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SOME REMARKS ON THE SOLUTIONS OF A FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSION OF STURM-LIOUVILLE TYPE

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ABSTRACT. We consider a Sturm-Liouville type integro-differential inclusion of fractional order and we establish some Filippov type existence existence results.

1. Introduction

In this paper we consider the following problem

$$D_C^q y(t) \in F(t, x(t), V(x)(t)) \quad a.e. \ ([0,T]), \quad x(0) = x_0, y(0) = y_0, \tag{1.1}$$

where $y(t) \equiv p(t)x'(t)$, $F(.,.,.) : [0,T] \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map, $V : C(I,\mathbf{R}) \to C(I,\mathbf{R})$ is a nonlinear Volterra integral operator, $p(.) : [0,T] \to (0,\infty)$ is a continuous function, $x_0, y_0 \in \mathbf{R}$ and D_C^q denotes Caputo's fractional derivative of order $q \in (0,1)$.

In the theory of ordinary differential equations it is wellknown that any linear real second-order differential equation may be written in the self adjoint form

$$-(r(t)x')' + q(t)x = 0. \tag{1.2}$$

Equation (1.2) together with boundary conditions of the form $a_1x(0) - a_2x'(0) = 0$, $b_1x(T) - b_2x'(T) = 0$ is called the Sturm-Liouville problem. For a complete disscusion on Sturm-Liouville problems we refer, for example, to [10]. This is the reason why differential inclusions of the form $(r(t)x')' \in F(t,x)$ are usually called Sturm-Liouville type differential inclusions, even if the boundary value problems associated are not as at the original Sturm-Liouville problem.

In the last years one may see a strong development of the theory of differential equations and inclusions of fractional order ([6], [9], [11] etc.). The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena. In the fractional calculus there are several fractional derivatives. From them, the fractional derivative introduced by Caputo in [3], allows to use Cauchy conditions which have physical meanings.

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The aim of our paper is to consider the extension of the Sturm-Liouville problem to the fractional framework, given by problem (1.1), and to present several existence results for problem (1.1). On one hand, we show that Filippov's ideas ([7]) can be suitably adapted in order to obtain the existence of solutions of problem (1.1). We recall that for a first order differential inclusion defined by a lipschitzian set-valued map with nonconvex values Filippov's theorem ([7]) consists in proving the existence of o solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion. On the other hand, we prove the existence of solutions continuously depending on a parameter for problem (1.1). This result may be seen as a continuous variant of Filippov's theorem. The key tool in the proof of this theorem is a result of Bressan and Colombo ([2]) concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values. We note that similar results for other classes of fractional differential inclusions may be found in our previous papers [4], [5]. We mention also that in [8], namely Theorem 2.4, it is provided a sufficient condition under which any nonoscilatory solution of problem (1.1), with F single-valued and not depending on the last variable, is bounded.

The paper is organized as follows: in Section 2 we recall some preliminary results that we use in the sequel, in Section 3 we obtain our Filippov type existence results and in Section 4 we treat the parameterized situation.

2. Preliminaries

In what follows $I = [0, T]$, X is a real separable Banach space with norm $|.|$ and with the corresponding metric $d(.,.)$. As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $|x(.)|_C = \sup_{t\in I} |x(t)|$ and by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $|x(.)|_1 = \int_0^T |x(t)|\mathrm{d}t$.

Denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A(.) : I \to \{0, 1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote by $\mathrm{cl}(A)$ the closure of A.

We recall that *the fractional integral of order* $p > 0$ of a Lebesgue integrable function $f : (0, \infty) \to \mathbf{R}$ is defined by

$$I^p f(t) = \int_0^t \frac{(t-s)^{p-1}}{\Gamma(p)} f(s) ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$ and $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt$.

Caputo's fractional derivative of order $p > 0$ of a function $f : [0, \infty) \to \mathbf{R}$ is defined by

$$D_c^p f(t) = \frac{1}{\Gamma(n-p)} \int_0^t (t-s)^{-p+n-1} f^{(n)}(s) ds,$$

where $n = [p] + 1$. It is assumed implicitly that f is n times differentiable whose n-th derivative is absolutely continuous.

In the sequel $V : C(I, \mathbf{R}) \to C(I, \mathbf{R})$ is a nonlinear Volterra integral operator defined by $V(x)(t) = \int_0^t k(t, s, x(s)) ds$, where $k(.,.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a given function and $F(.,.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map.

A continuous mapping $x(.) \in C(I, \mathbf{R})$ is called a (**mild**) solution of problem (1.1) if there exists an integrable function $f(.) \in L^1(I, \mathbf{R})$ such that

$$f(t) \in F(t, x(t), V(x)(t)) \quad a.e. \ (I), \tag{2.1}$$

$$x(t) = x_0 + y_0 \int_0^t \frac{1}{p(s)} ds + \int_0^t \frac{1}{p(s)} (\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f(u) du) ds \quad \forall t \in I. \tag{2.2}$$

This definition of the solution is justified by the fact that if $f(.) \in L^1(I, \mathbf{R})$ satisfies (2.1), then from the equality $D_C^q y(t) = f(t)$ it follows $p(t)x'(t) = y(t) = y_0 + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds$ and, then, integrating by parts we obtain (2.2).

We note that $x(.)$ in (2.2) may be written as

$$x(t) = x_0 + y_0 \int_0^t \frac{1}{p(s)} ds + \frac{1}{\Gamma(q)} \int_0^t (\int_u^t \frac{(s-u)^{q-1}}{p(s)} ds) f(u) du.$$

Since $p(.) : [0, T] \to (0, \infty)$ is continuous, we denote $M := \sup_{t \in I} \frac{1}{p(t)}$. We put also

$$a(t) = x_0 + y_0 \int_0^t \frac{1}{p(s)} ds, \quad S(t, u) = \frac{1}{\Gamma(q)} \int_u^t \frac{(s-u)^{q-1}}{p(s)} ds.$$

Note, that for all $t, u \in I$ we have

$$|S(t, u)| \leq \frac{1}{M\Gamma(q)} \int_u^t (s-u)^{q-1} ds \leq \frac{1}{Mq\Gamma(q)} (t-u)^q \leq \frac{T^q}{Mq\Gamma(q)} =: M_1.$$

We shall call $(x(.), f(.))$ a *trajectory-selection pair* of (1.1) if (2.1) and (2.2) are satisfied.

We make the following notation

$$\mathcal{S}(x_0, y_0) = \{x(.); \quad x(.) \text{ is a solution of } (1.1)\}.$$

Finally, we recall several preliminary results we shall use in the sequel.

Lemma 2.1. *Let X be a separable Banach space, let $H : I \to \mathcal{P}(X)$ be a measurable set-valued map with nonempty closed values and $g, h : I \to X, L : I \to (0, \infty)$ measurable functions. Then one has.*

i) The function $t \to d(h(t), H(t)$ is measurable.

ii) If $H(t) \cap (g(t) + L(t)B) \neq \emptyset$ a.e. (I) then the set-valued map $t \to H(t) \cap (g(t) + L(t)B)$ has a measurable selection.

Its proof may be found in [1].

A subset $D \subset L^1(I, X)$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I \backslash A$. We denote by $\mathcal{D}(I, X)$ the family of all decomposable closed subsets of $L^1(I, X)$.

Next (S, d) is a separable metric space; we recall that a set-valued map $G(\cdot) : S \to \mathcal{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset X$, the subset $\{s \in S; G(s) \subset C\}$ is closed. The proof of the next two lemmas may be found in [2].

Lemma 2.2. *Let $F^*(.,.) : I \times S \to \mathcal{P}(X)$ be a closed-valued $\mathcal{L}(I) \otimes \mathcal{B}(S)$ measurable set-valued map such that $F^*(t, .)$ is l.s.c. for any $t \in I$.*

Then the set-valued map $G(.) : S \to \mathcal{D}(I, X)$ defined by

$$G(s) = \{v \in L^1(I, X); \quad v(t) \in F^*(t, s) \quad a.e. \ (I)\}$$

is l.s.c. with nonempty closed values if and only if there exists a continuous mapping $p(.) : S \to L^1(I,X)$ *such that*

$$d(0, F^*(t,s)) \le p(s)(t) \quad a.e.\ (I),\ \forall s \in S.$$

Lemma 2.3. *Let* $G(.) : S \to \mathcal{D}(I,X)$ *be a l.s.c. set-valued map with closed decomposable values and let* $\phi(.) : S \to L^1(I,X)$, $\psi(.) : S \to L^1(I,\mathbf{R})$ *be continuous such that the set-valued map* $H(.) : S \to \mathcal{D}(I,X)$ *defined by*

$$H(s) = cl\{v \in G(s); \quad |v(t) - \phi(s)(t)| < \psi(s)(t) \quad a.e.\ (I)\}$$

has nonempty values.

Then H *has a continuous selection, i.e. there exists a continuous mapping* $h : S \to L^1(I,X)$ *such that* $h(s) \in H(s) \quad \forall s \in S$.

3. A Filippov type result

In order to establish our existence result for problem (1.1) we need the following hypotheses.

Hypothesis 3.1. i) $F(.,.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and is* $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \times \mathbf{R})$ *measurable.*

ii) *There exists* $L(.) \in L^1(I, \mathbf{R}_+)$ *such that, for almost all* $t \in I, F(t,.,.)$ *is L(t)-Lipschitz in the sense that for almost* $t \in I$

$$d(F(t,x_1,y_1), F(t,x_2,y_2)) \le L(t)(|x_1 - x_2| + |y_1 - y_2|)\ \forall\ x_1, x_2, y_1, y_2 \in \mathbf{R},$$

where d(A,B) is the Hausdorff distance

$$d(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B); a \in A\}.$$

iii) $p(.) : I \to (0,\infty)$ *is continuous and* $k(.,.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *satisfy:* $\forall x \in \mathbf{R},\ (t,s) \to k(t,s,x)$ *is measurable and* $|k(t,s,x) - k(t,s,y)| \le L(t)|x - y|\ \ a.e.(t,s) \in I \times I,\ \forall x,y \in \mathbf{R}$.

We shall use next the following notations

$$m(t) = \int_0^t L(u)du, \quad \alpha(x) = \frac{(x+1)^2 - 1}{2}, \quad x \in \mathbf{R}.$$

In what follows $x_1, y_1 \in \mathbf{R}$, $g(.) \in L^1(I,\mathbf{R})$ and $z(.) \in C(I,\mathbf{R})$ is a solution of the Cauchy problem

$$D_C^q w(t) = g(t), \quad p(t)z'(t) \equiv w(t), \quad z(0) = x_1, \quad w(0) = y_1.$$

Hypothesis 3.2. i) *Hypothesis 3.1 is satisfied.*

ii) *The function* $t \to q(t) := d(g(t), F(t, z(t), V(z)(t))$ *is integrable on* I.

Theorem 3.3. *Consider* $\delta \ge 0$ *and assume that Hypothesis 3.2 is satisfied. Then for any* $x_0, y_0 \in \mathbf{R}$ *with* $(|x_0 - y_0| + MT|x_1 - y_1|) \le \delta$ *there exists* $(x(.), f(.))$ *a trajectory-selection pair of (1.1) such that*

$$|x(t) - z(t)| \le \xi(t) \quad \forall\ t \in I,$$

$$|f(t) - g(t)| \le L(t)(\xi(t) + \int_0^t L(u)\xi(u)du) + q(t) \quad a.e.\ (I),$$

where

$$\xi(t)=\delta e^{M_1\alpha(m(t))}+\int_0^t q(u)e^{M_1\alpha(m(t)-m(u))}du.$$

Proof. Set $x_0(t)\equiv z(t)$, $f_0(t)\equiv g(t)$, $t\in I$ and for $n\geq 1$

$$q_n(t)=\int_0^t q(u)\frac{(\alpha(m(t)-m(u))^{n-1}}{(n-1)!}du+\frac{(\alpha(m(t))^{n-1}}{(n-1)!}(|x_0-y_0|+MT|x_1-y_1|).$$

We claim that is enough to construct the sequences $x_n(.)\in C(I,\mathbf{R})$, $f_n(.)\in L^1(I,\mathbf{R})$, $n\geq 1$ with the following properties

$$x_n(t)=a(t)+\int_0^t S(t,u)f_n(u)du,\quad \forall t\in I, \tag{3.1}$$

$$|x_1(t)-x_0(t)|\leq \delta+M_1\int_0^t q(u)du=:q_0(t)\quad \forall t\in I, \tag{3.2}$$

$$|f_1(t)-f_0(t)|\leq q(t)\quad a.e.\ (I), \tag{3.3}$$

$$f_n(t)\in F(t,x_{n-1}(t),V(x_{n-1})(t))\quad a.e.\ (I),\ n\geq 1, \tag{3.4}$$

$$|f_{n+1}(t)-f_n(t)|\leq L(t)(|x_n(t)-x_{n-1}(t)|+\int_0^t L(u)|x_n(u)-x_{n-1}(u)|du)\ a.e., \tag{3.5}$$

$$|x_n(t)-x_{n-1}(t)|\leq (M_1)^{n-1}q_n(t)\quad \forall t\in I. \tag{3.6}$$

Indeed, from (3.6) $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I,\mathbf{R})$. Thus, from (3.5) for almost all $t\in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbf{R}$. Moreover, from (3.2) and the last inequality we have

$$|x_n(t)-z(t)|\leq \sum_{i=0}^{n-1}|x_{i+1}(t)-x_i(t)|\leq \sum_{i=0}^{n-1}(M_1)^i q_{i+1}(t)\leq \xi(t) \tag{3.7}$$

On the other hand, from (3.3), (3.5) and (3.6) we obtain for almost all $t\in I$

$$\begin{aligned}&|f_n(t)-g(t)|\leq \textstyle\sum_{i=1}^{n-1}|f_{i+1}(t)-f_i(t)|+|f_1(t)-g(t)|\leq\\ &L(t)(\xi(t)+\textstyle\int_0^t L(u)\xi(u)du)+q(t).\end{aligned} \tag{3.8}$$

Let $x(.)\in C(I,\mathbf{R})$ be the limit of the Cauchy sequence $x_n(.)$. From (3.8) the sequence $f_n(.)$ is integrably bounded and we have already proved that for almost all $t\in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbf{R}$. Take $f(.)\in L^1(I,\mathbf{R})$ with $f(t)=\lim_{n\to\infty}f_n(t)$.

Passing to the limit in (3.1) and using Lebesgue's dominated convergence theorem we get (2.2). Finally, passing to the limit in (3.7) and (3.8) we obtained the desired estimations.

It remains to construct the sequences $x_n(.),f_n(.)$ with the properties in (3.1)-(3.6). The construction will be done by induction.

The set-valued map $t\to F(t,z(t),V(z)(t))$ is measurable with closed values and

$$F(t,z(t),V(z)(t))\cap\{g(t)+q(t)B\}\neq\emptyset\quad a.e.\ (I).$$

From Lemma 2.1 we find $f_1(.)$ a measurable selection of the set-valued map $H_1(t):=F(t,z(t),V(z)(t))\cap\{g(t)+q(t)B\}$. Obviously, $f_1(.)$ satisfy (3.3). Define $x_1(.)$ as in (3.1) with $n=1$. Therefore, we have

$$\begin{aligned}&|x_1(t)-z(t)|\leq|x_0-y_0|+MT|x_1-y_1|+|\textstyle\int_0^t S(t,u)(f_1(u)-g(u))du|\\ &\leq\delta+M_1\textstyle\int_0^t q(s)ds=q_0(t).\end{aligned}$$

Assume that for some $N \geq 1$ we already constructed $x_n(.) \in C(I, \mathbf{R})$ and $f_n(.) \in L^1(I, \mathbf{R}), n = 1, 2, ...N$ satisfying (3.1)-(3.6). We define the set-valued map

$$H_{N+1}(t) := F(t, x_N(t), V(x_N)(t)) \cap \{f_N(t) + L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(u)|x_N(u) - x_{N-1}(u)|du B\}, \quad t \in I.$$

The set-valued map $t \to F(t, x_N(t), V(x_N)(t))$ is measurable and from the lipschitzianity of $F(t, ., .)$ we have that for almost all $t \in I$ $H_{N+1}(t) \neq \emptyset$. We apply Lemma 2.1 and find a measurable selection $f_{N+1}(.)$ of $F(., x_N(.), V(x_N)(.))$ such that for almost $t \in I$

$$|f_{N+1}(t) - f_N(t)| \leq L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(u)|x_N(u) - x_{N-1}(u)|du).$$

We define $x_{N+1}(.)$ as in (3.1) with $n = N + 1$ and we get

$$|x_{N+1}(t) - x_N(t)| \leq M_1 \int_0^t |f_{N+1}(u) - f_N(u)|du \leq M_1 \int_0^t L(u)(|x_N(u) - x_{N-1}(u)| + \int_0^u L(s)|x_N(s) - x_{N-1}(s)|ds)du \leq M_1 \int_0^t L(u)(M_1^{N-1} q_N(u) + \int_0^u L(s) M_1^{N-1} q_N(r)dr)du.$$

We shall prove next that

$$\int_0^t L(u)(q_n(u) + \int_0^u L(r)q_n(r)dr)du \leq q_{n+1}(t) \tag{3.9}$$

and therefore (3.6) holds true with $n = N + 1$ which completes the proof.

One has

$$\int_0^t L(u)(q_n(u) + \int_0^u L(r)q_n(r)dr)du = \int_0^t (1 + m(t) - m(u))L(u)q_n(u)du$$
$$= \int_0^t (1 + m(t) - m(u))L(u)\frac{(\alpha(m(u))^{n-1}}{(n-1)!}|x_0 - u_0|du +$$
$$\int_0^t (1 + m(t) - m(u))L(u)(\int_0^u p(r)\frac{(\alpha(m(t) - m(r))^{n-1}}{(n-1)!}dr)du \leq$$
$$|x_0 - u_0| \int_0^t (1 + m(t) - m(u))L(u)\frac{(\alpha(m(u)))^{n-1}}{(n-1)!} +$$
$$\int_0^t (\int_r^t \frac{(\alpha(m(u) - m(r)))^{n-1}}{(n-1)!}(1 + m(t) - m(u))L(u))p(r)drdu.$$

According to the definition of $\alpha(.)$ we have

$$\int_0^t (1 + m(t) - m(u))L(u)\frac{\alpha(m(u)))^{n-1}}{(n-1)!}du = \int_0^t (2 + m(t))L(u)\frac{(\alpha(m(u))^{n-1}}{(n-1)!}du -$$
$$\frac{(\alpha(m(t))^n}{n!} \leq (m(t) + 2)\frac{(m(t)/2 + 1)^{n-1}}{(n-1)!}\int_0^t (m(u))^{n-1}L(u)du - \frac{(\alpha(m(t)))^n}{n!}$$
$$= \frac{(\alpha(m(t)))^n}{n!}.$$

As above we deduce that

$$\int_r^t \frac{(\alpha(m(u) - m(r))))^{n-1}}{(n-1)!}(1 + m(t) - m(u))L(u)du \leq \frac{(\alpha(m(t) - m(r)))^n}{n!}$$

and inequality (3.9) is proved.

4. Continuous family of solutions

In order to establish our continuous version of Filippov theorem for problem (1.1) we need the following hypotheses.

Hypothesis 4.1. i) *S is a separable metric space and $a(.) : S \to \mathbf{R}$, $b(.) : S \to \mathbf{R}$, $c(.) : S \to (0, \infty)$ are continuous mappings.*

(ii) *There exists the continuous mappings $g(.) : S \to L^1(I, \mathbf{R})$, $q(.) : S \to \mathbf{R}$, $z(.) : S \to C(I, \mathbf{R})$, $w(.) : S \to C(I, \mathbf{R})$ such that*

$$D_C^q(w(s))(t) = g(s)(t), \quad p(t)(z(s))'(t) \equiv w(s)(t), \quad \forall s \in S, t \in I$$

and

$$d(g(s)(t), F(t, y(s)(t), V(y(s))(t)) \leq q(s)(t) \quad a.e.\ (I),\ \forall\ s \in S.$$

Theorem 4.2. *Assume that Hypotheses 3.1 and 4.1 are satisfied.*

Then there exist the continuous mappings $x(.) : S \to C(I, \mathbf{R})$, $f(.) : S \to L^1(I, \mathbf{R})$ such that for any $s \in S$, $(x(s)(.), f(s)(.))$ is a trajectory-selection pair of problem

$$D_C^q y(t) \in F(t, x(t), V(x)(t)), \quad p(t)x'(t) \equiv y(t), \quad x(0) = a(s), \quad x'(0) = b(s)$$

and

$$|x(s)(t) - z(s)(t)| \leq \xi(s)(t) \quad \forall (t, s) \in I \times S, \tag{4.1}$$

$$|f(s)(t) - g(s)(t)| \leq L(t)(\xi(s,t) + \int_0^t L(u)\xi(s,u)du) + q(s)(t) + c(s)\ a.e.\ (I), \tag{4.2}$$

$\forall s \in S$, where

$$\xi(s,t) = e^{M_1 \alpha(m(t))}[M_1 tc(s) + |a(s) - y(s)(0)| + MT|b(s)- (y(s))'(0)|)] + M_1 \int_0^t q(s)(u) e^{M_1 \alpha(m(t) - m(u))} du.$$

Proof. We denote $\varepsilon_n(s) = c(s)\frac{n+1}{n+2}$, $n = 0, 1, ...,$ $d(s) = |a(s) - z(s)(0)| + MT|b(s) - (z(s))'(0)|$,

$$q_n(s)(t) = (M_1)^n \int_0^t q(s)(u) \frac{(\alpha(m(t)-m(u)))^{n-1}}{(n-1)!} du + (M_1)^{n-1} \frac{(\alpha(m(t)))^{n-1}}{(n-1)!}(M_1 t\varepsilon_n(s) + d(s)), \quad n \geq 1.$$

Set also $x_0(s)(t) = z(s)(t)$, $f_0(s)(t) = g(s)(t)$, $\forall s \in S$.

We consider the set-valued maps $G_0(.), H_0(.)$ defined, respectively, by

$$G_0(s) = \{v \in L^1(I, \mathbf{R}); \quad v(t) \in F(t, z(s)(t), V(z(s))(t)) \quad a.e.\ (I)\},$$

$$H_0(s) = \text{cl}\{v \in G_0(s); \quad |v(t) - g(s)(t)| < q(s)(t) + \varepsilon_0(s)\}.$$

Since $d(g(s)(t), F(t, z(s)(t), V(z(s))(t)) \leq q(s)(t) < q(s)(t) + \varepsilon_0(s)$, according with Lemma 2.1, the set $H_0(s)$ is not empty.

Set $F_0^*(t,s) = F(t, z(s)(t), V(z(s))(t))$ and note that

$$d(0, F_0^*(t,s)) \leq |g(s)(t)| + q(s)(t) = q^*(s)(t)$$

and $q^*(.) : S \to L^1(I, \mathbf{R})$ is continuous.

Applying now Lemmas 2.2 and 2.3 we obtain the existence of a continuous selection f_0 of H_0, i.e. such that

$$f_0(s)(t) \in F(t, z(s)(t), V(z(s))(t)) \quad a.e.\ (I),\ \forall s \in S,$$

$$|f_0(s)(t) - g(s)(t)| \leq q_0(s)(t) = q(s)(t) + \varepsilon_0(s) \quad \forall s \in S,\ t \in I.$$

We define $x_1(s)(t) = a(s) + b(s)\int_0^t \frac{1}{p(u)}du + \int_0^t S(t,u)f_0(s)(u)du$ and one has

$$|x_1(s)(t) - x_0(s)(t)| \leq |a(s) - z(s)(0)| + MT|b(s) - (z(s))'(0)| + M_1\cdot \int_0^t |f_0(s)(u) - g(s)(u)|du \leq d(s) + M_1\int_0^t (q(s)(u) + \varepsilon_0(s))du = q_1(s)(t).$$

We shall construct two sequences of approximations $f_n(.) : S \to L^1(I, \mathbf{R})$, $x_n(.) : S \to C(I, \mathbf{R})$ with the following properties

a) $f_n(.) : S \to L^1(I, \mathbf{R})$, $x_n(.) : S \to C(I, \mathbf{R})$ are continuous.

b) $f_n(s)(t) \in F(t, x_n(s)(t), V(x_n(s))(t))$, a.e. (I), $s \in S$.

c) $|f_n(s)(t) - f_{n-1}(s)(t)| \leq L(t)(q_n(s)(t) + \int_0^t L(u)q_n(s)(u)du)$, a.e. (I), $s \in S$.

d) $x_{n+1}(s)(t) = a(s) + b(s)\int_0^t \frac{1}{p(u)}du + \int_0^t S(t,u)f_n(s)(u)du$, $\forall t \in I, s \in S$.

Suppose we have already constructed $f_i(.), x_i(.)$, $i = 1, ..., n$ satisfying a)-c) and define $x_{n+1}(.)$ as in d). As in the proof of inequality (3.9) we have

$$\int_0^t L(u)(q_n(s)(u) + \int_0^u L(r)q_n(s)(r)dr)du \leq q_{n+1}(s)(t) - \frac{c(s)(\alpha(m(t)))^n t}{(n+2)(n+3)n!}. \quad (4.3)$$

From c) and d) one has

$$|x_{n+1}(s)(t) - x_n(s)(t)| \leq M_1\int_0^t |f_n(s)(u) - f_{n-1}(s)(u)|du \leq M_1\int_0^t L(u)(q_n(s)(u) + \int_0^u L(r)q_n(s)(r)dr)du < q_{n+1}(s)(t). \quad (4.4)$$

Consider the following set-valued maps, for any $s \in S$,

$$G_{n+1}(s) = \{v \in L^1(I, \mathbf{R}); \quad v(t) \in F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t)) \quad a.e.\,(I)\},$$

$$H_{n+1}(s) = \mathrm{cl}\{v \in G_{n+1}(s); \quad |v(t) - f_n(s)(t)| < L(t)(q_n(s)(t) + \int_0^t L(u)q_n(s)(u)du) \quad a.e.\,(I)\}.$$

To prove that $H_{n+1}(s)$ is nonempty we note first that the real function $t \to r_n(s)(t) = c(s)\frac{(M_1)^{n+1}tL(t)(m(t))^n}{(n+2)(n+3)n!}$ is measurable and strictly positive for any s. From (4.3) we get

$$d(f_n(s)(t), F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t)) \leq L(t)(|x_n(s)(t) - x_{n+1}(s)(t)| + \int_0^t L(u)|x_n(s)(u) - x_{n+1}(s)(u)|du) \leq L(t)(q_n(s)(t) + \int_0^t L(u)q_n(s)(u)du) - r_n(s)(t)$$

and therefore according to Lemma 2.1 there exists $v(.) \in L^1(I, \mathbf{R})$ such that $v(t) \in F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t))$ a.e. (I) and

$$|v(t) - f_n(s)(t)| < d(f_n(s)(t), F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t)) + r_n(s)(t)$$

and hence $H_{n+1}(s)$ is not empty.

Set $F^*_{n+1}(t,s) = F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t))$ and note that we may write

$$d(0, F^*_{n+1}(t,s)) \leq |f_n(s)(t)| + L(t)(q_{n+1}(s)(t) + \int_0^t L(u)q_{n+1}(s)(u)du) = q^*_{n+1}(s)(t) \quad a.e.\,(I)$$

and $q^*_{n+1}(.) : S \to L^1(I, \mathbf{R})$ is continuous.

By Lemmas 2.2 and 2.3 there exists a continuous map $f_{n+1}(.) : S \to L^1(I, \mathbf{R})$ such that for any $s \in S$

$$f_{n+1}(s)(t) \in F(t, x_{n+1}(s)(t), V(x_{n+1}(s))(t)) \quad a.e.\,(I),$$

$$|f_{n+1}(s)(t) - f_n(s)(t)| \leq L(t)(q_{n+1}(s)(t) + \int_0^t L(u)q_{n+1}(s)(u)du) \quad a.e.\,(I).$$

From (4.4) and d) we obtain

$$|x_{n+1}(s)(.) - x_n(s)(.)|_C \le M_1|f_{n+1}(s)(.) - f_n(s)(.)|_1 \le \frac{(M_1\alpha(m(T)))^n}{n!}(M_1|q(s)(.)|_1 + M_1Tc(s) + d(s)). \tag{4.5}$$

Therefore $f_n(s)(.)$, $x_n(s)(.)$ are Cauchy sequences in the Banach space $L^1(I, \mathbf{R})$ and $C(I, \mathbf{R})$, respectively. Let $f(.) : S \to L^1(I, \mathbf{R})$, $x(.) : S \to C(I, \mathbf{R})$ be their limits. The function $s \to M_1|q(s)(.)|_1 + M_1Tc(s) + d(s)$ is continuous, hence locally bounded. Therefore (4.5) implies that for every $s' \in S$ the sequence $f_n(s')(.)$ satisfies the Cauchy condition uniformly with respect to s' on some neighborhood of s. Hence, $s \to f(s)(.)$ is continuous from S into $L^1(I, \mathbf{R})$.

From (4.5), as before, $x_n(s)(.)$ is Cauchy in $C(I, \mathbf{R})$ locally uniformly with respect to s. So, $s \to x(s)(.)$ is continuous from S into $C(I, \mathbf{R})$. On the other hand, since $x_n(s)(.)$ converges uniformly to $x(s)(.)$ and

$$d(f_n(s)(t), F(t, x(s)(t), V(x(s))(t)) \le L(t)(|x_n(s)(t) - x(s)(t)| + \int_0^t L(u)|x_n(s)(u) - x(s)(u)|du) \quad a.e.\ (I),\ \forall s \in S$$

passing to the limit along a subsequence of $f_n(.)$ converging pointwise to $f(.)$ we obtain

$$f(s)(t) \in F(t, x(s)(t), V(x(s))(t)) \quad a.e.\ (I),\ \forall s \in S.$$

Passing to the limit in d) we obtain

$$x(s)(t) = a(s) + b(s)\int_0^t \frac{1}{p(u)}du + \int_0^t S(t,u)f(s)(u)du.$$

By adding inequalities c) for all n and using the fact that $\sum_{i\ge 1} q_i(s)(t) \le \xi(s)(t)$ we obtain

$$|f_{n+1}(s)(t) - g(s)(t)| \le \sum_{l=0}^n |f_{l+1}(s)(u) - f_l(s)(u)| + |f_0(s)(t) - g(s)(t)| \le \sum_{l=0}^n L(t)q_{l+1}(s)(t) + q(s)(t) + \varepsilon_0(s) \le L(t)\xi(s)(t) + q(s)(t) + c(s). \tag{4.6}$$

Similarly, by adding (4.4) we get

$$|x_{n+1}(s)(t) - z(s)(t)| \le \sum_{l=0}^n q_l(s)(t) \le \xi(s)(t). \tag{4.7}$$

By passing to the limit in (4.6) and (4.7) we obtain (4.1) and (4.2), respectively.

Theorem 4.2 allows to obtain the next corollary which is a general result concerning continuous selections of the solution set of problem (1.1).

Hypothesis 4.3 *Hypothesis 3.1 is satisfied and there exists* $q_0(.) \in L^1(I, \mathbf{R}_+)$ *such that* $d(0, F(t, 0, V(0)(t))) \le q_0(t)$ *a.e.* (I).

Theorem 4.4. *Assume that Hypothesis 4.3 is satisfied.*
Then there exists a function $x(.,.) : I \times \mathbf{R}^2 \to \mathbf{R}$ *such that*
a) $x(., (\xi, \eta)) \in \mathcal{S}(\xi, \eta)$, $\forall (\xi, \eta) \in \mathbf{R}^2$.
b) $(\xi, \eta) \to x(., (\xi, \eta))$ *is continuous from* $\mathbf{R}^2$ *into* $C(I, \mathbf{R})$.

Proof. We take $S = \mathbf{R} \times \mathbf{R}$, $a(\xi, \eta) = \xi$, $b(\xi, \eta) = \eta$ $\forall (\xi, \eta) \in \mathbf{R} \times \mathbf{R}$, $c(.) : \mathbf{R} \times \mathbf{R} \to (0, \infty)$ an arbitrary continuous function, $g(.) = 0$, $z(.) = 0$, $q(\xi, \eta)(t) = q_0(t)$ $\forall (\xi, \eta) \in \mathbf{R} \times \mathbf{R}$, $t \in I$ and we apply Theorem 4.2 in order to obtain the conclusion of the theorem.

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