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DYNAMICAL ANALYSIS OF A PREY-PREDATOR FRACTIONAL ORDER MODEL

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ABSTRACT. In this paper, we are concerned with the fractional order of a two-prey one-predator model without help between each team of preys against the predator. Existence and uniqueness of the solutions of the model are proved. Also, the stability of the equilibrium points are studied. Some numerical simulations are given. Finally, we gave an example of the equilibrium point which is a centre for the integer order system but locally asymptotically stable for its fractional order counterpart.

1. INTRODUCTION

Fractional calculus is the field of mathematics which is a generalization of ordinary differentiation and integration of arbitrary order. In recent years, fractional differential equations are increasingly used to model many real phenomena. It is well recognized that the fractional differential equations arise in a lot of systems in science and engineering such as fluid mechanics [14, 26, 27], elasticity theory [3, 23, 24] , signal and image processing [17, 25] and is also widely used in modeling biological problems [5, 15, 20].

The importance of using the fractional differential equations due to the advantages of using fractional derivative versus the integer derivative such as the fractional derivative is non local. The fractional differential equations is very useful in modeling systems with memory and also equations which involve delay. The fractional differential equations are closely related to fractals which are numerous in biological systems. The analysis of solutions obtained using the fractional differential equations are as stable as their integer order counter part.

Several definitions of fractional differential operator have been proposed such as Grunwald-Letnikov, Riemann-Liouville, Caputo and Hadamard. The Riemann-Liouville is widely used in mathematical analysis. The fractional derivative is defined via a fractional integral operator. First the integral operator of order $n \in N$ of a function $f : R^+ \to R$ is defined according to the Cauchy formula as:

$$J^n f(t) = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s)\, ds.$$

The fractional integral of order $\alpha > 0$, is defined by using Gamma function as:

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$$J^{\alpha} f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds.$$

The Riemann-Liouville derivative of order $\alpha(> 0)$ is given by:

$$D^{\alpha} f(t) = D^m \left(J^{m-\alpha} f(t)\right),$$

where $m = \lceil \alpha \rceil$ and $D = \frac{d}{dt}$.

In this paper, we used Caputo approach to define the fractional derivative. It is a modification to the Riemann-Liouville definition and has the advantage that it is not necessary to define the fractional order initial conditions. The Caputo fractional derivative of order $\alpha(> 0)$ is denoted by D_*^{α} and is given in the following form:

$$D_*^{\alpha} f(t) = J^{n-\alpha} \left(D^n f(t)\right).$$

A very important property of Caputo fractional derivative is that if $f(t)$ is absolutely continuous on $[a, b]$, then

$$\lim_{\alpha \to 1} D_*^{\alpha} f(t) = D f(t) = \frac{df}{dt}.$$

More details about fractional calculus and fractional differential equations see [6,16,21, 28].

El-Mesiry et. al, gave a numerical scheme to solve a nonlinear multi-term fractional order differential equation [9]. El-Sayed et. al, studied some methods which are used to solve multi-term nonlinear fractional differential equation [10]. Ahmed et. al, studied the fractional order predator-prey system and the fractional order rabies model. They proved the existence and the uniqueness of solutions and studied the stability of the equilibrium points [1, 2, 8–12].

The Lotka-Volterra predator-prey model was proposed by Alfred J. Lotka [18]. The model was later extended to include density dependent prey growth and a functional response by Crawford Stanley Holling [22]. In 1980, an alternative model to the Lotka-Volterra model is emerged [4]. A system of two teams of preys, with densities $x_1(t),\ x_2(t)$, respectively, interacting with one team of predator with density $x_3(t)$, was proposed by Elettreby [13].

We are interested in the fractional order of a two-prey one-predator model. In section 2, we proved the existence and the uniqueness of the solutions of the model. In section 3, we studied the stability of the equilibrium points of the proposed system. The numerical solution of the fractional order two-prey one-predator model is given in section 4.

2. Existence and uniqueness

In this section, we will consider the following fractional order two-prey one-predator without help system;

$$\begin{aligned}
D_*^{\alpha} x_1(t) &= a\, x_1(t)(1 - x_1(t)) - x_1(t)\, x_3(t), \quad t \in (0, T], \\
D_*^{\alpha} x_2(t) &= b\, x_2(t)(1 - x_2(t)) - x_2(t)\, x_3(t), \ \ t \in (0, T], \\
D_*^{\alpha} x_3(t) &= -c\, x_3^2(t) + d\, x_1(t)\, x_3(t) + e\, x_2(t)\, x_3(t), \quad t \in (0, T],
\end{aligned} \tag{1}$$

with the initial values;

$$x_1(t)\,|_{t=0} = x_1(0), \quad x_2(t)\,|_{t=0} = x_2(0) \quad \text{and} \quad x_3(t)\,|_{t=0} = x_3(0), \tag{2}$$

where $0 < \alpha \leq 1,\quad x_1(t) \geq 0,\quad x_2(t) \geq 0$ and $x_3(t) \geq 0$ are the two-prey and predator densities, respectively. Thel constants $a,\ b,\ c,\ d$ and e are positive.

The initial value problem (1, 2) can be written as the following matrix form;

$$D_*^{\alpha}\, X(t) = A\, X(t) + x_1(t)\, B\, X(t) + x_2(t)\, C\, X(t) + x_3(t)\, D\, X(t),\ X(0) = X_0, \quad (3)$$

where

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix},\ X_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix},\ A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and}$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ d & e & -c \end{bmatrix}.$$

Lemma 1. *[7] Consider $E = [0,T] \times [x_0^{(0)} - \epsilon, x_0^{(0)} + \epsilon]$, with some $T > 0$ and $\epsilon > 0$, and let $f : E \mapsto \mathbb{R}$. Then the fractional differential equation;*

$$D_*^{\alpha} X(t) = f(t, X(t)), \quad \alpha > 0 \text{ and } X^{(k)}(0) = X_0, \quad k = 0, 1, 2, ..., m-1,$$

has a solution and this solution is unique if the following two conditions are hold togather;

(1) *f is bounded and continuous on E.*
(2) *f satisfies Lipschitz condition with respect to the second variable, implies that,*

$$|f(t, X(t)) - f(t, Y(t))| \leq L|X(t) - Y(t)|,$$

with some constant $L > 0$ independent of t, X and Y.

Definition 1. *Let $C[0,T]$ be the class of continuous functions $X(t)$ on the interval $[0,T]$ such that* $\sup |X(t)| < \infty, \quad t \in [0,T]$.

Definition 2. *Let $C^*[0,T]$ be the class of continuous column vector $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$ where $x_i(t) \in C[0,T],\ i = 1, 2, 3$, the norm of $X \in C^*[0,T]$ is given by;*

$$\|X\| = \sum_{i=1}^{3} \sup |x_i(t)|, \quad t \in [0,T].$$

Theorem 1. *The initial value problem (1, 2) has a unique solution.*

Proof. Let $F(X(t)) = A\, X(t) + x_1(t)\, B\, X(t) + x_2(t)\, C\, X(t) + x_3(t)\, D\, X(t) \qquad t \in [0,T]$,

be a continuous and bounded function on $[X_0 - \epsilon, X_0 + \epsilon]$ for any $\epsilon > 0$. Suppose that $X(t)$ and $Y(t)$ are two distinct solutions of the initial value problem (1, 2) such that $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}$. and $X, Y \in C^*[0,T]$, then

$$\begin{aligned} \|F(X) - F(Y)\| &= \|A\ X(t) + x_1(t)\ B\ X(t) + x_2(t)\ C\ X(t) + x_3(t)\ D\ X(t) \\ &- (A\, Y(t) + y_1(t)\ B\ Y(t) + y_2(t)\ C\ Y(t) + y_3(t)\ D\ Y(t)\| \\ &\leq \|A\ (X(t) - Y(t))\| + \|x_1(t)\ B\ (X(t) - Y(t))\| + \|(x_1(t) - y_1(t))\ B\ Y(t)\| \\ &+ \|x_2(t)\ C\ (X(t) - Y(t))\| + \|(x_2(t) - y_2(t))\ C\ Y(t)\| \\ &+ \|x_3(t)\ D\ (X(t) - Y(t))\| + \|(x_3(t) - y_3(t))\ D\ Y(t)\|. \end{aligned}$$

Since $|x_i(t) - y_i(t)| \leq \|X(t) - Y(t)\|$ for each $i = 1, 2, 3$.

Then, we have

$$\begin{aligned} \|F(X) - F(Y)\| &\leq [\|A\| + \|B\| \,(|x_1(t)| + \|Y(t)\|) + \|C\| \,(|x_2(t)| + \|Y(t)\|) \\ &+ \|D\| \,(|x_3(t)| + \|Y(t)\|)] \, \| X(t) - Y(t)\|. \end{aligned}$$

Let

$$L = [\|A\| + \|B\| \,(|x_1(t)| + \|Y(t)\|) + \|C\| \,(|x_2(t)| + \|Y(t)\|) + \|D\| \,(|x_3(t)| + \|Y(t)\|)$$

It is clear that $L > 0$. Then

$$\|F(X) - F(Y)\| \leq L \,\|X(t) - Y(t)\|.$$

By applying Lemma 1, and since $F(X(t))$ is continuous and satisfying Lischitz condition, the initial value problem (1, 2) has a unique solution . ☐

3. The asymptotic stability of the equilibrium

Consider the following system;

$$\begin{aligned} D_*^\alpha x_1(t) &= f_1(x_1, x_2, x_3), \\ D_*^\alpha x_2(t) &= f_2(x_1, x_2, x_3), \\ D_*^\alpha x_3(t) &= f_3(x_1, x_2, x_3), \end{aligned} \tag{4}$$

with the initial values $x_1(0) = x_{10}$, $x_2(0) = x_{20}$, $x_3(0) = x_{30}$ and $\alpha \in (0, 1]$. To evaluate the equilibrium points of system (4), let $D_*^\alpha \bar{x}_i(t) = 0$, then $f_i(\bar{x}_1, \bar{x}_2, \bar{x}_3) = 0$, $i = 1, 2, 3$, where $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ is the equilibrium point of system (4). Then the equilibrium points are:

$$E_0(0, 0, 0),\ E_1(1, 0, 0),\ E_2(0, 1, 0),\ E_3(1, 1, 0),\ E_4(0, \frac{bc}{bc + e}, \frac{be}{bc + e}),\ E_5(\frac{ac}{ac + d}, 0, \frac{ad}{ac + d}),$$

and

$$E_6(\frac{abc + ae - be}{abc + bd + ae}, \frac{abc + bd - ad}{abc + bd + ae}, \frac{abd + abe}{abc + bd + ae}),$$

which exists under the conditions:

$$b\,e \leq a\,b\,c + a\,e,$$

$$a\,d \leq a\,b\,c + b\,d.$$

The Jacobian matrix of the system (1) is:

$$J = \begin{bmatrix} a\,(1 - 2\,x_1) - x_3 & 0 & -x_1 \\ 0 & b\,(1 - 2\,x_2) - x_3 & -x_2 \\ d\,x_3 & e\,x_3 & -2\,c\,x_3 + d\,x_1 + e\,x_2 \end{bmatrix}.$$

Substituting by the equilibrium point $E_0(0, 0, 0)$ in the above Jacobian matrix, we get $J_0 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Which has the eigenvalues $\lambda = 0,\, a\,(> 0),\, b\,(> 0)$. Since, we have two positive eigenvalues, then the equilibrium point E_0 is unstable. Similarly, the Jacobian matrix for the point $E_1(1, 0, 0)$, is $J_1 = \begin{bmatrix} -a & 0 & -1 \\ 0 & b & 0 \\ 0 & 0 & d \end{bmatrix}$. Their eigenvalues are $\lambda = -\,a,\ d\,(>0),\ b\,(>0)$. Then the equilibrium point E_1 is unstable. The eigenvalues of the equilibrium point $E_2(0,1,0)$ are $\lambda = -\,b,\ a\,(>0),\ e\,(>0)$. So, it is unstable. Also, the eigenvalues for the equilibrium point $E_3(1,1,0)$ are $\lambda = -\,a,\ -\,b,\ d+e\,(>0)$. Which mean that the point E_3 is unstable. For the equilibrium point $E_4(0,\ \frac{b\,c}{b\,c+e},\ \frac{b\,e}{b\,c+e})$, the Jacobian matrix is:

$$J_4 = \begin{bmatrix} a - \frac{b\,e^2}{(b\,c+e)^2} & 0 & 0 \\ \frac{b^2\,c\,e}{(b\,c+e)^2} & \frac{-b^2\,c}{b\,c+e} & \frac{-\,b\,c}{b\,c+e} \\ \frac{b\,d\,e}{b\,c+e} & \frac{b\,e^2}{b\,c+e} & \frac{-\,b\,c\,e}{b\,c+e} \end{bmatrix},$$

which has the following characteristic polynomial:

$$\left(a - \frac{b\,e^2}{(b\,c+e)^2} - \lambda\right)\,\left(\lambda^2 + a_1\,\lambda + a_2\right) = 0, \tag{5}$$

where $a_1 = \frac{b\,c\,(b+e)}{b\,c+e}$ and $a_2 = \frac{b^2\,c\,e}{b\,c+e}$.

A sufficient condition to say that an equilibrium point is a locally asymptotically stable is that all eigenvalues λ satisfies $|arg(\lambda)| > \frac{\alpha\,\pi}{2}$ [19]. This condition implies that the characteristic polynomial of that point should satisfies the Routh-Huwitz conditions [1]. Since a_1, and a_2 are both positive, then the stability of the point E_4 depends on the first braket in (5). Thus E_4 is locally asymptotically stable if $\lambda = a - \frac{b\,e^2}{(b\,c+e)^2} < 0$. Similarly, the equilibrium point $E_5(\frac{a\,c}{a\,c+d},\ 0,\ \frac{a\,d}{a\,c+d})$ has the Jacobian matrix:

$$J_5 = \begin{bmatrix} \frac{-\,a^2\,c}{a\,c+d} & \frac{a^2\,c\,d}{(a\,c+d)^2} & \frac{-\,a\,c}{a\,c+d} \\ 0 & b\,-\,\frac{a\,d^2}{(a\,c+e)^2} & 0 \\ \frac{a\,d^2}{a\,c+d} & \frac{a\,d\,e}{a\,c+d} & \frac{-\,a\,c\,d}{a\,c+d} \end{bmatrix},$$

which has the characteristic polynomial:

$$\left(b - \frac{a\,d^2}{(a\,c+d)^2} - \lambda\right)\,\left(\lambda^2 + a_1\,\lambda + a_2 = 0\right), \tag{6}$$

where $a_1 = \frac{a\,c\,(a+d)}{a\,c+d}$ and $a_2 = \frac{a^2\,c\,d}{a\,c+d}$. Since a_1 and a_2 are both positive, then E_5 is locally asymptotically stable if $\lambda = b - \frac{a\,d^2}{(a\,c+d)^2} < 0$.

For the equilibrium point E_6, the Jacobian matrix is given by:

$$J_6 = \begin{bmatrix} -a\,\bar{x}_1 & 0 & -\bar{x}_1 \\ 0 & -b\,\bar{x}_2 & -\bar{x}_2 \\ d\,\bar{x}_3 & e\,\bar{x}_3 & -c\,\bar{x}_3 \end{bmatrix},$$

where $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (\frac{a\,b\,c+a\,e-b\,e}{a\,b\,c+b\,d+a\,e}, \frac{a\,b\,c+b\,d-a\,d}{a\,b\,c+b\,d+a\,e}, \frac{a\,b\,d+a\,b\,e}{a\,b\,c+b\,d+a\,e})$. The above matrix has the following characteristic polynomial:

$$\lambda^3 + a_1\,\lambda^2 + a_2\,\lambda + a_3 = 0,$$

where

$$\begin{aligned} a_1 &= a\,\bar{x}_1 + b\,\bar{x}_2 + c\,\bar{x}_3, \\ a_2 &= (a\,b\,\bar{x}_1\,\bar{x}_2 + a\,c\,\bar{x}_1\,\bar{x}_3 + b\,c\,\bar{x}_2\,\bar{x}_3 + e\,\bar{x}_2\,\bar{x}_3 + d\,\bar{x}_1\,\bar{x}_3), \\ a_3 &= \bar{x}_1\,\bar{x}_2\,\bar{x}_3\,(a\,b\,c + a\,e + b\,d). \end{aligned}$$

Since $a_1 > 0$, and $a_1\,a_2 >\ a_3$, that means $|arg(\lambda)| > \frac{\alpha\pi}{2}$. Then the equilibrium point E_6 is locally asymptotically stable.

4. Numerical Results

The Adams-type predictor-corrector method for the numerical solution of the fractional differential equations was discussed in [8]. This method can be used for both linear and nonlinear problems. It may be extended to multi-term equations (involving more than one differential operator) too [8].

In this paper, we used Adams-type predictor-corrector method for the numerical solution of our system. First, we will give the Adams-type predictor-corrector method for solving general initial value problem with Caputo derivative,

$$D_*^\alpha\, y(t) = f(t,\, y(t)),$$

with the initial condition $y(0) = y_0$ and $t \in (0, T]$. We assumed a set of points $\{t_j,\, y_j\}$, where $y_j = y(t_j)$, are the points used for our approximation and $t_j = j\,h$, $j = 0,\,1,\,.....,N\,(integer)$, $h = \frac{T}{N}$. The general formula for Adams-type predictor-corrector method is;

$$y_{n+1} = \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{t_{n+1}^k}{k!}\, y_0^{(k)} + \frac{h^\alpha}{\Gamma\,(\alpha+2)} \sum_{j=0}^{n} \sigma_{j,\,n+1}\, f(t_j,\, y_j) + \frac{h^\alpha}{\Gamma\,(\alpha+2)}\, \sigma_{n+1,\,n+1}\, f(t_{n+1},\, y_{n+1}^P),$$

where

$$\sigma_{j,\,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)\,(n+1)^\alpha, & \text{if } j=0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2\,(n-j+1)^{\alpha+1}, & \text{if } 1 \le j \le n, \\ 1, & \text{if } j = n+1 \end{cases}$$

and

$$y_{n+1}^P = \sum_{k=0}^{\lceil\alpha\rceil-1} \frac{t_{n+1}^k}{k!}\, y_0^{(k)} + \frac{1}{\Gamma\,(\alpha)} \sum_{j=0}^{n} \rho_{j,\,n+1}\, f(t_j,\, y_j),$$

$$rho_{j,\,n+1} = \frac{h^\alpha}{\alpha}\, \left((n+1-j)^\alpha - (n-j)^\alpha\right).$$

Applying the above algorithm for the system (1), we have the following;

$$x_{1,\,n+1} = x_{1,\,0} + \tfrac{h^\alpha}{\Gamma(\alpha+2)} \textstyle\sum_{j=0}^n \sigma_{1,\,j,\,n+1}\, \left(a\,x_{1,\,j}\,(1-x_{1,\,j}) - x_{1,\,j}\,x_{3,\,j}\right)$$
$$+\tfrac{h^\alpha}{\Gamma(\alpha+2)}\, \sigma_{1,\,n+1,\,n+1}\, \left(a\,x_{1,\,n+1}^P\,(1-x_{1,\,n+1}^P) - x_{1,\,n+1}^P\,x_{3,\,n+1}^P\right),$$

$$x_{2\,n+1} = x_{2,\,0} + \tfrac{h^\alpha}{\Gamma(\alpha+2)} \textstyle\sum_{j=0}^n \sigma_{2,\,j,\,n+1}\, \left(b\,x_{2,\,j}\,(1-x_{2,\,j}) - x_{2,\,j}\,x_{3,\,j}\right)$$
$$+\tfrac{h^\alpha}{\Gamma(\alpha+2)}\, \sigma_{2,\,n+1,\,n+1}\, \left(b\,x_{2,\,n+1}^P\,(1-x_{2,\,n+1}^P) - x_{2,\,n+1}^P\,x_{3,\,n+1}^P\right),$$

$$x_{3,\,n+1} = x_{3,\,0} + \tfrac{h^\alpha}{\Gamma(\alpha+2)} \textstyle\sum_{j=0}^n \sigma_{3,\,j,\,n+1}\, \left(-\,c\,x_{3,\,j}^2 + d\,x_{1,\,j}\,x_{3,\,j} + e\,x_{1,\,j}\,x_{3,\,j}\right)$$
$$+\tfrac{h^\alpha}{\Gamma(\alpha+2)}\, \sigma_{3,\,n+1,\,n+1}\, \left(-\,c\,x_{3,\,n+1}^{2^P} + d\,x_{1,\,n+1}^P\,x_{3,\,n+1}^P + e\,x_{2,\,n+1}^P\,x_{3,\,n+1}^P\right),$$

where

$$x_{1,n+1}^P = x_{1,0} + \tfrac{1}{\Gamma(\alpha)} \textstyle\sum_{j=0}^n \rho_{1,\,j,\,n+1}\,[a\,x_{1,\,j}\,(1-x_{1,\,j}) - x_{1,\,j}\,x_{3,\,j}],$$

$$x_{2,\,n+1}^P = x_{2,\,0} + \tfrac{1}{\Gamma(\alpha)} \textstyle\sum_{j=0}^n \rho_{2,\,j,\,n+1}\,[b\,x_{2,\,j}\,(1-x_{2,\,j}) - x_{2,\,j}\,x_{3,\,j}],$$

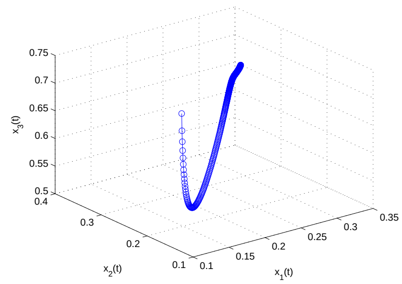
$$x_{3,\,n+1}^P = x_{3,\,0} + \tfrac{1}{\Gamma(\alpha)} \textstyle\sum_{j=0}^n \rho_{3,\,j,\,n+1}\,[-\,c\,x_{3,\,j}^2 + d\,x_{1,\,j}\,x_{3,\,j} + e\,x_{2,\,j}\,x_{3,\,j}].$$

Therefore, for $i = 1, 2, 3$

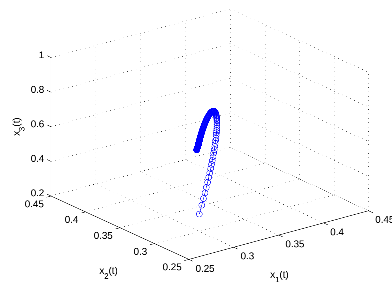
$$\sigma_{i,j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & \text{if } j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & \text{if } 1 \le j \le n, \\ 1, & \text{if } j = n+1 \end{cases}$$

and

$$\rho_{i,\,j,\,n+1} = \frac{h^{\alpha}}{\alpha}\left((n+1-j)^{\alpha} - (n-j)^{\alpha}\right).$$

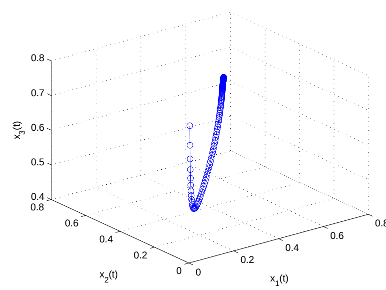


(A)



(B)

FIGURE 1. $\alpha = .8,\ a = 1.0, b = 1.0, c = 1.0, d = 1.5, e = 1.0$



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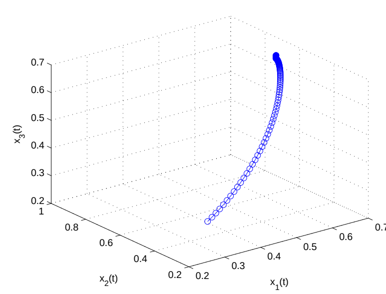

(B)

FIGURE 2. $\alpha = 0.9,\ a = 2.0,\ b = 2.0,\ c = 2.0,\ d = 1,\ e = 1.0$

In Figures 1 and 2, we plot the approximate solutions between x_1 , x_2 and x_3.In figure 1, the constant values are $\alpha = .8,\ a = 1.0, b = 1.0, c = 1.0, d = 1.5, e = 1.0$. In figure 2, the constant values are $\alpha = 0.9$, $a = 2.0$, $b = 2.0$, $c = 2.0$, $d = 1$, $e = 1.0$. In Figures 1a and 2a , we used the initial values $x_1(0) = 0.1$, $x_2(0) = 0.125$, $x_3(0) = .75$., which satisfied the existence conditions of the locally asymptotically stable equilibrium point E_6. In figures 1b and 2b, we used the initial values $x_1(0) = 0.3$, $x_2(0) = 0.3$, $x_3(0) = .3$., which also satisfied the existence conditions of the locally asymptotically stable equilibrium point E_6 and we showed that the solutions are stable.

5. CONCLUSION

We studied existence, uniqueness, stability of the equilibrium points and numerical solutions of a fractional order two-prey one-predator without help between each team of preys model. The equilibrium point E_6 , was stable equilibrium point under some conditions at the ordinary differential equation form of the model. But in our fractional form, we found that the same point is stable without any conditions. This is an example of the equilibrium point which is a centre for the integer order system but locally asymptotically stable for its fractional-order counterpart. This means that the fractional-order differential equations are ,at least, as stable as their integer order counterpart.

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