t) + a \int \frac{v(s)}{(t-s)^\beta} ds,$$

*then there exists a constant $K = K(\beta)$ such that*

$$v(t) \leq w(t) + Ka \int_0^t \frac{w(s)}{(t-s)^\beta} ds \text{ for every } t \in [0, b].$$

We begin with some Definitions from the theory of fractional calculus.

**Definition 2.5.** [15, 16]. *The fractional (arbitrary) order integral of the function $h \in L^1([a,b], R_+)$ of order $\alpha \in R_+$ is defined by*

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

*where $\Gamma$ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where $\delta$ is the delta function.*

**Definition 2.6.** [16]. *For a function $h$ given on the interval $[0, b]$, the Riemann-Liouville fractional derivative of $h$ of order $\alpha \in R_+$ is defined by*

$$D_{0^+}^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} (\int_0^t (t-s)^{n-\alpha-1} h(s)ds).$$

## 3. Main Results

Denote by $S(f, c_0)$ the set of all solutions of problem $(1) - (3)$.
In order to define a solution of problem (1)-(3) , we shall consider the space

$$\begin{aligned} PC_*([0,T], R) \quad &= \Big\{ y : [0,T] \to R : y_k \in C(t_k, t_{k+1}], k = 0, \dots, m \quad \text{and there exist} \\ &\quad y(t_k^-), y_*(t_k^+), k = 1, \dots, m \text{ with } y(t_k) = y(t_k^-) \Big\}, \end{aligned}$$

which is a Banach space with the norm

$$\|y\|_{PC_*} = \max_{k=1,\dots m.} \|y_k\|_*,$$

where $y_k$ is the restriction of $y$ to $J_k = (t_k, t_{k+1}],\ k = 0, \dots, m.$
$\|y_k\|_* = \sup_{t \in [t_k, t_{k+1}]} |(t-t_k)^{1-\alpha} y_k(t)|, \quad \text{for every } k = 1 \dots m.$
For $A$ a subset of the space $PC_*([0,T], R)$, define $\mathcal{A}_\alpha$ by

$$\mathcal{A}_\alpha = \{y_\alpha,\ y \in A\},$$

where

$$y_\alpha(t) = \begin{cases} t^{1-\alpha} y(t), & \text{if } t \in (t_k, t_k + 1] \\ \lim_{t \to t_k} t^{1-\alpha} y(t), & \text{if } t = t_k. \end{cases}$$

**Theorem 3.1.** *Let $A$ be a bounded set in $PC_*([0,T], R)$. Assume that $\mathcal{A}_\alpha$ is equicontinuous on $PC([0,T], R)$, then $A$ is relatively compact in $PC_*([0,T], R)$.*

Let $\{y_n\}_{n=1}^\infty \subset A$, then $\{(y_\alpha)_n\}_{n=1}^\infty \subset PC([0,T], R)$, from Arzela-Ascoli theorem, the set

$$K_0 = \{(y_\alpha)_n : n \in N^*\}$$

is relatively compact in $PC([0,T], R)$, thus there exists a subsequence of $(y_\alpha)_{n \in N}$, still denoted by $(y_\alpha)_{n \in N}$, which converges to $y \in (PC([0,T], R), \|.\|_{PC})$.
Hence

$$\|(y_\alpha)_n - y\|_* = \sup_{t \in [t_k, t_{k+1}]} (t-t_k)^{1-\alpha} |y_\alpha)_n(t) - y(t)| \to 0 \text{ as } n \to +\infty.$$

Therefore

$$\{y_n\}_{n=1}^\infty \longrightarrow y \text{ on } PC_*([0,T], R).$$

Let us define what we mean by a solution of problem (1)-(3).
Set

$$J' := J \backslash \{t_1, \dots, t_m\}.$$

**Definition 3.1.** *A function $y \in PC_*([0,T], R)$ is said to be a solution of problem $(1)-(3)$ if $y$ satisfies the equation $D^\alpha y(t) = f(t, y(t))$ on $J'$ and conditions (2)-(3) hold.*

For the existence of solutions for the problem (1)-(3), we need the following auxiliary lemmas.

**Lemma 3.1.** [30] *Let $\alpha > 0$, then the differential equation*

$$^{RL}D^\alpha_{a^+} h(t) = 0,$$

*has solutions $h(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \ldots + c_n(t-a)^{\alpha-n}$ for some $c_i \in R,\ i = 1 \ldots n,\ \ where\ \ n = [\alpha] + 1$.*

**Lemma 3.2.** [30] *Let $\alpha > 0$, then*

$$I^{\alpha RL} D^\alpha_{a^+} h(t) = h(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \ldots + c_n(t-a)^{\alpha-n}$$

*for some $c_i \in R,\ \ i = 0, \dots, n,\ \ where\ \ n = [\alpha] + 1$.*

**Lemma 3.3.** *Let $0 < \alpha < 1$ and let $h$ be a continuous function. A function $y$ is a solution of the fractional integral equation*

$$y(t) = \begin{cases} t^{\alpha-1} c_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds & \text{if } t \in [0, t_1], \\ (t-t_1)^{\alpha-1} t_1^{\alpha-1} c_0 + \frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} h(s) ds & \\ + \frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)} I_1(y(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} h(s) ds & \text{if } t \in (t_1, t_2], \\ (t-t_k)^{\alpha-1} \prod_{i=1}^{i=k} (t_i - t_{i-1})^{\alpha-1} c_0 & \\ + \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} \Big[ \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} h(s) ds & \\ + \sum_{i=1}^{k-1} \prod_{j=1}^{k-i} (t_{k-j+1} - t_{k-j})^{\alpha-1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s) ds \Big] & \\ + \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} \Big[ I_k(y(t_k^-)) & \\ + \sum_{i=1}^{k-1} \prod_{j=1}^{k-i} (t_{k-j+1} - t_{k-j})^{\alpha-1} I_i(y(t_i^-)) \Big] & \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^t (t-s)^{\alpha-1} h(s) ds, & \text{if } t \in (t_k, t_{k+1}] \\ & k = 2, \dots, m \end{cases} \tag{4}$$

*if and only if $y$ is a solution of the fractional initial-value problem*

$$D^\alpha y(t) = h(t), \quad t \in J' \tag{5}$$

$$\Delta^* y|_{t_k} = I_k(y(t_k^-)),\ k = 1 \ldots m, \tag{6}$$

$$\lim_{t \to 0} t^{1-\alpha} y(t) = c_0. \tag{7}$$

Assume $y$ satisfies (5)-(7). If $t \in [0, t_1]$ then ${}^{RL}D^{\alpha}y(t) = h(t)$. Lemma 3.3 implies

$$y(t) = t^{\alpha-1}c_1 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds.$$

Hence $c_1 = c_0$. Thus

$$y(t) = t^{\alpha-1}c_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds.$$

If $t \in (t_1, t_2]$, then Lemma 3.3 implies

$$\begin{aligned} y(t) &= (t-t_1)^{\alpha-1}y^*(t_1^+) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}h(s)ds \\ &= (t-t_1)^{\alpha-1}\left(I_1(y(t_1^-) + y(t_1^-)\right) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}h(s)ds \\ &= (t-t_1)^{\alpha-1}t_1^{\alpha-1}c_0 + \frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}h(s)ds + (t-t_1)^{\alpha-1}I_1(y(t_1^-)). \end{aligned}$$

If $t \in (t_2, t_3]$, then Lemma 3.3 implies

$$\begin{aligned} y(t) &= (t-t_2)^{\alpha-1}y^*(t_2^+) + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t (t-s)^{\alpha-1}h(s)ds \\ y(t) &= (t-t_2)^{\alpha-1}[y(t_2^-) + I_2(y(t_2^-))] + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t (t-s)^{\alpha-1}h(s)ds \\ &= (t-t_2)^{\alpha-1}(t_2-t_1)^{\alpha-1}t_1^{\alpha-1}c_0 + \frac{(t-t_2)^{\alpha-1}(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)ds \\ &+ \frac{(t-t_2)^{\alpha-1}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}h(t)ds + \frac{1}{\Gamma(\alpha)}\int_{t_2}^t (t-s)^{\alpha-1}h(s)ds \\ &+ (t-t_2)^{\alpha-1}\left[(t_2-t_1)^{\alpha-1}I_1(y(t_1^-)) + I_2(y_2(t_2^-))\right]. \end{aligned}$$

If $t \in (t_k, t_{k+1}]$, then again from Lemma 3.3 we get (4).
Conversely, assume that $y$ satisfies the impulsive fractional integral equation (4). If $t \in [0, t_1]$ then $\lim_{t\to 0} t^{1-\alpha}y(t) = c_0$ and using the fact that ${}^{RL}D^{\alpha}$ is the left inverse of $I^{\alpha}$ we get

$${}^{RL}D^{\alpha}y(t) = h(t), \quad \text{for each } t \in [0, t_1].$$

If $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$ and using the fact that ${}^{RL}D^{\alpha}C = 0$, where $C$ is a constant, we get

$${}^{RL}D^{\alpha}y(t) = h(t), \text{for each } t \in [t_k, t_{k+1}).$$

Also, we can easily show that

$$\Delta^* y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m.$$

Denote by $S(f, c_0)$ the set of all solutions of the problem $(1)-(3)$ .
We have to prove that solution sets $S(f, c_0)$ is contractible.
Set

$$d = \min_{i=1\dots m}(t_i - t_{i-1})$$

We give the proof in two Claims.

**Claim 3.1.** *We consider the following problem*

$$^{RL}D^{\alpha}x(t) = g(t, x(t)), \quad t \in J = [0, T],\, t \neq t_k, \tag{8}$$

$$\Delta^* x\big|_{t=t_k} = J_k(x(t_k^-)), \tag{9}$$

$$\lim_{t \longrightarrow 0^+} t^{1-\alpha}x(t) = c_0. \tag{10}$$

**Theorem 3.2.** *Assume that*

(H1) *There exists a constant $l > 0$ such that $|g(t, u) - g(t, \overline{u})| \leq l|u - \overline{u}|$, for each $t \in J$, and each $u, \overline{u} \in \mathbb{R}$.*

(H2) *There exists a constant $l^* > 0$ such that $|I_k(u) - I_k(\overline{u})| \leq l^*|u - \overline{u}|$, for each $u, \overline{u} \in \mathbb{R}$ and $k = 1, \ldots, m$.*

*If*

$$\frac{d^* B(\alpha, \alpha) + d^{**} + l^* \ d^{\alpha-1}}{\Gamma(\alpha)} < 1, \tag{11}$$

*then* (8)-(10) *has a unique solution $\overline{x}$ on $J$,*

*where* $d^* = (l\ m\ d^{m(\alpha-1)T^{2\alpha-1}} + l\ T^{\alpha} + lT^{2\alpha-1}),\ \ d^{**} = l^*\ m\ d^{(m+1)(\alpha-1)}$.
*We transform the problem* $(8) - (10)$ *into a fixed point problem. Consider the operator $F : PC_*([0, T], R) \longrightarrow PC_*([0, T], R)$ defined by*

$$\begin{aligned}
F(y)(t) &= (t - t_k)^{\alpha-1} \prod_{t_0 < t_i < t} (t_i - t_{i-1})^{\alpha-1} c_0 \\
&+ \frac{(t - t_k)^{\alpha-1}}{\Gamma(\alpha)} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s, y(s))ds \\
&+ \frac{(t - t_k)^{\alpha-1}}{\Gamma(\alpha)} \sum_{0 < t_i < t} ( \prod_{t_i < t_j < t} (t_j - t_{j-1})^{\alpha-1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} g(s, y(s))ds) \\
&+ \frac{(t - t_k)^{\alpha-1}}{\Gamma(\alpha)} \left( I_k(y(t_k^-)) + \sum_{0 < t_i < t} ( \prod_{t_i < t_j < t} (t_j - t_{j-1})^{\alpha-1} I_i(y(t_i^-))) \right) \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} g(s, y(s))ds.
\end{aligned}$$

*Clearly, the fixed points of the operator $F$ are solution of the problem* (8)-(10)*. We shall use the Banach contraction principle to prove that $F$ has a fixed point. We shall show that $F$* is a contraction. *Let $y_1, y_2 \in PC_*([0, T], R)$. Then, for each*

*$t \in J$ we have*

$$\begin{aligned}
&(t-t_k)^{1-\alpha}|F(y_2)(t)-F(y_1)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|g(s,y(s))-g(s,y(s))|ds \\
&+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|g(s,y_2(s))-g(s,y_1(s))|ds) \\
&+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\left(|I_k(y_2(t_k^-))-I_k(y_1(t_k^-))|+\sum_{0<t_i<t}I_i(y_2(t_i^-)-y_1(t_i^-))\right) \\
&+\frac{(t-t_k)^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|g(s,y(s))-g(s,y(s))|ds \\
&\leq \frac{l\|y_2-y_1\|_{PC_*}}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}(s-t_{k-1})^{\alpha-1}ds \\
&+\frac{l\ m\ d^{m(\alpha-1)}\|y_2-y_1\|}{\Gamma(\alpha)}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}(s-t_{i-1})^{\alpha-1}ds \\
&+\frac{1}{\Gamma(\alpha)}\left(l^*\ d^{\alpha-1}\|y_2-y_1\|_{PC_*}+l^*\ m\ d^{(m+1)(\alpha-1)}\|y_2-y_1\|_{PC_*}\right) \\
&+\frac{l\ T^{1-\alpha}\|y_2-y_1\|_{PC_*}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}(s-t_k)^{\alpha-1}ds
\end{aligned}$$

*Finally*

$$\|Fy_2-F(y_1\|_{PC_*}\leq\frac{d^*\ B(\alpha,\alpha)+d^{**}+l^*\ d^{\alpha-1}}{\Gamma(\alpha)}\|y_2-y_1\|_{PC_*}.$$

*Consequently by (11), $F$ is a contraction. As a consequence of Banach fixed point theorem, we deduce that $F$ has a fixed point which is a solution of the problem* $(8)-(10)$.

**Claim 3.2.** *Define the homotopy $H : S(f,c_0)\times[0,T]\longrightarrow S(f,c_0)$ by*

$$H(y,\lambda)(t)=\begin{cases} y(t), & 0<t\leq\lambda T \\ \overline{x}(t) & \lambda T<t\leq T. \end{cases}$$

*where $\overline{x}=S(g,c_0)$ is the unique solution of problem (8) − (10). In particular*

$$H(y,\lambda)=\begin{cases} y, & \textit{for } \lambda=1, \\ \overline{x}, & \textit{for } \lambda=0. \end{cases}$$

*We prove that $H$ is a continuous homotopy. Let $(y_n,\lambda_n)\in S(f,c_0)\times[0,T]$ be such that $(y_n,\lambda_n)\longrightarrow(y,\lambda)$, as $n\longrightarrow+\infty$.*
*We shall prove that $H(y_n,\lambda_n)\longrightarrow H(y,\lambda)$, we have*

$$H(y_n,\lambda_n)(t)=\begin{cases} y_n(t), & \textit{for } t\in(0,\lambda_nT], \\ \overline{x}(t), & \textit{for } t\in(\lambda_nT,t]. \end{cases}$$

*We consider several cases,*

**(a)** *if* $\lim_{n\longrightarrow+\infty}\lambda_n=0$,

$$\begin{aligned}
&|H(y_n,\lambda_n)(t)-H(y,\lambda)(t)| \le |H(y_n,\lambda_n)(t)-H(y,\lambda)(t)|_{[0,\lambda\ T]}\\
&+|H(y_n,\lambda_n)(t)-H(y,\lambda)(t)|_{[\lambda\ T,\lambda_n\ T]}+|H(y_n,\lambda_n)(t)-H(y,\lambda)(t)|_{[\lambda_n\ T,T]}\\
&\le |(y_n(t)-y(t)|_{[0,\lambda\ T]}+|(y_n(t)-\overline{x}(t)|_{[\lambda\ T,\lambda_n\ T]}+|\overline{x}(t)-\overline{x}(t)|_{[\lambda_n\ T,T]}\\
&\le |(y_n(t)-y(t)|_{[0,\lambda\ T]}+|(y_n(t)-\overline{x}(t)|_{[\lambda\ T,\lambda_n\ T]}\\
&\le d^{\alpha-1}\|(y_n-y\|_{PC_*}+t^{\alpha-1}|t^{1-\alpha}(y_n(t)-t^{1-\alpha}\overline{x}(t)|_{[\lambda\ T,\lambda_n\ T]},
\end{aligned}$$

*which tends to 0 as* $n\longrightarrow+\infty$.

**(b)** *If* $\lim_{n\longrightarrow\infty}\lambda_n=1$,

*it's treated similarly.*
*If* $\lambda_n\neq 0$ *and* $0<\lim_{n\longrightarrow\infty}\lambda_n<1$,
*two cases must be treated,*

- $t\in(0,\lambda_n]$,

*then* $H(y_n,\lambda_n)(t)=H(y,\lambda)(t)=y_n(t)-y(t)$,

$$\begin{aligned}
y_n(t) &= (t-t_k)^{\alpha-1}\prod_{t_0<t_i<t}(t_i-t_{i-1})^{\alpha-1}c_0\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}g(s,y(s))ds\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\sum_{0<t_i<t}\left(\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}g(s,y_n(s))ds\right)\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left(I_k(y_n(t_k^-))+\sum_{0<t_i<t}\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}I_i(y_n(t_i^-))\right)\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}g(s,y_n(s)),n\ge 1,
\end{aligned}$$

*since* $f$ *is continuous function and continuity of* $I_k$ *one has for* $t\in(0,\lambda]$,

$$\begin{aligned}
y(t) &= (t-t_k)^{\alpha-1}\prod_{t_0<t_i<t}(t_i-t_{i-1})^{\alpha-1}c_0\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}[\sum_{0<t_i<t}\left(\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}f(s,y(s))ds\right)\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left(I_k(y(t_k^-))+\sum_{0<t_i<t}\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}I_i(y(t_i^-))\right)\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,y(s))ds,
\end{aligned}$$

$$\|H(y_n,\lambda_n)-H(y,\lambda)\|_{PC_*}=\|y_n-y\|_{PC_*}\longrightarrow 0\ \text{as}\ n\longrightarrow+\infty,$$

- $t\in(\lambda_n,1]$,

*then* $H(y_n,\lambda_n)(t)=H(y,\lambda)(t)=\overline{x}(t),$ *thus*

$$\|H(y_n,\lambda_n)-H(y,\lambda)\|_{PC_*}\longrightarrow 0 \ as\ n\longrightarrow+\infty.$$

Therefore $H$ is a continuous function, proving that $S(f,c_0)$ is contractible to the point $\overline{x}$ .
Our result is based on the nonlinear alternative of Leray-Schauder type. We assume the following hypotheses :

(H3) $F : J\times R\to\mathbb{R}$ is a continuous function.
(H4) There exist $p\in C(J,R_+)$ and $q\in C(J,\mathbb{R}^+)$ continuous such that

$$|f(t,u)|\leq p(t)|u|+q(t)$$

for $t\in J$ and $u\in\mathbb{R}$.

(H5) There exist constants $a_k,b_k\in R^+$ such that

$$|I_k(u)|\leq a_k|u|+b_k \quad \text{for } u\in\mathbb{R}.$$

**Theorem 3.3.** *Under assumptions* $(H3)-(H5)$*, the solution set* $S(f,c_0)$ *is an* $R_\delta$*- set .*

Transform the problem (1)-(3) into a fixed point problem. Consider the operator $N : PC_*([0,T],R)\longrightarrow PC_*([0,T],R)$ defined by

$$\begin{aligned}
N(y)(t) &= (t-t_k)^{\alpha-1}\prod_{t_0<t_i<t}(t_i-t_{i-1})^{\alpha-1}c_0\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,y(s))ds\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\sum_{0<t_i<t}(\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}f(s,y(s))ds)\\
&+\frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)}\left(I_k(y(t_k^-))+\sum_{0<t_i<t}(\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}I_i(y(t_i^-)))\right)\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}f(s,y(s))ds.
\end{aligned}$$

Clearly, from Lemma 2.3, fixed points of $N$ are solutions to (1)-(3). We shall show that $N$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type [18]. The proof will be given in several steps.
**Step 1:** $N$ is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in $PC_*([0,T],R)$, then

$$
\begin{aligned}
&|(t-t_k)^{1-\alpha}N(y_n)(t)-(t-t_k)^{1-\alpha}N(y)(t)| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|f(s,y_n(s))-f(s,y(s))|ds \\
&+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\Big(\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|f(s,y_n(s))-f(s,y(s))|ds \\
&+\frac{1}{\Gamma(\alpha)}|I_i(y_n(t_i^-))-I_i(y(t_i^-))| \\
&+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\Big(\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}|I_i(y_n(t_i^-))-I_i(y(t_i^-))| \\
&+\frac{(t-t_k)^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|f(s,y_n(s))-f(s,y(s))|ds, \\
&\leq \frac{1}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|f(s,y_n(s))-f(s,y(s))|ds \\
&+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|f(s,y_n(s))-f(s,y(s))|ds \\
&+\frac{1}{\Gamma(\alpha)}|I_i(y_n(t_i^-))-I_i(y(t_i^-))|+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\sum_{0<t_i<t}|I_i(y_n(t_i^-))-I_i(y(t_i^-))| \\
&+\frac{T^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|f(s,y_n(s))-f(s,y(s))|ds, \\
&\leq \frac{1}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}(s-t_k)^{\alpha-1}\|f(.,y_n(.))-f(.,y(.))\|_{PC_*}ds \\
&+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}(s-t_i)^{\alpha-1}\|f(.,y_n(.))-f(.,y(.))\|_{PC_*}ds \\
&+\frac{d^{\alpha-1}}{\Gamma(\alpha)}\|I_i(y_n(.))-I_i(y(.))\|_{PC_*}+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\sum_{0<t_i<t}(s-t_{i-1})^{\alpha-1}\|I_i(y_n(.))-I_i(y(.))\|_{PC_*} \\
&+\frac{T^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}(s-t_k)^{\alpha-1}\|f(.,y_n(.))-f(.,y(.))\|_{PC_*}ds,
\end{aligned}
$$

from $(H_3)$ and the continuity of $I_k$ we have

$$
\begin{aligned}
&\|N(y_n)(.)-N(y)(.)\|_{PC_*} \\
&\leq \frac{md^{m(\alpha-1)T^{2\alpha-1}}+T^{\alpha}+T^{2\alpha-1}}{\Gamma(\alpha)}B(\alpha,\alpha)\|f(.,y_n(.))-f(.,y(.))\|_{PC_*} \\
&+\frac{md^{(m+1)(\alpha-1)}+d^{\alpha-1}}{\Gamma(\alpha)}\|I_i(y_n(.))-I_i(y(.))\|_{PC_*},
\end{aligned}
$$

hence

$$\|N(y_n)(.)-N(y)(.)\|_{PC_*}\longrightarrow 0 \text{ as } n\longrightarrow +\infty.$$

**Step 2:** $N$ maps bounded sets into bounded sets in $PC_*([0,T],R)$.
Indeed, it is enough to show that there exists a positive constant $l$ such that for each $y \in B_\eta = \{y \in PC_*([0,T],R) : \|y\|_{PC_*} \leq \eta\}$ one has $\|N(y)\|_{PC_*} \leq l$.
Let $y \in B_\eta$ then for each $t \in J$, then from $(H_4)$ and $(H_5)$, one has

$$
\begin{aligned}
&|(t-t_k)^{1-\alpha}N(y)(t)| \leq \prod_{t_0<t_i<t}(t_i-t_{i-1})^{\alpha-1}c_0\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}f(s,y(s))ds\\
&+\frac{1}{\Gamma(\alpha)}\sum_{0<t_i<t}\Big(\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|f(s,y(s))|ds\\
&+\frac{1}{\Gamma(\alpha)}\left(|I_k(y(t_k^-)|+\sum_{0<t_i<t}\Big(\prod_{t_i<t_j<t}(t_j-t_{j-1})^{\alpha-1}|I_i(y(t_i^-))|\Big)\right)\\
&+\frac{(t-t_k)^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|f(s,y(s))|ds\\
&\leq d^{m(\alpha-1)}c_0+\frac{1}{\Gamma(\alpha)}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|f(s,y(s))|ds\\
&+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|f(s,y(s))|ds\\
&+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\sum_{0<t_i<t}|I_i(y(t_i^-))|+\frac{1}{\Gamma(\alpha)}|I_k(y(t_k^-)|+\frac{T^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|f(s,y(s))|ds\\
&\leq d^{m(\alpha-1)}c_0+\frac{\|q\|_\infty T^\alpha}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)}\left(\|p\|_\infty\|y(.)\|_{PC_*}\right)\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}(s-t_{k-1})^{\alpha-1}ds\\
&+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\left[\frac{mT^\alpha\|q\|_\infty}{\alpha}+\left(\|p\|_\infty\|y(.)\|_{PC_*}\right)\sum_{0<t_i<t}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}(s-t_{i-1})^{\alpha-1}ds\right]\\
&+\frac{d^{m(\alpha-1)}}{\Gamma(\alpha)}\sum_{0<t_i<t}\left((s-t_i)^{\alpha-1}a_i\|y(.)\|_{PC_*}+b_i\right)+\frac{1}{\Gamma(\alpha)}\left(a_i(s-t_i)^{\alpha-1}\|y(.)\|_{PC_*}+b_i\right)\\
&+\frac{T\|q\|_\infty}{\Gamma(\alpha+1)}+\frac{T^{1-\alpha}}{\Gamma(\alpha)}\left(\|p\|_\infty\|y(.)\|_{PC_*}\right)\int_{t_k}^{t}(t-s)^{\alpha-1}(s-t_k)^{\alpha-1}ds,
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|N(y)\|_{PC_*} \leq T^{m(\alpha-1)}c_0 &+ \frac{md^{m(\alpha-1)T^{2\alpha-1}}+T^\alpha+T^{2\alpha-1}}{\Gamma(\alpha)}B(\alpha,\alpha)\\
&+\frac{T^\alpha(md^{m(\alpha-1)}+1)+T}{\Gamma(\alpha+1)}\|q\|_\infty\\
&+\frac{(md^{m(\alpha-1)}+1)(d^{\alpha-1}a^*\eta+b^*)}{\Gamma(\alpha)} := l
\end{aligned}
$$

where $a^* = \max_{i=\overline{1,m}} a_i$ and $b^* = \max_{i=\overline{1,m}} b_i$.
**Step 3:**

$N$ maps bounded set into equicontinuous sets of $PC_*([0,T],R)$.
Let $\tau_1,\tau_2 \in (0,1],\ \tau_1<\tau_2$ and $B_\eta$ be a bounded set of $PC_*([0,1],R)$ as in step 2. Let $y\in B_\eta$ so one has

$$
\begin{aligned}
& |(\tau_2-t_k)^{1-\alpha}N(y)(\tau_2)-(\tau_1-t_k)^{1-\alpha}N(y)(\tau_1)| \\
\leq\ & \prod_{t_0<t_i<\tau_2-\tau_1}(t_i-t_{i-1})^{\alpha-1}c \\
+\ & \frac{1}{\Gamma(\alpha)}\sum_{0<t_i<\tau_2-\tau_1}\Big(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j-t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}|f(s,y(s))|ds\Big) \\
+\ & \frac{|(\tau_2-t_k)^{1-\alpha}-(\tau_1-t_k)^{1-\alpha}|}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}(\tau_2-s)^{\alpha-1}|f(s,y(s))|ds \\
+\ & \frac{(\tau_1-t_k)^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}]|f(s,y(s))|ds \\
+\ & \frac{(\tau_2-t_k)^{1-\alpha}}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}|f(s,y(s))|ds \\
+\ & \frac{1}{\Gamma(\alpha)}\sum_{0<t_i<\tau_2-\tau_1}\Big(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j-t_{j-1})^{\alpha-1}|I_i(y(t_i^-))|\Big) \\
\leq\ & \prod_{t_0<t_i<\tau_2-\tau_1}(t_i-t_{i-1})^{\alpha-1}c_0 \\
+\ & \frac{\|q\|_\infty}{\Gamma(\alpha)}\sum_{0<t_i<\tau_2-\tau_1}\Big(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j-t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}ds\Big) \\
+\ & \frac{\|p\|_\infty\|y\|_{PC_*}}{\Gamma(\alpha)}\sum_{0<t_i<\tau_2-\tau_1}\Big(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j-t_{j-1})^{\alpha-1}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}(s-t_i)^{\alpha-1}ds\Big) \\
+\ & \frac{[(\tau_2-t_k)^{1-\alpha}-(\tau_1-t_k)^{1-\alpha}]\|q\|_\infty}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}(\tau_2-s)^{\alpha-1}ds \\
+\ & \frac{[(\tau_2-t_k)^{1-\alpha}-(\tau_1-t_k)^{1-\alpha}]\|p\|_\infty\|y\|_{PC_*}}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}(\tau_2-s)^{\alpha-1}(s-t_k)^{\alpha-1}ds \\
+\ & \frac{(\tau_1-t_k)^{1-\alpha}\|q\|_\infty}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}]ds \\
+\ & \frac{(\tau_1-t_k)^{1-\alpha}\|p\|_\infty\|y\|_{PC_*}}{\Gamma(\alpha)}\int_{t_k}^{\tau_1}[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}](s-t_k)^{\alpha-1}ds \\
+\ & \frac{(\tau_2-t_k)^{1-\alpha}\|q\|_\infty}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds \\
+\ & \frac{(\tau_2-t_k)^{1-\alpha}\|p\|_\infty\|y\|_{PC_*}}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}(s-t_k)^{\alpha-1}ds \\
+\ & \frac{d^{\alpha-1}}{\Gamma(\alpha)}(a^*\eta+b^*)\sum_{0<t_i<\tau_2-\tau_1}\Big(\prod_{t_i<t_j<\tau_2-\tau_1}(t_j-t_{j-1})^{\alpha-1}\Big) \\
\leq\ & \prod_{t_0<t_i<\tau_2-\tau_1}(t_i-t_{i-1})^{\alpha-1}c_0
\end{aligned}
$$

$$+ \quad \frac{(t_i - t_{i-1})^{2\alpha-1} B(\alpha,\alpha)}{\Gamma(\alpha)} (\|p\|_\infty \eta + T^{1-\alpha}\|q\|_\infty) \sum_{0<t_i<\tau_2-\tau_1} ( \prod_{t_i<t_j<\tau_2-\tau_1} (t_j - t_{j-1})^{\alpha-1})$$

$$+ \quad \frac{|(\tau_2 - t_k)^{1-\alpha} - (\tau_1 - t_k)^{1-\alpha}|(\tau_1 - t_k)^{2\alpha-1} B(\alpha,\alpha)}{\Gamma(\alpha)} (\|p\|_\infty \eta + T^{1-\alpha}\|q\|_\infty)$$

$$+ \quad \frac{|(\tau_2 - t_k)^{1-\alpha} - (\tau_1 - t_k)^{1-\alpha}|(\tau_2 - t_k)^{2\alpha-1} B(\alpha,\alpha)}{\Gamma(\alpha)} (\|p\|_\infty \eta + T^{1-\alpha}\|q\|_\infty)$$

$$+ \quad \frac{(\tau_2 - \tau_1)^\alpha}{\Gamma(\alpha+1)} (\|p\|_\infty \eta + T^{1-\alpha}\|q\|_\infty)$$

$$+ \quad \frac{d^{\alpha-1}}{\Gamma(\alpha)} (a^* \eta + b^*) \sum_{0<t_i<\tau_2-\tau_1} ( \prod_{t_i<t_j<\tau_2-\tau_1} (t_j - t_{j-1})^{\alpha-1})$$

As $\tau_2 \longrightarrow \tau_1$ the right-hand side of the above inequality tends to zero, then $N(B_\eta)$ is equicontinuous. As a consequence of steps 1 to 3 together with the Ascoli-Arzela theorem (3.1), we can conclude that $N : PC_*([0,T],R) \longrightarrow PC_*([0,T],R)$ is completely continuous.

**Step 4:** A priori bounds on solutions.
Let $y = \lambda N(y)$ for some $0 < \lambda < 1$. This implies by $(H_4)$ and $(H_5)$, for $t \in [0,t_1]$ one has

$$t^{1-\alpha}|y(t)| \leq |c_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} \left(s^{1-\alpha}\|p\||y(s)| + T^{1-\alpha}\|q\|\right) ds,$$

and then

$$\begin{aligned} t^{1-\alpha}|y(t)| &\leq |c_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} \left(s^{1-\alpha}\|p\||y(s)| + T^{1-\alpha}\|q\|\right) ds \\ &\leq |c_0| + \frac{T^{1-\alpha}\|q\|\Gamma(\alpha)}{\Gamma(2\alpha)} + \frac{\|p\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} \left(s^{1-\alpha}|y(s)|\right) ds. \end{aligned}$$

From lemma 2.3 there exists $K_0(\alpha)$ such that

$$t^{1-\alpha}|y(t)| \leq L_0 + \frac{\|p\|K_0(\alpha)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} L ds,$$

where

$$L_0 = |c_0| + \frac{T^{1-\alpha}\|q\|\Gamma(\alpha)}{\Gamma(2\alpha)},$$

then

$$\sup_{t\in[0,t_1]} t^{1-\alpha}|y(t)| \leq L_0 + \frac{L\|p\|K(\alpha)\Gamma(\alpha)}{\Gamma(2\alpha)} =: M_0.$$

We continue this process taking into account that $t \in (t_m, T]$, then

$$\begin{aligned}
&(t-t_m)^{1-\alpha}|y(t)| \leq \\
&|y^*(t_m^+)| + \frac{(t-t_m)^{1-\alpha}}{\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}|f(s,y(s))|ds \\
&\leq |y(t_m^-)| + |I_m(y(t_m^-))| + \frac{(t-t_m)^{1-\alpha}}{\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}|f(s,y(s))|ds \\
&\leq T^{\alpha-1}(a_m+1)M_0 + b_m + \frac{T\|q\|\Gamma(\alpha)}{\Gamma(\alpha+1)} \\
&+ \frac{T^{1-\alpha}\|p\|_\infty}{\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}(s-t_m)^{\alpha-1}(s-t_m)^{1-\alpha}|y(s)|)ds.
\end{aligned}$$

From lemma 2.3 there exists $K_m(\alpha)$ such that

$$t^{1-\alpha}|y(t)| \leq L_m + \frac{\|p\|K_m(\alpha)}{\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}(s-t_m)^{\alpha-1}L_m ds,$$

where

$$L_m = T^{\alpha-1}(a_m+1)M_0 + b_m + \frac{T\|q\|}{\Gamma(\alpha+1)},$$

then

$$\sup_{t\in(t_m,T]} t^{1-\alpha}|y(t)| \leq L_m + \frac{L_m\|p\|K_m(\alpha)\Gamma(\alpha)}{\Gamma(2\alpha)} =: M_m.$$

Define $M$ by

$$M = \max_{k=1,\dots,m.} M_k,$$

let

$$U = \{y \in PC_*([0,1],R) : \|y\|_{PC_*} < M+1\},$$

and consider the operator $N:\ \overline{U} \longrightarrow PC_*([0,T],R)$. From the choice of $U$, there is no $y \in \partial U$ such that $y = \lambda N(y)$ for some $\lambda \in (0,1)$.
As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that $N$ has a fixed point $y$ in $U$ which is a solution of the problem $(1)-(3)$.

3.1. **Compactness of Solution Set.** Now we show that the set

$$S = \{y \in PC_*([0,T],R) : y \text{ is a solution of } (1)-(3)\} \text{ is compact.}$$

Let $(y_n)_{n\in N}$ be a sequence in $S$. We put $B = \{y_n : n \in N\} \subseteq PC_*([0,T],R)$. Then from earlier parts of the proof of this theorem, we conclude that $B$ is bounded and equicontinuous. Then from the Ascoli-Arzela theorem, we can conclude that $B$ is compact.
Consider the equation

$$D^\alpha y(t) = f(t,y(t)), \quad \text{a.e} \ \ t \in J = (0,t_1], \tag{12}$$

$$\lim_{t\longrightarrow 0^+} t^{1-\alpha}y(t) = c_0. \tag{13}$$

Recall that $J_0 = [0,t_1]$ and $J_k = (t_k,t_{k+1}]$, $k = 1,\dots,m$,

$$C_{k,*}([t_k,t_{k+1}],R) = \{y \in C(J_k,R) \ \text{ with } y^*(t_k^+) \text{ exists}\}.$$

Hence:
 $y_n|_{J_0}$ has a subsequence $(y_{n_m})_{n_m\in N}$ converges to $y$ with

$(y_{n_m})_{n_m\in N} \subset S_1 = \{y \in C([0,t_1],R) : y \text{ is a solution of } (12)-(13)\}$.
Let

$$z_0(t) = t^{\alpha-1}c_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds,$$

and

$$|y_{n_m}(t) - z_0(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,y_{n_m}(s)) - f(s,y(s))|ds.$$

As $n_m \to +\infty$, $y_{n_m}(t) \to z_0(t)$, and then

$$y(t) = t^{\alpha-1}c_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds.$$

Consider now

$$D^\alpha y(t) = f(t,y(t)), \quad \text{a.e } \ t \in J_1 = (t_1,t_2], \tag{14}$$

$$y^*(t_k^+) = y(t_k^-) + I_k(t_k^-), \tag{15}$$

$y_n|_{J_1}$ has a subsequence relabeled as $(y_{n_m}) \subset S_2$ converging to $y$ in $C_{1,*}([t_1,t_2],R)$ where

$$S_2 = \{y \in C_{1,*}([t_1,t_2],R) : y \text{ is a solution of } (12)-(13)\}.$$

Let

$$z_1(t) = (t-t_1)^{\alpha-1}t_1^{\alpha-1}c_0 + \frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)ds$$
$$+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}f(s,y(s))ds + (t-t_1)^{\alpha-1}I_1(y(t_1^-)),$$

, and

$$|y_{n_m}(t) - z_1(t)| \leq \frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}|f(s,y_{n_m}(s)) - f(s,y(s))|ds$$
$$+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}|f(s,y_{n_m}(s)) - f(s,y(s))|ds$$
$$+ (t-t_1)^{\alpha-1}|I_1(y_{n_m}(t_1^-) - y(t_1^-))|.$$

As $n_m \to +\infty$, $y_{n_m}(t) \to z_1(t)$, and then

$$y(t) = (t-t_1)^{\alpha-1}t_1^{\alpha-1}c_0 + \frac{(t-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}h(s)ds$$
$$+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}h(s)ds + (t-t_1)^{\alpha-1}I_1(y(t_1^-)).$$

We continue this process, and we conclude that $\{y_n \mid n \in N\}$ has subsequence converging to

$$
\begin{aligned}
&y(t) = (t-t_m)^{\alpha-1} \prod_{t_0<t_i<t} (t_i - t_{i-1})^{\alpha-1} c_0 \\
&+ \frac{(t-t_m)^{\alpha-1}}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1} f(s, y(s)) ds \\
&+ \frac{(t-t_m)^{\alpha-1}}{\Gamma(\alpha)} \sum_{0<t_i<t} \Big( \prod_{t_i<t_j<t} (t_j - t_{j-1})^{\alpha-1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} f(s, y(s)) ds \Big) \\
&+ \frac{(t-t_m)^{\alpha-1}}{\Gamma(\alpha)} \left( I_m(y(t_m^-)) + \sum_{0<t_i<t} \Big( \prod_{t_i<t_j<t} (t_j - t_{j-1})^{\alpha-1} I_i(y(t_i^-)) \Big) \right) \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1} f(s, y(s)) ds.
\end{aligned}
$$

Hence $S(f, c_0)$ is compact.
Define

$$
\tilde{f}(t, y(t)) = \begin{cases} f(t, y(t)), & |y(t)| \leq M, \\ f(t, y(\frac{My(t)}{|y(t)|})), & |y(t)| \geq M, \end{cases}
$$

Since $f$ is continuous, the function $\widetilde{f}$ is continuous and it is bounded by $(H_4)$ so there exists $M_* > 0$ such that

$$
|\widetilde{f}(t, y)| \leq M_*, \; for \; a.e. \; t \text{ and all } \; y \in R. \tag{16}
$$

We consider the following modified problem,

$$
\begin{cases} D^\alpha y(t) = \tilde{f}(t, y(t)), & t \in J = (0, 1], \; t \neq t_k, \quad 0 < \alpha \leq 1, \\ \Delta^* y|_{t_k} = \tilde{I}_k(y(t_k^-)), & k = 1, \dots, m, \\ \lim\limits_{t \longrightarrow 0^+} t^{1-\alpha} y(t) = c_0. \end{cases}
$$

We can easily prove that $S(f, c_0) = S(\tilde{f}, c_0) = Fix\tilde{N}$, where

$$
\tilde{N} : PC_*([0, T], R) \longrightarrow PC_*([0, T], R)
$$

is defined by

$$
\begin{aligned}
&\tilde{N}(y)(t) = (t-t_k)^{\alpha-1} \prod_{t_0<t_i<t} (t_i - t_{i-1})^{\alpha-1} c_0 \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} \sum_{0<t_i<t} \left( \prod_{t_i<t_j<t} (t_j - t_{j-1})^{\alpha-1} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \tilde{f}(s, y(s)) ds \right) \\
&+ \frac{(t-t_k)^{\alpha-1}}{\Gamma(\alpha)} \left( I_k(y(t_k^-) + \sum_{0<t_i<t} \prod_{t_i<t_j<t} (t_j - t_{j-1})^{\alpha-1} \tilde{I}_i(y(t_i^-)) \right) \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \tilde{f}(s, y(s)) ds,
\end{aligned}
$$

$$(t-t_k)^{1-\alpha}|\tilde{N}(y)(t)| \leq d^{m(\alpha-1)}|c_0| + \frac{M_* d^{m(\alpha-1)}}{\Gamma(\alpha)} + \int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}ds + \frac{M_*(d^{m(\alpha-1)}+1)}{\Gamma(\alpha)} + \frac{M_* T^\alpha}{\Gamma(\alpha+1)},$$

$$\|\tilde{N}y\| \leq d^{m(\alpha-1)}|c_0| + \frac{M_*\ T^\alpha\ (d^{m(\alpha-1)}+1)}{\Gamma(\alpha+1)} + \frac{M_*(d^{m(\alpha-1)}+1)}{\Gamma(\alpha)} =: M_{**}.$$

Finally we have

$$\|\tilde{N}(y)\|_{PC_*} \leq M_{**},$$

then $\tilde{N}$ is uniformly bounded, as in steps 1 to 2 we can prove that

$$\tilde{N} : PC_*([0,T],R) \longrightarrow PC_*([0,T],R),$$

is compact which allows us to define the compact perturbation of the identity $\widetilde{G}(y) = y - \tilde{N}(y)$ which is a proper map. From the compactness of $\widetilde{N}$, we can easily prove that all conditions of Theorem (2.1) are satisfied. Therefore the solution set $S(\tilde{f}, c_0) = \tilde{G}^{-1}(0)$ is an $R_\delta$-set so $S(f, c_0)$ is an $R_\delta$-set.

## 4. Conclusion

In the first leg of the paper we have considered our problem with some regulates i.e the non linearity and $I_k$ are lipschitz function so the problem has a unique solution, and in the first leg using the construction and compactness of the operator $\widetilde{F}$ and Lasota-Yorke approximation lemma 2.1 we can approximate the problem by this regular differential equation and the conditions of Theorem 2.1 are satisfied so $F^{-1}(0)$ is an $R_\delta$-set.

## 5. Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the fractional impulsive differential equation,

$$D^\alpha y(t) = \frac{\Gamma(\alpha)}{\Gamma(\beta-\alpha)(t+a)^\alpha}y(t) + \frac{k_{\alpha,\beta}}{(t+a)^\alpha},\ t \in J = (0,1],\ t \neq \frac{1}{2},\ 0 < \alpha, \beta \leq 1, \tag{17}$$

$$\lim_{t \longrightarrow 0} t^{1-\alpha} y(t) = 0, \tag{18}$$

$$\Delta^* y|_{t=\frac{1}{2}} = \frac{1}{3}|y(\frac{1}{2}^-)| + 1, \tag{19}$$

where

$$k_{\alpha,\beta} = \frac{1}{\Gamma(1-\alpha)} - \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)},\ a > 0.$$

set

$$f(t,u) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)(t+a)^\alpha}u + \frac{k_{\alpha,\beta}}{(t+a)^\alpha},\ \text{for } (t,u) \in J \times R$$

$$I_1(u) = \frac{1}{3}|u| + 1 \text{ for all } u \in \mathbb{R}.$$

We have $t_1 = \frac{1}{2}$ and $c_0 = 0$, it clear that $f$ is a continuous function and so condition $(H_3)$ is satisfied.
Set

$$P(t) = \frac{\Gamma(\alpha)}{\Gamma(\beta-\alpha)(t+a)^{\alpha}}, \quad \|P\|_{\infty} = \frac{\Gamma(\alpha)}{\Gamma(\beta-\alpha)a^{\alpha}},$$

$$q(t) = k_{\alpha,\beta}, \quad \|q\|_{\infty} = \frac{k_{\alpha,\beta}}{a^{\alpha}},$$

we can easily prove that condition $(H_4)$ yields.
Set also $a_1 = \frac{1}{3}$, $b_1 = 1$ and so condition $(H_5)$ is satisfied.
Note that

$$y(t) = k(t+a)^{\beta-1} + 1,$$

where $k$ is real constant are solutions of the problem $(17)-(19)$.
Therefore, the solution set of the problem $(17)-(19)$ is not empty and it is interesting to study the topological properties of the solution sets in this case.
From Theorem (3.2) the solution set of $(17)-(19)$ is an $R_\delta$-set.

**Acknowledgements** The authors are grateful to the referees for the helpful remarks.

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