

APPROXIMATE CONTROLLABILITY OF AN IMPULSIVE NEUTRAL DIFFERENTIAL EQUATION WITH DEVIATING ARGUMENT AND BOUNDED DELAY

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ABSTRACT. In this paper we prove the approximate controllability of an impulsive neutral differential equation with deviated argument and control parameter included in the nonlinear term. We use Schuader fixed point theorem and fundamental assumptions on system operators to prove the result. Thereby we remove the need to assume the invertibility of a controllability operator, which fails to exist in infinite dimensional space, if the generated semigroup is compact. We also give an example to illustrate our result.

1. INTRODUCTION

Impulsive dynamical behaviour due to abrupt jumps at certain time instants in the evolution process is exhibited by various physical and biological systems. It is interesting to study impulsive control systems which is based on the theory of impulsive differential equations due to its theoretical and practical significance. The study of impulsive control systems in Banach spaces is stimulated by its numerous applications in nanoelectronics, pharmacokinetics, population dynamics, etc. Neutral differential equations are functional differential equations in which the highest order derivative of the unknown function appear both with and without deviations. We refer the papers of Benchohra et al. [11] and Chang [12] which discuss the exact controllability of impulsive functional systems with infinite delay. However, in these papers the invertibility of a controllability operator is assumed. As a consequence their approach fails in infinite dimensional spaces whenever the semigroup is compact. Also it is practically difficult to verify their condition directly. This is one of the motivations of our paper.

Although with a different approach Zhou [9] established approximate controllability of an abstract semilinear control system. Naito [1] proved that under range condition on the control action operator, the same semilinear system in [9] is approximately controllable. Mahmudov [10] established that approximate controllability of semilinear system is implied by approximate controllability of its linear part.

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In certain real world problems, delay depends not only on the time but also on the unknown quantity. The differential equations with deviated arguments are generalization of delay differential equations in which the unknown quantity and its derivative appear in different values of their arguments. It is interesting to note that approximate controllability problem for impulsive nonlinear dynamical systems with deviated argument has not been investigated thoroughly in literature. In an attempt to fill this gap we study the approximate controllability of the following control system using semigroup theory and fixed point approach which removes the above restrictions.

Our aim is to study the semilinear problem (1.1) with the nonlinear term containing the control parameter coupled with impulsive conditions since it was not studied before. This work is an extension of our previous works [13, 14, 15]. We study the nonlinear term with deviated argument and control parameter in a neutral differential equation with impulsive conditions and finite delay. In this paper we study the approximate controllability of the following problem

$$\begin{aligned} \frac{d[x(t) + g(t, x_t)]}{dt} &= A[x(t) + g(t, x_t)] + Bu(t) + f(t, x(a(x(t), t)), u(t)), t \in J \\ x_0(\theta) &= \phi(\theta), \theta \in [-r, 0] \\ x(t_k^+) - x(t_k^-) &= I_k(x(t_k)), k = 1, \dots, m, \end{aligned} \tag{1.1}$$

where A is the infinitesimal generator of C_0 semigroup $S(t)$. The state variable $x(\cdot)$ takes values in the Hilbert space X and the control parameter $u \in L_2(J, U)$, where U is a Hilbert space. B is a bounded linear operator from U to X . The function $x : [-r, 0] \cup J \rightarrow X$ is defined as $x_t(\theta) = x(t + \theta)$, $-r \leq \theta \leq 0$, $t \in J$. $D = t_1, t_2, \dots, t_m \subset J = [0, T]$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $I_k (k = 1, 2, \dots, m)$ is a nonlinear map and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$. $x(t_k^+), x(t_k^-)$ represents the jump in the state x at time t_k with I_k determining the size of the jump.

2. PRELIMINARIES

Now for convenience, let us introduce the notation

$$\begin{aligned} M &= \sup\{\|S(t)\| : 0 \leq t \leq T\}, M_1 = \|B\|, \|g(\cdot, \phi)\| \leq M_2 \\ \sup_{t \in J} \|S(t)(x_0 + g(0, \phi))\| &\leq M_0 \\ \|\lambda\| &= \int_0^b |\lambda_i(s)| ds. \\ k &= \max\{1, MM_1, MM_1T\}. \\ a_i &= 3kM^2M_1\|\lambda_i\|_1, b_i = 3M\|\lambda_i\|_1, c_i = \max\{a, b\} \\ d_1 &= 3kMM_1(\|x_T\| + M_0 + M_2 + M\Sigma_{k=1}^m d_k), \\ d_2 &= (3M\|x_0\| + M_2 + M\Sigma_{k=1}^m d_k) \\ d &= \max\{d_1, d_2\} \end{aligned}$$

Let us define the following operators:

$$\begin{aligned} \text{Let } \Gamma_0^T &= \int_0^T S(T-s)BB^*S^*(T-s)ds \\ \text{Let } R(\alpha, \Gamma_0^T) &= (\alpha I + \Gamma_0^T)^{-1} \end{aligned}$$

- (H1) The semigroup $S(t)$, $t > 0$, is compact.
- (H2) The function $f : J \times X \times U \rightarrow X$ is continuous and there exists function $\lambda(\cdot) \in L_1(I, R^+)$ and a non decreasing function $g_i \in L_1(C \times U, R^+)$, $i =$

1, 2, ..., q, such that

$$\|f(t, x, u)\| \leq \sum_{i=1}^q \lambda_i(t) g_i(x, u)$$

for all $(t, \phi, x) \in I \times X \times U$.

(H3) For each $\alpha > 0$

$$\limsup_{r \rightarrow \infty} (r - \sum_{i=1}^q \frac{c_i}{\alpha} \sup\{g_i(x, u) : \|(x, u)\| \leq r\}) = \infty$$

(H4) $I_k C(X, X)$ and there exists a constant d_k such that

$$\|I_k(x)\| \leq d_k$$

for each $x \in X (k = 1, 2, \dots, m)$.

(H5) $g : J \times X$ is completely continuous and uniformly bounded $\|g(\cdot, \phi)\| \leq M_2$.

(H6) The function $f : I \times X \times U \rightarrow X$ is continuous and there exists a constant $L > 0$ such that $\|f(\cdot, x(a(x(\cdot), \cdot)), u)\| \leq L$ for all $t \in I, u \in U$

(H7) $\alpha R(\alpha, \Gamma_0^T) \rightarrow 0$ as $\alpha \rightarrow 0^+$

(H8) $\|I_k(x(t_k)) - I_k(y(t_k))\| \leq L_I (\|x(t_k) - y(t_k)\|), \forall x(t_k), y(t_k) \in J_k = J\{t_1, t_2, \dots, t_m\}$.

(H9) $a : X \times J \rightarrow J$ such that $|a(x(s), s)| < s$.

Remark The assumption (H7) holds iff the following linear system is approximately controllable.

$$x'(t) = Ax(t) + (Bu)(t), t \in [0, T]$$

$$x(0) = x_0$$

Definition 1. We define the mild solution of (1.1) as $x(\cdot) \in PC(J, X)$, $x_0 = \phi$ and which satisfies the following integral equation

$$\begin{aligned} x(t) &= S(t)[\phi(0) + g(0, \phi)] + g(t, x_t) + \int_0^t S(t-s)[f(s, x(s), x(a(x(s), s)))] \\ &+ Bu(s)] ds + \sum_{0 < t_k < t} S(t-t_k) I_k x(t_k), \quad 0 \leq t \leq T \end{aligned} \quad (2.1)$$

For $\alpha > 0$ define an operator $F_\alpha(x, u) = (z, v)$ on $PC \times C(J, U)$ where $PC = PC([-r, T], X) = \{x : [-r, T] \rightarrow X : x_k \in C(J_k, X), k = 1, \dots, m\}$, where $J_k = J \{t_1, t_2, \dots, t_m\}$

$$v(t) = B^* T^* (T-t) R(\alpha, \Gamma_0^T) p(x, u) \quad (2.2)$$

$$\begin{aligned} z(t) &= S(t)(x_0 + g(0, \phi)) - g(t, x_t) + \int_0^t S(t-s)[f(s, x(a(x(s), s)), u) \\ &+ Bv(s)] ds + \sum_{0 < t_k < t} S(t-t_k) I_k(x(t_k)) \end{aligned} \quad (2.3)$$

$$\begin{aligned} p(x(\cdot)) &= x_T - S(T)(x_0 + g(0, \phi)) + g(T, x_T) \\ &- \int_0^T S(T-s) f(s, x(a(x(s), s)), u) - \sum_{k=1}^m S(T-t_k) I_k(x(t_k)) \end{aligned} \quad (2.4)$$

Let $Y_{r_0} = \{x(\cdot) \in PC([-r, b], X) \times C(J \times U) : \|x_t\| + \|v(t)\| \leq r_0\}$ and r_0 is a positive constant.

3. MAIN RESULT

Theorem 3.1. For arbitrary $h \in X$, the control

$$u(t) = B^*S^*(T-t)R(\alpha, \Gamma_0^T)p(x, u) \quad (3.1)$$

where

$$\begin{aligned} p(x, u) = h & - S(T)(x_0 + g(0, \phi(0))) + g(T, x_T) \\ & - \int_0^T S(T-s)f(s, x(a(x(s), s)), u(s))ds \end{aligned} \quad (3.2)$$

transfers initial state x_0 to

$$\begin{aligned} z_T = h & - \alpha(\alpha I + \Gamma_0^T)^{-1}(h - S(T)[g(0, \phi(0)) + \phi(0)] + g(T, x(T)) \\ & - \int_0^T S(T-r)f(r, x(a(x(r), r)), u(r))dr \end{aligned} \quad (3.3)$$

Proof. By substituting (2.2), (2.3) in the mild solution

$$\begin{aligned} z(t) & = S(t)(x_0 + g(0, \phi(0)) - g(t, x_t)) \\ & + \int_0^t S(t-s)[f(s, x(a(x(s), s)), u(s)) + Bv(s)]ds \end{aligned} \quad (3.4)$$

and writing the obtained equation at $t = T$

$$\begin{aligned} z(T) & = S(T)(x_0 + g(0, \phi(0)) - g(T, (x_T))) \\ & + \int_0^T S(T-s)[f(s, x(a(x(s), s)), u(s)) \\ & + BB^*S^*(T-s)(\alpha I + \Gamma_0^T)^{-1}p(x, u)]ds \\ & = S(T)(x_0 + g(0, \phi(0)) - g(T, x_T)) \\ & + \int_0^T S(T-s)f(s, x(a(x(s), s)), u(s))ds + \Gamma_0^T R p(x) \end{aligned}$$

Using $\Gamma_0^T(\alpha I + \Gamma_0^T)^{-1} = I - \alpha(\alpha I + \Gamma_0^T)^{-1}$ We get

$$\begin{aligned} z(t) & = S(T)(x_0 + g(0, \phi(0)) - g(T, x_T)) \\ & + \int_0^T S(T-s)f(s, x(a(x(s), s)), u(s))ds \\ & + p(x, u) - \alpha(\alpha I + \Gamma_0^T)^{-1}p(x, u) \\ & = h - p(x, u) + p(x, u) - \alpha(\alpha I + \Gamma_0^T)^{-1}p(x, u) \\ & = h - \alpha(\alpha I + \Gamma_0^T)^{-1}p(x, u) \end{aligned} \quad (3.5)$$

Thus the control $u(t) = B^*S^*(T-t)R(\alpha, \Gamma_0^T)p(x, u)$ transfers initial state x_0 to

$$\begin{aligned} z_T & = h - \alpha(\alpha I + \Gamma_0^T)^{-1}(h - S(T)[g(0, \phi(0)) + \phi(0)] \\ & - \int_0^T S(T-r)f(r, x(a(x(r), r)), u(r))dr] + g(T, x(T)). \end{aligned}$$

□

Theorem 3.2. Assume that hypotheses (H1) – (H9) hold, then for all $0 < \alpha \leq 1$ the system (1.1) has a solution on J .

Proof. Step1: For $0 < \alpha \leq 1$, there is a positive constant $r_0 = r_0(\alpha)$ such that $F^\alpha : Y_{r_0} \rightarrow Y_{r_0}$. Let

$$\mu_i(r) = \sup\{g_i(x, v) : \|(x, v)\| \leq r, (x, v) \in C \times U\}.$$

By the assumption (H3) there exists $r_0 > 0$ such that

$$\frac{d}{\alpha} + \sum_{i=1}^q \frac{c_i}{\alpha} \mu_i(r_0) \leq r_0$$

If $(x, y) \in Y_{r_0}$

$$\begin{aligned} \|v(t)\| &\leq \frac{1}{\alpha} [MM_1(\|x_T\| + M_0 + M_2) \\ &+ M \int_0^T \left\{ \sum_{i=1}^m \lambda_i(s) g_i(x(a(x(s), s)), u) \right\} ds + M \sum_{i=1}^m d_k] \\ &\leq \frac{1}{\alpha} MM_1(\|x_T\| + M_0 + M_2 + M \sum_{i=1}^m d_k) \\ &+ \frac{1}{\alpha} M^2 M_1 \sum_{i=1}^q \|\lambda\|_1 \mu_i(r_0) \\ &\leq \frac{1}{\alpha} \left[\frac{d}{3k} + \frac{1}{3k} \sum_{i=1}^q c_i \mu_i(r_0) \right] \\ &= \frac{1}{\alpha 3k} (d + \sum_{i=1}^q c_i \mu_i(r_0)) \\ &\leq \frac{r_0}{3k} \end{aligned} \tag{3.6}$$

$$\begin{aligned} \|z(t)\| &= \|S(t)(x_0 + g(0, \phi)) - g(t, x_t) \\ &+ \int_0^t S(t-s)[Bv(s) + f(s, x(a(x(s), s)), u)] ds \\ &+ \sum_{0 < t_k < t} S(t-t_k) I_k(x(t_k))\| \\ &\leq \frac{d}{3} + MM_1 T \|v\| + M \int_0^t \sum_{i=1}^q \lambda_i(s) g_i(x(a(x(s), s)), u(s)) ds \\ &\leq \frac{d}{3} + MM_1 T \|v\| + M \int_0^t \sum_{i=1}^q \lambda_i(s) g_i(x(s), s), u(s)) ds \\ &\leq \frac{d}{3} + k \|v\| + \frac{1}{3} \sum_{i=1}^q \mu_i(r_0) c_i \\ &\leq \frac{1}{3} \left[d + \sum_{i=1}^q c_i \mu_i(r_0) \right] + k \|v\| \\ &\leq \frac{2r_0}{3} \end{aligned} \tag{3.7}$$

We get

$$\|F^\alpha(x, u)(t)\| = \|z(t)\| + \|v(t)\| \leq r_0.$$

Therefore, F^α maps Y_{r_0} into itself.

Step 2 : As per infinite-dimensional version of Arzela-Ascoli theorem and step 1 we need to prove that:

(i) for arbitrary $t \in J$ the set $V(t) = \{F^\alpha(x, u)(t) : (x, u) \in Y_{r_0}\}$ is relatively compact,

(ii) for an arbitrary $\epsilon > 0$ there exists $\delta > 0$ such that $\|F^\alpha(x, u)(t_1) - F^\alpha(x, u)(t_2)\| < \epsilon$ if $(x, u) \in Y_{r_0}$, $|t_1 - t_2| \leq \delta$, for all $t_1, t_2 \in J$.

In the case, $t = 0$ it is trivial, since $V(0) = \phi(0)$. So let t be a fixed real no. and let be a given real number satisfying $0 < \tau < t$.

Define

$$\begin{aligned} F_\tau^\alpha(x, u)(t) &= S(\tau)F_1^\alpha(t - \tau), B^*S^*(T - s)R(\alpha, \Gamma_0^T)p(x, u) \\ F_1^\alpha(x, u)(t - \eta) &= S(t - \eta)(x_0 + g(0, x_0) + \int_0^{t-\eta} S(t - s - \eta)Bvds \\ &+ \int_0^{t-\eta} S(t - s - \eta)f(s, x(a(x(s), s)), u(s))ds \end{aligned}$$

Since $S(t)$ is compact and $z(t - \tau)$ is bounded on Y_{r_0} , the set

$$V_\tau(t) = \{F_\tau^\alpha(x, u)(t) + g(t, x_t) : (x, u) \in Y_{r_0}\}$$

is relatively compact in X . i.e. there exists a finite set $\{y_i, 1 \leq i \leq n\}$ in $PC \times U$ s.t.

$$V_\tau(t) \subset \cup_{i=1}^m B(y_i, \epsilon/2),$$

where $B(y_i, \epsilon/2)$ is an open ball in $PC \times U$ with centre at y_i and radius $\epsilon/2$. Also,

$$\begin{aligned} &\|(F_1^\alpha x)(t) - (F_\tau^\alpha x)(t)\| \\ &\leq \left\| \int_{t-\eta}^t S(t-s)BB^*S^*(T-s)(\alpha I + \Gamma_0^T)^{-1}p(x)ds \right. \\ &\quad \left. + \int_{t-\eta}^t S(t-s)f(s, x(a(x(s), s)), u(s))ds \right\| \\ &\leq \frac{1}{\alpha}M^2M_1^2P\eta + M\Sigma_{i=1}^q \int_0^t \lambda_i(s)ds\mu_i(r_0) \\ &\leq \epsilon/2 \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} P &= \|x_T\| + M\|x_0 + g(0, x_0)\| + M_2 + M \int_0^T \Sigma_{i=1}^q \lambda_i(s)\mu_i(r_0)ds \\ &\quad + \Sigma_{i=1}^q S(T - t_k)I_k(x(t_k)) \end{aligned}$$

So,

$$V(t) = \{F_1^\alpha(x, u) + g(t, x_t) : (x, u) \in Y_{r_0}\} \subset \cup_{i=1}^m B(y_i, \epsilon)$$

Hence for each $t \in [0, T]$, $V(t)$ is relatively compact in $C \times U$.

Step 3: We prove $V = \{F^\alpha(x, u)(\cdot) | (x, u) \in Y_{r_0}\}$ is equicontinuous on $[0, T]$. For $0 < t_a + \theta < t_b + \theta \leq T$

$$\begin{aligned}
\|v(t_a) - v(t_b)\| &\leq \|B^* S^*(T - t_a) - B^* S^*(T - t_b)\| \\
&\quad \times \frac{1}{\alpha} [\|x_T\| + M_0 + M_2 \\
&\quad + M \int_0^T (\sum_{i=1}^q \lambda_i(s) g_i(x(a(x(s), s)), u)) ds + M \sum_{i=1}^m d_k] \\
&\leq \|B^* S^*(T - t_a) - B^* S^*(T - t_2)\| \\
&\quad \times \frac{1}{\alpha} [\|x_T\| + M_0 + M_2 + M \sum_{i=1}^m d_k \\
&\quad + M^2 M_1 \sum_{i=1}^q \|\lambda\|_1 \mu_i(r_0)] \\
\|z_{t_b} - z_{t_a}\| &\leq \|(S(t_b + \theta) - S(t_a + \theta))(\phi(0) + g(0, x_0))\| \\
&\quad + M \int_0^{t_1 + \theta} \|[S(t_b + \theta - s) - S(t_a + \theta - s)] \\
&\quad \times \sum_{i=1}^q \lambda_i(s) g_i(x(a(x(s), s)), u(s))\| ds \\
&\quad + \int_{t_b + \theta}^{t_a + \theta} S(t_b + \theta - s) \sum_{i=1}^q \lambda_i(s) g_i(x(a(x(s), s)), u(s)) ds \\
&\quad + \int_0^{t_a + \theta} \|(S(t_b + \theta - s) - S(t_a + \theta - s)) B v(s)\| ds \\
&\quad + M M_1 \int_{t_a + \theta}^{t_b + \theta} \|v(s)\| ds + \int_{t_a + \theta}^{t_b + \theta} \|g(s, x_s)\| ds \\
&\quad + \sum_{0 < t_k < t_a + \theta} \|S(t_b + \theta - t_k) - S(t_a + \theta - t_k)\| \|I_k x(t_k)\| \\
&\quad + \sum_{t_a + \theta \leq t_k < t_b + \theta} \|S(t_b + \theta - t_k)\| \|I_k x(t_k)\|
\end{aligned}$$

(3.9)

$$\begin{aligned}
\|z(t_b + \theta) - z(t_a + \theta)\| &\leq \|S(t_a + \theta) - S(t_b + \theta)\| \|x_0 + g(0, x_0)\| \\
&\quad + \sum_{i=1}^q \int_0^{t_a + \theta} \|[S(t_a + \theta - s) - S(t_b + \theta - s)] \lambda_i(s) ds\| \mu_i(r_0) \\
&\quad + M \sum_{i=1}^q \int_{t_a + \theta}^{t_b + \theta} \lambda_i(s) ds \mu_i(r_0) \\
&\quad + M_1 \int_{t_a + \theta}^{t_b + \theta} \|S(t_a + \theta - s) - S(t_b + \theta - s)\| \|v(s)\| ds \\
&\quad + M M_1 \int_{t_a + \theta}^{t_b + \theta} \|v(s)\| ds + M_2 \int_{t_a + \theta}^{t_b + \theta} ds \\
&\quad + \sum_{0 < t_k < t_a + \theta} \|S(t_b + \theta - t_k) - S(t_a + \theta - t_k)\| d_k \\
&\quad + \sum_{t_a + \theta \leq t_k < t_b + \theta} \|S(t_b + \theta - t_k)\| d_k \\
&= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8
\end{aligned}$$

Thus RHS doesn't depend on particular choices of (x, u) . It is clear that $I_2 \rightarrow 0$, $I_4 \rightarrow 0$ and $I_6 \rightarrow 0$ as $t_1 - t_2 \rightarrow 0$. Since the semigroup $S(\cdot)$ is compact, so $\|S(t_2 + \theta - s) - S(t_1 + \theta - s)\| \rightarrow 0$ as $t_1 - t_2 \rightarrow 0$. Then $I_1 \rightarrow 0$ and by Lebesgue

Dominated Convergence theorem $I_3, I_5, I_7, I_8 \rightarrow 0$ as $t_1 - t_2 \rightarrow 0$. So $F_1^\alpha[Y_{r_0}]$ is equicontinuous and bdd. Equicontinuity of g follows from (H5). So $F^\alpha[Y_{r_0}]$ is equicontinuous and bdd. So equicontinuity of V is shown. By Arzela-Ascoli, $F^\alpha[Y_{r_0}]$ is relatively compact in $PC(J, X)$. To apply Schauder fixed point theorem it remains to show that F^α is continuous on $PC[J, X] \times C[J, U]$.

Let $(y^n(s), u^n(s)) \in PC \times U$ s.t. $(y^n(s), u^n(s)) \rightarrow (y(s), u(s))$ then for all

$$\{y^n(a(y^n(s), s)), u^n(s)\} \in PC(J, X) \times C(J, U)$$

$(y^n(a(y^n(s), s)), u^n(s)) \rightarrow (y(a(y(s), s)), u(s))$ in $PC(I, X) \times C(I, U)$. since

$$\begin{aligned} & \|y^n(a(y^n(s), s)) - y(a(y(s), s))\| \\ & \leq \|y^n(a(y^n(s), s)) - y(a(y^n(s), s))\| \\ & \quad + \|y(a(y^n(s), s)) - y(a(y(s), s))\| \\ & \leq \sup \|y^n - y\| + \|y(a(y(s), s)) - y(a(y(s), s))\| \quad (3.10) \\ & \rightarrow 0 \end{aligned}$$

as

$$n \rightarrow \infty.$$

Since f and g are continuous hence

$$f(s, y^n(a(y^n(s), s)), u^n(s)) \rightarrow f(s, y(a(y(s), s)), u(s))$$

and

$$g(s, y_s^n) \rightarrow g(s, y_s)$$

for each $s \in J$ and

$$\|f(s, y^n(a(y^n(s), s)), u^n(s)) - f(s, y(a(y(s), s)), u(s))\| \leq 2 \sum_{i=1}^q \lambda_i(s) \mu(r_0),$$

and

$$\|g(s, y_s^n) - g(s, y_s)\| \leq 2M_2$$

By Lebesgue Dominated Convergence theorem

$$\begin{aligned} & \|(F^\alpha(y^n))(t) - (F^\alpha y)(t)\| \\ & \leq \|v^n(t) - v(t)\| + \|z_t^n(\theta) - z_t(\theta)\| \\ & \leq \|B^* S^* R(\alpha, \Gamma_0^T)[p(y^n, u) - p(y, u)]\| \\ & \quad + \|g(t, y_t^n) - g(t, y_t)\| + \int_0^t S(t-s) \|Bv_n(s) - Bv(s)\| \\ & \quad + \|f(s, y^n(a(y^n(s), s)), u^n(s)) - f(s, y(a(y(s), s)), u(s))\| ds \quad (3.11) \\ & \quad + \sum_{0 < t_k < t} \|S(t-t_k)(I_k(y^n(t_k)) - I_k(y(t_k)))\| \\ & \leq a_1 \|g(s, y_s^n) - g(s, y_s)\| + a_2 \|f(s, y^n(a(y^n(s), s)), u^n(s)) \\ & \quad - f(s, y(a(y(s), s)), u(s))\| + mML_I \|y^n(t_k) - y(t_k)\| \\ & \rightarrow 0 \end{aligned}$$

where a_1, a_2 are appropriate constants. Hence F^α is a compact continuous operator on Y_{r_0} and from Schauder's fixed point theorem, F^α has a fixed point. \square

Theorem 3.3. Assume (H1), (H2), (H3) and (BA1) are satisfied then the system

$$\frac{d(x(t) + g(t, x(t)))}{dt} = A[x(t) + g(t, x(t))] + Bu(t) + f(t, x(t), x(a(x(t), t)), u(t)) \quad (3.12)$$

is approximately controllable on $[0, T]$

Proof. Let x^α be fixed point of F in Y_r where

$$(Fx)(t) = S(t)(x_0 + g(0, x_0)) - g(t, x(t)) + \int_0^t S(t-s)[f(s, x(a(x(s), s)), u(s)) + Bu(s)]ds \quad (3.13)$$

By previous theorem any fixed point of F is a mild solution of (3.12 on $[0, T]$ under the control

$$u^\alpha(t) = B^* S^*(T-t)R(\alpha, \Gamma_0^T)p(x^\alpha, u^\alpha)$$

and satisfies $x^\alpha(T) = h - \alpha R(\alpha, \Gamma_0^T)p(x, u)$.

By using hypothesis (H2) we get $\int_0^T \|f(s, x^\alpha(a(x^\alpha(s), s)), u^\alpha(s))\| ds \leq L^2 T$. Consequently, the sequence $f(s, x^\alpha(a(x^\alpha(s), s)), u^\alpha(s))$ is bounded in $L_2(J, X)$. Thus there are subsequences, still denoted by $f(s, x^\alpha(a(x^\alpha(s), s)), u^\alpha(s))$ that converge weakly to say $f(s, x(a(x(s), s)), u(s))$.

Define

$$q = h - S(T)(x_0 + g(0, x_0)) - g(T, x(T)) - \int_0^T S(T-s)f(s, x(a(x(s), s)), u(s))ds - \sum_{k=1}^m S(T-t_k)I_k(x(t_k)).$$

It follows that

$$\|p(x^\alpha) - q\| \leq \left\| \int_0^T S(T-s)[f(s, x^\alpha(a(x^\alpha(s), s)), u^\alpha(s)) - f(s, x(a(x(s), s)), u(s))]ds \right\| + \left\| \sum_{i=1}^m S(T-t_k)(I_k(x^\alpha(t_k)) - I_k(x(t_k))) \right\| \quad (3.14)$$

By the compactness of operators of the operators

$k(t) \rightarrow \int_0^t S(t-s)k(s)ds : L_2([0, T], X) \rightarrow C([0, T], X)$ and [H5] the RHS of (3.3) tends to 0 as $\alpha \rightarrow 0^+$ By (3.3)

$$\begin{aligned} \|x^\alpha(T) - h\| &= \|\alpha R(\alpha, \Gamma_0^T)p(x^\alpha, u^\alpha)\| \\ &= \|\alpha R(\alpha, \Gamma_0^T)(p(x^\alpha, u^\alpha) - q + q)\| \\ &\leq \|\alpha R(\alpha, \Gamma_0^T)q\| + \|\alpha R(\alpha, \Gamma_0^T)(p(x^\alpha, u^\alpha) - q)\| \\ &\leq \|\alpha R(\alpha, \Gamma_0^T)q\| + \|\alpha R(\alpha, \Gamma_0^T)\| \|p(x^\alpha, u^\alpha) - q\| \\ &\leq \|\alpha R(\alpha, \Gamma_0^T)q\| + \|p(x^\alpha, u^\alpha) - q\| \rightarrow 0 \end{aligned}$$

as $\alpha^+ \rightarrow 0$. This proves the approximate controllability of (1.1) \square

4. EXAMPLE

Let us consider the following controlled neutral system with impulses

$$\begin{aligned} \frac{\partial}{\partial t}[x(t, \xi) - \zeta(t, x(t-h, \xi))] &= \frac{\partial^2}{\partial \xi^2}[x(t, \xi) - \zeta(t, x(t-h, \xi))] + u(t, \xi) \\ &+ f(t, x(a(x(t, \xi), t), \xi), u(t, \xi)), 0 < y < 1 \\ x(t_k^+, \xi) - x(t_k^-, \xi) &= \bar{I}_k(x(t_k^-, \xi)), k = 1, \dots, m. \\ x(t, 0) &= x(t, 1) = 0, t > 0 \\ x(t, \xi) &= \phi(t, \xi), -h \leq t \leq 0; \end{aligned} \tag{4.1}$$

Here ϕ is continuous and $I_k \in C(\mathbb{R}, \mathbb{R})$.

Let $g(t, x_t)(\xi) = \zeta(t, x(t-h, \xi))$,

$$F(t, x(a(x(t), t), u(t)))(\xi) = f(t, x(a(x(t, \xi), t), \xi), u(t, \xi))$$

and $(Bu)(t)(\xi) = u(t, \xi)$, Taking $X = L_2(0, 1)$ and we define $A : X \rightarrow X$ by $Ax = \frac{d^2x}{d\xi^2}$ where domain of A is

$$\begin{aligned} D(A) = \{x \in X, x, \frac{dx}{d\xi} \text{ are absolutely continuous,} \\ \frac{d^2x}{d\xi^2} \in X, \frac{dx}{dy}(0) = \frac{dx}{dy}(1) = 0\} \end{aligned} \tag{4.2}$$

Then $Ax = \sum_{n=1}^{\infty} (-n^2\pi^2) \langle x, e_n \rangle e_n, x \in D(A)$.

where $e^n(\theta) = \sqrt{2}\cos(n\pi\theta) 0 < x < 1, n = 1, 2, \dots$

The operator A generates a compact semigroup

$$\begin{aligned} S(t)x &= \sum_{n=1}^{\infty} 2e^{-n^2\pi^2t} \cos(n\pi\xi) \int_0^1 \cos(n\pi\xi) \int_0^1 \cos(n\pi\xi)x(\psi)d\psi \\ &+ \int_0^1 x(\psi)d\psi, x \in X \end{aligned} \tag{4.3}$$

Further, the functions f, ζ are continuous and there exists constants k_1, k_2 such that $f(t, x(a(x(t, \xi), t), \xi), u(t, \xi)) \leq k_1, \zeta(t, x(t-h, \xi)) \leq k_2$ and there exists constants d_k such that $\|I_k(x)\| \leq d_k$.

Hence (4.1) can be expressed as (1.1) with A, g, I_k and F as defined above. The linear system corresponding to (4.1) is approximately controllable, and by Theorem 3.3, the system (4.1) is approximately controllable.

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