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CLASSIFICATION OF PHASE PORTRAITS OF LINEAR AND NONLINEAR FRACTIONAL ORDER SYSTEMS

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Abstract. In this paper, we study the equilibrium points classification of linear and nonlinear fractional order differential equations defined by differential operators of Caputo type. We consider both cases commensurate and incommensurate. We discuss about the qualitative type of phase portrait and stabiity of equilibrium points. Also, we attempt to generalize the Hartman-Grobman theorem, which is fundamental for the nonlinear dynamics of ordinary differential equations, to show that the dynamics of nonlinear fractional order systems are topologically equivalent to those of linear fractional systems locally. Simulation results are demonstrated for some examples of fractional order systems to illustrate the effectiveness of the analytical results.

1. Introduction

Fractional calculus is more than 300 years old, but it did not attract enough interest at the early stage of development [42]-[41]. An approach to geometric and physical interpretation of fractional integration and differentiation has been suggested by [37]. In the last three decades, fractional calculus has become popular among scientists in order to many systems in interdisciplinary fields can be described by the fractional differential equations, such as diffusion waves [20], nonlinear oscillation of earthquakes [18], viscoelastic material models [32], robotic manipulating systems [21], hydrologic models [7], wave propagation in nonlocal elastic continua [43], world economies models [46], gyros systems [2], energy supply-demand equations [1] and muscular blood vessel model [3]. Moreover, the fractional order equations are naturally related to systems with memory which exists in most biological systems. Also, they are closely related to fractals which are abundant in biological systems. Hence, fractional order equations are more suitable than integer order ones in biological, economic and social systems where memory effects are important [5]. Recent advances in fractional calculus have been reported in [28].

The solution of differential equations of fractional order has been much involved. Some analytical methods have been presented, such as the popular Laplace transform method [38, 40], the Fourier transform method [33], the iteration method

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[42], Green function method [44, 29] and the fractional functional variable method [27]. Numerical schemes for solving fractional differential equations have been introduced, for example, in [17, 35].

Recently, study on the dynamics of fractional order differential systems has greatly attracted the interest of many researchers. For instance, dynamics of the fractional order Hastings-Powell food chain model has been studied in [31]. The stability results of the fractional order differential equations systems have been a main goal in researches. For example, Matignon [30] considered the stability of fractional order differential equations system in control processing. Also, Ahmed et al. [6] considered the stability of fractional order predator-prey and rabies models. The stability of fractional order differential equations system with time delay and rational orders has been studied in [14, 36]. The stability in the sense of Lyapunov has also been studied by using Gronwall's lemma and Schwartz inequality [34]. In [25, 24], the Mittag-Leffler stability and the fractional Lyapunov's second method were proposed. At last, some stability and asymptotic stability results of nonlinear fractional order differential equations were proposed in [13, 8]. In [4] it has been shown that a limit cycle can be generated in the fractional order Wien bridge oscillator. Existence of a limit cycle for the fractional Brusselator has been shown in [48]. El-Saka et al [19] suggested some conditions on existence of Hopf bifurcation in fractional order dynamical systems. The pitchfork bifurcation and vibrational resonance in a fractional-order Duffing oscillator have been studied in [49]. Also, numerous fractional order chaotic systems have already been introduced and their chaotic behaviours have been investigated in detail, such as the fractional order Duffing system [22], fractional-order Chen system [11], fractional order memristor-based system [10], fractional order financial system [9], fractional order neural network [23] and so on. Also, some investigations are devoted to achieve chaos stabilization and synchronization in fractional order chaotic or hyperchaotic systems [26, 47].

In this paper, we study the phase portraits classification of linear and nonlinear fractional order differential equations. We consider both cases commensurate and incommensurate. We discuss about the qualitative type of phase portrait and stability of equilibrium points. Also, the Hartman-Grobman theorem is one of the most powerful tools used in dynamical systems. The Hartman-Grobman theorem allows us to represent the local phase portrait about certain types of equilibrium point in a nonlinear system of ordinary differential equations by a similar, simpler linear system that we can find by computing the system's Jacobian matrix at the equilibrium point. If this theorem is established for fractional order differential equations? In this paper, we will answer to this question.

The paper is organized as follows. In Section 2, we first recall some definitions and theorems used throughout the paper. In Section 3, the equilibrium points of linear fractional differential systems are classified. The equilibrium points classification of nonlinear fractional differential systems and generalization of the Hartman-Grobman theorem to fractional order dynamical systems are studied in Sections 4. Numerical simulations are given in Section 5. Conclusions are included in Section 6.

2. Preliminaries

In this section, we recall the most commonly used definitions and theorems of fractional order systems.

Definition 1 Let $\alpha \in \mathbb{R}^+$. The operator J_a^α, defined on $L_1[a,b]$ by

$$J_a^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt$$

for $a \leq x \leq b$, is called the Riemann-Liouville fractional integral operator of order α.

Definition 2 Let $\alpha \in \mathbb{R}^+$. The Caputo differential operator of order α is defined by

$${}_a^C D_t^\alpha f(t) := J_a^{m-\alpha} D^m f(t), \tag{1}$$

whenever $D^m f \in L_1[a,b]$ and $m := \lceil \alpha \rceil = \min\{z \in \mathbb{Z} : z \geq \alpha\}$.

Definition 3 Let $\alpha > 0$. The function E_α defined by

$$E_\alpha(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\alpha+1)},$$

whenever the series converges is called the Mittag-Leffler function of order α.

Theorem 1 Let $\alpha > 0$. The Mittag-Leffler function E_α behaves as follows:

(a) $E_\alpha(re^{i\phi}) \to 0$ for $r \to \infty$ if $\mid \phi \mid > \frac{\alpha\pi}{2}$,
(b) $E_\alpha(re^{i\phi})$ remains bounded for $r \to \infty$ if $\mid \phi \mid = \frac{\alpha\pi}{2}$,
(c) $\mid E_\alpha(re^{i\phi}) \mid \to \infty$ for $r \to \infty$ if $\mid \phi \mid < \frac{\alpha\pi}{2}$.

Proof. Refer to [15].

Consider the Caputo fractional autonomous system [39, 45]

$${}_{t_0}^C D_t^\alpha x(t) = f(x), \tag{2}$$

with initial condition $x(t_0)$, and $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n]$ indicates the fractional orders, i.e. $D^\alpha = [D^{\alpha_1}, D^{\alpha_2}, \ldots, D^{\alpha_n}]^T$ where $\alpha_i \in (0,1)$, $f : \Omega \to \mathbb{R}^n$ is locally Lipschitz in x on Ω, and $\Omega \in \mathbb{R}^n$ is a domain that contains the origin $x = 0$. If $\alpha_1 = \alpha_2 = \ldots = \alpha_n$, system (2) is called a commensurate order, otherwise system (2) indicates an incommensurate order system. The equilibrium point of system (2) is defined as follows:

Definition 4 The constant x_0 is an equilibrium point of Caputo fractional dynamic system (2), if and only if $f(x_0) = 0$.

Remark 1 When $\alpha \in (0,1)$, it follows from (1) that the Caputo fractional system (2) has the same equilibrium points as the integer order system $\dot{x}(t) = f(x)$.

Remark 2 For convenience,we state all definitions and theorems for the case when the equilibrium point is the origin of $\mathbb{R}^n$; i.e. $x_0 = 0$. There is no loss of generality in doing so because any equilibrium point can be shifted to the origin via a change of variables. Suppose the equilibrium point for (2) is $\bar{x} \neq 0$ and consider the change of variable $y = x - \bar{x}$. The αth order derivative of y is given by

$${}_{t_0}^C D_t^\alpha y = {}_{t_0}^C D_t^\alpha (x - \bar{x}) = f(x) = f(y + \bar{x}) = g(y),$$

where $g(0) = 0$ and in the new variable y, the system has equilibrium at the origin.

Remark 3 Until now, we considered the fractional derivative ${}_a^C D_t^\alpha$ with fixed lower terminal a and moving upper terminal t. For simplicity of notation, in the following we eliminate the left subscript a and the right subscript t. Thus, we use the notation ${}^{C}D^{\alpha}$ for Caputo fractional derivative.

Definition 5 The zero solution of fractional differential system (2) is said to be stable if, for any initial values $x_k \in \mathbb{R}^n$, there exists $\epsilon > 0$ such that any solution $x(t)$ of (2) satisfied $\| x(t) \| < \epsilon$ for all $t > t_0$. The zero solution is said to be asymptotically stable if, in addition to being stable, $\| x(t) \| \to 0$ as $t \to +\infty$.

Theorem 2 Let $\alpha > 0$, $m = \lceil \alpha \rceil$ and $\lambda \in \mathbb{R}$. The solution of the initial value problem

$$ {}^{C}D^{\alpha}x(t) = \lambda x(t) + q(t), \quad x^{(k)}(0) = x_0^{(k)}, \quad k = 0, 1, \ldots, m-1, \tag{3} $$

where $q \in C[0,h]$ is a given function, can be expressed in the form

$$ x(t) = \sum_{k=0}^{m-1} x_0^{(k)} u_k(t) + \tilde{x}(t) $$

with

$$ \tilde{x}(t) = \begin{cases} J_0^{\alpha} q(t) & \lambda = 0, \\ \\ \frac{1}{\lambda} \int_0^t q(t-\tau) u_0'(\tau) d\tau & \lambda \neq 0, \end{cases} $$

where $u_k(t) := J_0^k e_{\alpha}(t)$, $k = 0, 1, \ldots, m-1$, and $e_{\alpha}(t) := E_{\alpha}(\lambda t^{\alpha})$.

Proof. Refer to [15].

Remark 4 In the case $0 < \alpha < 1$, we can rewrite the solution of the initial value problem (3) in the form

$$ x(t) = x_0^{(0)} E_{\alpha}(\lambda t^{\alpha}) + \alpha \int_0^t q(t-\tau) \tau^{\alpha-1} E_{\alpha}'(\lambda \tau^{\alpha}) d\tau. $$

Theorem 3 Consider the commensurate fractional differential equation

$$ {}^{C}D^{\alpha}x(t) = \Lambda x(t) + q(t), $$

with $0 < \alpha < 1$, an $N \times N$ matrix Λ, a given function $q : [0,h] \to \mathbb{C}^N$ and an unknown solution $x : [0,h] \to \mathbb{C}^N$. For each k-fold eigenvalue λ of the matrix Λ we have k linearly independent solutions of the homogeneous linear differential equation ${}^{C}D^{\alpha}x(t) = \Lambda x(t)$ that can be represented in the form

$$ x_l(t) = \pi^{(l)}(t), \quad l = 1, 2, \ldots, k, $$

where the $\pi^{(l)}(t)$ are N-dimensional vectors whose component functions $\pi_j^{(l)}$, $j = 1, 2, \ldots, N$, are of the form

$$ \pi_j^{(l)}(t) = \sum_{\mu=0}^{l-1} c_j^{(l,\mu)} t^{\mu\alpha} D^{\mu} E_{\alpha}(\lambda t^{\alpha}), $$

where the vectors $c_j^{(l,\mu)}$ can be obtained as the eigenvectors and suitable multiples of the generalized eigenvectors of Λ.

The combination of these solutions for all eigenvalues leads to N linearly independent solutions of the system ${}^{C}D^{\alpha}x(t) = \Lambda x(t)$, i.e. to a basis of the space of all solutions of this system.

Proof. Refer to [15].

As it was noted in [15], in the context of single-order fractional differential equations whose order is greater than 1, we may rewrite the given an initial value problem of the form

$$^{C}D^{\alpha}x(t) = f(t, x(t)), \quad x^{(k)}(0) = x_0^{(k)}, \quad k = 0, 1, \ldots, n-1,$$

with some non-integer $\alpha > 1$, $n - 1 < \alpha \le n$ and n is integer, in the equvalent form

$$\begin{aligned} D^1 x_1(t) =& x_2(t), \\ D^1 x_2(t) =& x_3(t), \\ &\vdots \\ D^1 x_{n-1}(t) =& x_n(t), \\ {}^{C}D^{\beta} x_n(t) =& f(t, x_1(t)), \end{aligned}$$

where $\alpha = n - 1 + \beta$ and thus $0 < \beta \le 1$, with initial conditions

$$x_k(0) = x_0^{(k-1)}, \quad k = 1, 2, \ldots, n,$$

i.e. as multi-order system with orders less than or equal to 1. Therefore, without loss of generality, we consider fractional order systems of order $0 < \alpha \le 1$.

3. Linear fractional order systems

In this section, we consider the commensurate and incommensurate linear fractional order systems. Consider the following linear fractional order system

$$^{C}D^{\bar{\alpha}}x(t) = Ax(t), \tag{4}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$, matrix $A \in \mathbb{R}^{n\times n}$, $\bar{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$, $^{C}D^{\bar{\alpha}}x(t) = [^{C}D^{\alpha_1}x(t), ^{C}D^{\alpha_2}x(t), \ldots, ^{C}D^{\alpha_n}x(t)]^T$ and $0 < \alpha_i \le 1$, for $i = 1, 2, \ldots, n$. In particular, if $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha$, then fractional differential system (4) can be written as the following same order linear system

$$^{C}D^{\alpha}x(t) = Ax(t). \tag{5}$$

We can divide the system (4) or (5) into two systems: simple systems and non-simple systems.

3.1. Simple systems.

We first consider simple linear fractional order systems.

Definition 6 Linear fractional order system (4) or (5) is simple if $\det(A) \neq 0$.

For commensurate fractional order system (5) and $0 < \alpha \le 1$, Matignon firstly gave a well-known stability result by an algebraic approach combined with the use of asymptotic results, where the necessary and sufficient conditions have been derived, the specific result is as follows [30, 6].

Theorem 4 The autonomous same order system (5) with Caputo derivative and initial value $x_0 = x(0)$, where $0 < \alpha \le 1$, is

- asymptotically stable if and only if $|\arg(eig(A))| > \frac{\alpha\pi}{2}$. In this case the components of the state decay towards 0 like $t^{-\alpha}$.
- stable if and only if either it is asymptotically stable, or those critical eigenvalues which satisfy $|\arg(eig(A))| = \frac{\alpha\pi}{2}$ have geometric multiplicity one.

Proof. Refer to [30, 6].

The equilibrium points may be classified depending upon the type of egienvalues. Here, we consider the case in $\mathbb{R}^2$, the case for higher dimension will be the same.

Using linear algebra, the phase portrait of any linear fractional systems of the form (4) can be transformed to a canonical form $^{C}D^{\bar{\alpha}}y(t) = \mathbb{J}y(t)$ by applying a transformation $x = Py$, where P is to be determined and $\mathbb{J} = P^{-1}AP$ is of one of the following forms:

$$\begin{aligned} &a)\quad \mathbb{J}_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad b)\quad \mathbb{J}_2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \\ &c)\quad \mathbb{J}_3 = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}, \qquad d)\quad \mathbb{J}_4 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad b > 0; \end{aligned} \tag{6}$$

where $\lambda_{1,2}$, a and b are real constants and $\mathbb{J}$ is Jordan canonical form of the matrix A. Matrix $\mathbb{J}_1$ has two real distinct eigenvalues, matrices $\mathbb{J}_2$ and $\mathbb{J}_3$ have repeated eigenvalues, and matrix $\mathbb{J}_4$ has complex eigenvalues. The qualitative type of phase portrait is determind from each of these canonical forms. Now, we state the main theorem of this subsection.

Theorem 5 Let $0 < \alpha \le 1$, and $\bar{x}$ be the equilibrium point of the commensurate linear autonomous system (5). The type of phase portrait of system (5) depends on the eigenvalues of matrix A, λ_1 and λ_2, as summarized below:

I. If the eigenvalues are distinct, real, and they have the same signe, then the equilibrium point is a node (asymptotically stable if $\lambda_{1,2} < 0$; unstable if $\lambda_{1,2} > 0$).

II. If one eigenvalue is positive and the other negative, then the equilibrium point is a saddle point or col.

III. If the eigenvalues are equal and real, then the equilibrium point is a node (asymptotically stable if $\lambda < 0$; unstable if $\lambda > 0$).
- If there are two linearly independent eigenvectors, $\mathbb{J}$ is diagonal (i.e. (6(b))), then the equilibrium point is a star node.
- If there is one linearly independent eigenvector, $\mathbb{J}$ is not diagonal (i.e. (6(c))), then the equilibrium point is a improper node.

IV. If the eigenvalues are complex; $\lambda = a \pm ib$, and $a \ne 0$, then the equilibrium point is a focus (asymptotically stable if $|\arg(\lambda)| > \frac{\alpha\pi}{2}$; unstable if $|\arg(\lambda)| < \frac{\alpha\pi}{2}$).

V. If the eigenvalues are complex; $\lambda = a \pm ib$, and $a = 0$, then
- the equilibrium point is a center for $\alpha = 1$;
- the equilibrium point is a focus for $0 < \alpha < 1$ (asymptotically stable if $|\arg(\lambda)| > \frac{\alpha\pi}{2}$; unstable if $|\arg(\lambda)| < \frac{\alpha\pi}{2}$).

Proof. In the case $\alpha = 1$, we recover a standard result from the theory of ordinary differential equations (of integer order). Therefore, here we consider only the case $0 < \alpha < 1$.

I. Suppose matrix A has two real and distinct eigenvalues. In this case, $\mathbb{J}$ is given by (6(a)) with λ_1 and λ_2 non-zero. Thus the canonical form of system (5) is

$$\begin{aligned} {}^{C}D^{\alpha} y_1 &= \lambda_1 y_1, \\ {}^{C}D^{\alpha} y_2 &= \lambda_2 y_2. \end{aligned} \tag{7}$$

According to Remark 4, system (7) has the solution

$$\begin{aligned} y_1(t) &= y_1(0) E_\alpha(\lambda_1 t^\alpha), \\ y_2(t) &= y_2(0) E_\alpha(\lambda_2 t^\alpha). \end{aligned}$$

In Theorem 1, let $r = t \in \mathbb{R}$ and arguments of λ_1 and λ_2 correspond to ϕ. First, let $\lambda_i < 0$, $i = 1, 2$. Thus, $|\arg(\lambda_i)| = \pi$, $i = 1, 2$, and $\lim_{t\to\infty} E_\alpha(\lambda_i t^\alpha) \to 0$, $i = 1, 2$.

Therefore, $\lim_{t\to\infty} y_i(t) = 0$, $i = 1, 2$, and the equilibrium point is asymptptically stable.

If $\lambda_i > 0$, $i = 1, 2$, then $|\arg(\lambda_i)| = 0$, $i = 1, 2$, and $\lim_{t\to\infty} E_\alpha(\lambda_i t^\alpha) \to \infty$, $i = 1, 2$. Therefore, $\lim_{t\to\infty} y_i(t) = \infty$, $i = 1, 2$, and the equilibrium point is unstable.

The solution curves near the equilibrium point may be found by solving the differential equation given by

$$\frac{dy_2}{dy_1} = \frac{d(y_2(0)E_\alpha(\lambda_2 t^\alpha))}{d(y_1(0)E_\alpha(\lambda_1 t^\alpha))} = \frac{y_2(0)\lambda_2 E_\alpha(\lambda_2 t^\alpha)}{y_1(0)\lambda_1 E_\alpha(\lambda_1 t^\alpha)} = \frac{\lambda_2}{\lambda_1}\frac{y_2}{y_1}$$

which is integrable. The solution curves are given by $|\ y_2\ |^{\lambda_1} = k\ |\ y_1\ |^{\lambda_2}$ or $y_2 = K y_1^{\frac{\lambda_2}{\lambda_1}}$. The equilibrium point at the origin of the y_1y_2-plane is a node.

II. If λ_1 and λ_2 have opposite signs, without loss of generality, let $\lambda_1 > 0$ and $\lambda_2 < 0$. Then, $|\arg(\lambda_1)| = 0$ and $|\arg(\lambda_2)| = \pi$. Thus, $\lim_{t\to\infty} y_1(t) = \infty$ and $\lim_{t\to\infty} y_2(t) = 0$. The canonical form of system (5) is the same system (7).

The cordinate axes (excluding the origin) are the unions of special trajectories, called separatrices. These are only trajectories that are radial straight lines. A particular coordinate axis contains a pair of separatrices (remember the origin is a trajectory in its own right) which are directed towards (away from) the origin if the corresponding eigenvalue is negative (positive). The remaining trajectories, they are given by $y_2 = K y_1^{\frac{\lambda_2}{\lambda_1}}$, have the separatrices as asymptotes; first approaching the equilibrium point as t increases from $-\infty$, passing through a point of closet approach and finally moving away again. In this case the equilibrium point of origin is a saddle point.

III. Let $\lambda_1 = \lambda_2 = \lambda_0 \neq 0$. Then, the canonical matrices are of the form $\mathbb{J}_3$ or $\mathbb{J}_4$. If $\mathbb{J}$ is diagonal, the canonical system has solutions given by

$$\begin{aligned} y_1(t) &= y_1(0)E_\alpha(\lambda_0 t^\alpha), \\ y_2(t) &= y_2(0)E_\alpha(\lambda_0 t^\alpha). \end{aligned}$$

The solution curves near the origin can be obtained by solving the differential equation

$$\frac{dy_2}{dy_1} = \frac{y_2(0)}{y_1(0)} := k.$$

Therefore, the slope of solutions is constant and the non-trivial trajectories are all radial straight lines, $y_2(t) = k y_1(t)$. Thus, the equilibrium point is a star node.

If $\lambda_0 < 0$, then $\lim_{t\to\infty} y_i(t) = 0$; $i = 1, 2$, and the equilibrium point is asymptotically stable star node. If $\lambda_0 > 0$, then $\lim_{t\to\infty} y_i(t) = \infty$; $i = 1, 2$, and the equilibrium point is unstable star node.

Now, soppose $\mathbb{J}$ is not diagonal (i.e. (6(c))). Then, we must consider

$$\begin{aligned} {}^C D^\alpha y_1 &= \lambda_0 y_1 + y_2, \\ {}^C D^\alpha y_2 &= \lambda_0 y_2. \end{aligned}$$

Using Theorem 3, this system has solutions

$$\begin{aligned} Y(t) &= y_1(0)u^{(1)}E_\alpha(\lambda_0 t^\alpha) + y_2(0)[u^{(2,0)}E_\alpha(\lambda_0 t^\alpha) + u^{(2,1)}t^\alpha \lambda_0 E_\alpha(\lambda_0 t^\alpha)] \\ &= E_\alpha(\lambda_0 t^\alpha)[y_1(0)u^{(1)} + y_2(0)(u^{(2,0)} + u^{(2,1)}\lambda_0 t^\alpha)]. \end{aligned}$$

Thus, the origin is said to be an improper node (asymptotically stable $\lambda_0 < 0$; unstable $\lambda_0 > 0$).

IV. Let $\lambda_{1,2} = a \pm ib$. The canonical system is ${}^C D^\alpha y(t) = \mathbb{J}y(t)$, where $\mathbb{J}$ is given by (6(d)). Note that if $V = [v_1 \quad v_2]^T$ is an eigenvector associated to $a+ib$, then the vector $\bar{V} = [\bar{v}_1 \quad \bar{v}_2]^T$ (where $\bar{v}$ is the conjugate of v) is an eigenvector associated to $a - ib$. On the other hand, we know that $E_\alpha((a+ib)t^\alpha)$ and $E_\alpha((a-ib)t^\alpha)$ are solutions. We have

$$E_\alpha(\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{\alpha k}}{\Gamma(\alpha k+1)}.$$

On the other hand, if we write $\lambda = re^{i\theta}$, where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(\frac{b}{a})$, then $\lambda^k = r^k e^{ik\theta} = r^k(\cos k\theta + i \sin k\theta)$. Therefore, we can obtain

$$E_\alpha(\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{r^k t^{\alpha k}}{\Gamma(\alpha k+1)}(\cos k\theta + i \sin k\theta).$$

Note that these solutions are complex functions. In order to find real solutions, we set

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + iw_1 \\ u_2 + iw_2 \end{pmatrix} = U + iW.$$

Then, we have

$$\begin{aligned} E_\alpha((a+ib)t^\alpha)v_1 =& \sum_{k=0}^{\infty} \frac{r^k t^{\alpha k}}{\Gamma(\alpha k+1)}(\cos k\theta + i \sin k\theta)(u_1 + iw_1) \\ =& \sum_{k=0}^{\infty} \frac{r^k t^{\alpha k}}{\Gamma(\alpha k+1)}\{(u_1 \cos k\theta - w_1 \sin k\theta) \\ &+ i(w_1 \cos k\theta + u_1 \sin k\theta)\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} E_\alpha((a+ib)t^\alpha)v_2 =& \sum_{k=0}^{\infty} \frac{r^k t^{\alpha k}}{\Gamma(\alpha k+1)}\{(u_2 \cos k\theta - w_2 \sin k\theta) \\ &+ i(w_2 \cos k\theta + u_2 \sin k\theta)\}. \end{aligned}$$

Putting everything together we get

$$E_\alpha((a+ib)t^\alpha)V = \sum_{k=0}^{\infty} \frac{r^k t^{\alpha k}}{\Gamma(\alpha k+1)} \begin{pmatrix} (u_1 \cos k\theta - w_1 \sin k\theta) + i(w_1 \cos k\theta + u_1 \sin k\theta) \\ (u_2 \cos k\theta - w_2 \sin k\theta) + i(w_2 \cos k\theta + u_2 \sin k\theta) \end{pmatrix}$$

Clearly, this implies $E_\alpha((a+ib)t^\alpha)V = Y_1 + iY_2$ where

$$\begin{aligned} Y_1 &= \sum_{k=0}^{\infty} \frac{r^k t^{\alpha k}}{\Gamma(\alpha k+1)} \begin{pmatrix} u_1 \cos k\theta - w_1 \sin k\theta \\ u_2 \cos k\theta - w_2 \sin k\theta \end{pmatrix}, \\ Y_2 &= \sum_{k=0}^{\infty} \frac{r^k t^{\alpha k}}{\Gamma(\alpha k+1)} \begin{pmatrix} w_1 \cos k\theta + u_1 \sin k\theta \\ w_2 \cos k\theta + u_2 \sin k\theta \end{pmatrix}. \end{aligned}$$

It is easy to see that we have $E_\alpha((a-ib)t^\alpha)\bar{V} = Y_1 - iY_2$. If $Y_1 + iY_2$ is a complex solution of the system ${}^C D^\alpha y(t) = \mathbb{J}y(t)$, then its real and imaginary parts, Y_1 and

Y_2, are solutions to this system. These are real solutions. It is possible to show that they are linearly independent. Then, the general solution is

$$Y = c_1 Y_1 + c_2 Y_2,$$

where c_1 and c_2 are arbitrary constants. Therefore, we have

$$\begin{aligned} Y &= \sum_{k=0}^{\infty} \frac{r^k t^{\alpha k}}{\Gamma(\alpha k+1)} \left\{ c_1 \begin{pmatrix} u_1 \cos k\theta - w_1 \sin k\theta \\ u_2 \cos k\theta - w_2 \sin k\theta \end{pmatrix} + c_2 \begin{pmatrix} w_1 \cos k\theta + u_1 \sin k\theta \\ w_2 \cos k\theta + u_2 \sin k\theta \end{pmatrix} \right\} \\ &= \sum_{k=0}^{\infty} \frac{r^k t^{\alpha k}}{\Gamma(\alpha k+1)} \{ c_1 (U \cos k\theta - W \sin k\theta) + c_2 (W \cos k\theta + U \sin k\theta) \} \\ &= \sum_{k=0}^{\infty} \frac{r^k t^{\alpha k}}{\Gamma(\alpha k+1)} \begin{pmatrix} c_1 & c_2 \end{pmatrix} \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix} \begin{pmatrix} U \\ W \end{pmatrix}. \end{aligned}$$

Note that the matrix

$$R := \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}$$

represents a rotation through $k\theta$ radians. Therefore, the trajectories of (5) spiral in the equilibrium point if and only if $|\arg(\lambda)| > \frac{\alpha\pi}{2}$, and the trajectories of (5) spiral away from the origin if and only if $|\arg(\lambda)| < \frac{\alpha\pi}{2}$.

V. This case is special case of IV for $a = 0$.

Corollary 1 Let $0 < \alpha \leq 1$ and $\lambda_{1,2} = a \pm ib$. If $a < 0$, then the equilibrium point of system (5) is asymptotically stable.

Proof. When $a < 0$, we have $\frac{\pi}{2} < |\arg(\lambda)| < \pi$. On the other hand, since $0 < \alpha \leq 1$ then $0 < \frac{\alpha\pi}{2} \leq \frac{\pi}{2}$. Therefore, we have $|\arg(\lambda)| > \frac{\alpha\pi}{2}$ and the equilibrium point is asymptotically stable.

Corollary 2 Let $\lambda_{1,2} = \pm ib$. If $0 < \alpha < 1$, then the equilibrium point of system (5) is asymptotically stable and if $1 < \alpha < 2$, then the equilibrium point of system (5) is unstable.

Proof. When $\lambda_{1,2} = \pm ib$, then $|\arg(\lambda)| = \frac{\pi}{2}$. If $0 < \alpha < 1$ then $0 < \frac{\alpha\pi}{2} < \frac{\pi}{2}$. Therefore, we have $|\arg(\lambda)| > \frac{\alpha\pi}{2}$ and the equilibrium point is asymptotically stable. On the other hand, if $1 < \alpha < 2$ then $\frac{\pi}{2} < \frac{\alpha\pi}{2} < \pi$. Therefore, we have $|\arg(\lambda)| < \frac{\alpha\pi}{2}$ and the equilibrium point is unstable.

Now we consider incommensurate linear fractional order system (4). The above two theorems deal with the same order fractional differential system. For the multi-order linear fractional differential system, Deng et al. firstly studied the case that α_i's are rational numbers between 0 and 1, for $i = 1, 2, \ldots, n$, where the following result [14] was introduced.

Theorem 6 Consider incommensurate linear fractional order system (4) where α_i's are rational numbers between 0 and 1, for $i = 1, 2, \ldots, n$. Let M be the lowest commen multiple (LCM) of the denominators u_i of α_i's, where $\alpha_i = \frac{v_i}{u_i}$, the greatest common divisor of u_i and v_i is 1, i.e. $(u_i, v_i) = 1$, $u_i, v_i \in \mathbb{Z}^+$, $i = 1, 2, \ldots, n$ and set $\gamma = \frac{1}{M}$. Then the zero solution of system (4) with initial value $x_0 = x(0)$ is

- asymptotically stable if and only if any zero solution of the polynomial

$$\Delta(\lambda) = \det(diag(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, \ldots, \lambda^{M\alpha_n}) - A) \tag{8}$$

  satisfies $|\arg(\lambda)| > \frac{\gamma\pi}{2}$, the components of the state variable $(x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ decay towards 0 like $t^{-\alpha_1}, t^{-\alpha_2}, \ldots, t^{-\alpha_n}$, respectively.

- stable if and only if either it is asymptotically stable or those critical zero solutions λ of the polynomial (8) satisfy $|\arg(\lambda)| = \frac{\gamma\pi}{2}$ have geometric multiplicity one.

Proof. Refer to [14].

Theorem 7 Assume the hypotheses of Theorem 6. Moreover, assume $\bar{x}$ be the equilibrium point of the incommensurate linear system (4). The type of phase portrait of system (4) depends on the eigenvalues of matrix A, λ_1 and λ_2, as summarized below:

I. If the eigenvalues are distinct, real, and they have the same signe, then the equilibrium point is a node.

II. If one eigenvalue is positive and the other negative, then the equilibrium point is a saddle point or col.

III. If the eigenvalues are equal and real, then the equilibrium point is a node.
   - If there are two linearly independent eigenvectors, $\mathbb{J}$ is diagonal (i.e. (6(b))), then the equilibrium point is a star node.
   - If there is one linearly independent eigenvector, $\mathbb{J}$ is not diagonal (i.e. (6(c))), then the equilibrium point is a improper node.

IV. If the eigenvalues are complex; $\lambda = a \pm ib$, and $a \neq 0$, then the equilibrium point is a focus.

V. If the eigenvalues are complex; $\lambda = a \pm ib$, and $a = 0$, then
   - the equilibrium point is a center for $\alpha = 1$;
   - the equilibrium point is a focus for $0 < \alpha < 1$.

Proof. Note that since α_i's are not integer, $i = 1, \ldots, n$, thus we have $M \geq 2$ and hence $0 < \gamma \leq \frac{1}{2}$. The proof proceed along the same lines as the proof of Theorem 5. We omit the details.

3.2. **Non-simple systems.** Now, We consider non-simple linear fractional order systems.

Definition 7 Linear Fractional order system (4) is non-simple if the matrix A is singular (i.e. $\det(A) = 0$, and at least one of the eigenvalues is zero).

It follows that there are non-trival solutions to $Ax = 0$ and the system has equilibrium points other than $x = 0$. For linear systems in the plane, there are only two possibilities: either the rank of A is one; or A is null. In the first case there is a line of equilibrium points passing through the origin; in the second, every point in the plane is a equilibrium point. Of cours, the rank of J is equal to the rank of A, so that the canonical systems exhibit corresponding non-simple behavior.

4. Nonlinear fractional order systems

In this section, we consider the commensurate and incommensurate nonlinear fractional order systems. Consider the following nonlinear fractional order system

$$^{C}D^{\bar{\alpha}}x(t) = f(x(t)), \tag{9}$$

where f is a continuously differentiable and nonlinear function, $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$, $\bar{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$, $^{C}D^{\bar{\alpha}}x(t) = [^{C}D^{\alpha_1}x(t), ^{C}D^{\alpha_2}x(t), \ldots, ^{C}D^{\alpha_n}x(t)]^T$ and $0 < \alpha_i \leq 1$, for $i = 1, 2, \ldots, n$. Suppose that system (9) has an equilibrium point at $\bar{x}$. The linearized system is then of the form

$$^{C}D^{\bar{\alpha}}x(t) = J|_{\bar{x}}x(t), \tag{10}$$

where $J = \frac{\partial f}{\partial x}$ is the Jacobian matrix.

In particular, if $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha$, then fractional differential system (9) can be written as the following same order nonlinear system

$$ {}^{C}D^{\alpha}x(t) = f(x(t)), \tag{11}$$

and the linearized system (10) can be written as

$$ {}^{C}D^{\alpha}x(t) = J|_{\bar{x}}x(t). \tag{12}$$

Definition 8 An equilibrium point of nonlinear system (9) or (11) is called simple if its linearized system is simple.

Definition 9 An equilibrium point of nonlinear system (9) or (11) is called hyperbolic if the real part of the eigenvalues of the Jacobian matrix $J = \frac{\partial f}{\partial x}$ are nonzero. If the real part of either of the eigenvalues of the Jacobian are equal to zero, then the equilibrium point is called nonhyperbolic.

Theorem 8 Consider the commensurate fractional order system (11) with $0 < \alpha \leq 1$ and $x \in \mathbb{R}^n$. The equilibrium points of system (11) are calculated by solving the following equation: $f(x) = 0$. These points are locally asyptotically stable if all eigenvalues λ_i of the Jacobian matrix $J = \frac{\partial f}{\partial x}$ evaluated at the equilibrium points satisfy: $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$.

Proof. Refer to [6, 16].

Now, we state the main theorem of this section about commensurate nonlinear fractional order systems.

Theorem 9 Let $0 < \alpha \leq 1$, and $\bar{x}$ be the equilibrium point of the nonlinear commensurate system (11). The type of phase portrait of system (11) depends on the eigenvalues of Jacobian matrix J, λ_1 and λ_2, as summarized below:

I. If the eigenvalues are distinct, real, and they have the same signe, then the equilibrium point is a node (asymptotically stable if $\lambda_{1,2} < 0$; unstable if $\lambda_{1,2} > 0$).

II. If one eigenvalue is positive and the other negative, then the equilibrium point is a saddle point or col.

III. If the eigenvalues are equal and real, then the equilibrium point is a node (asymptotically stable if $\lambda < 0$; unstable if $\lambda > 0$).
   - If there are two linearly independent eigenvectors, $\mathbb{J}$ is diagonal (i.e. (6(b))), then the equilibrium point is a star node.
   - If there is one linearly independent eigenvector, $\mathbb{J}$ is not diagonal (i.e. (6(c))), then the equilibrium point is a improper node.

   (Note that here $\mathbb{J}$ is Jordan canonical form of the Jacobian matrix J.)

IV. If the eigenvalues are complex; $\lambda = a \pm ib$, and $a \neq 0$, then the equilibrium point is a focus (asymptotically stable if $|\arg(\lambda)| > \frac{\alpha\pi}{2}$; unstable if $|\arg(\lambda)| < \frac{\alpha\pi}{2}$).

V. If the eigenvalues are complex; $\lambda = a \pm ib$, and $a = 0$, then
   - the equilibrium point is either a center or a focus for $\alpha = 1$;
   - the equilibrium point is a focus for $0 < \alpha < 1$ (asymptotically stable if $|\arg(\lambda)| > \frac{\alpha\pi}{2}$; unstable if $|\arg(\lambda)| < \frac{\alpha\pi}{2}$).

Proof. This theorem can be proved by using Theorems 8 and 5, so it is omitted here.

Remark 5 Corollaries 1 and 2 are satisfied for commensurate nonlinear fractional order systems.

Now, we consider incommensurate nonlinear fractional order systems.

Theorem 10 Consider incommensurate nonlinear fractional order system (9) where α_i's are rational numbers between 0 and 1, for $i = 1, 2, \dots, n$. Let M be the lowest commen multiple (LCM) of the denominators u_i of α_i's, where $\alpha_i = \frac{v_i}{u_i}$, $(u_i, v_i) = 1$, $u_i, v_i \in \mathbb{Z}^+$, $i = 1, 2, \dots, n$ and set $\gamma = \frac{1}{M}$. Then the zero solution of system (9) with initial value $x_0 = x(0)$ is

- asymptotically stable if and only if any zero solution of the polynomial
$$\Delta(\lambda) = \det(diag(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, \dots, \lambda^{M\alpha_n}) - J) \tag{13}$$
satisfies $|\arg(\lambda)| > \frac{\gamma\pi}{2}$.
- stable if and only if either it is asymptotically stable or those critical zero solutions λ of the polynomial (13) satisfy $|\arg(\lambda)| = \frac{\gamma\pi}{2}$ have geometric multiplicity one.

Proof. Refer to [14].

Here, we state the main theorem of this section about incommensurate nonlinear fractional order systems.

Theorem 11 Assume the hypotheses of Theorem 10. Moreover, assume $\bar{x}$ be the equilibrium point of the nonlinear incommensurate system (9). The type of phase portrait of system (9) depends on the eigenvalues of Jacobian matrix J, λ_1 and λ_2, as summarized below:

I. If the eigenvalues are distinct, real, and they have the same signe, then the equilibrium point is a node.
II. If one eigenvalue is positive and the other negative, then the equilibrium point is a saddle point or col.
III. If the eigenvalues are equal and real, then the equilibrium point is a node.
   - If there are two linearly independent eigenvectors, $\mathbb{J}$ is diagonal (i.e. (6(b))), then the equilibrium point is a star node.
   - If there is one linearly independent eigenvector, $\mathbb{J}$ is not diagonal (i.e. (6(c))), then the equilibrium point is a improper node.

   (Note that here $\mathbb{J}$ is Jordan canonical form of the Jacobian matrix J.)
IV. If the eigenvalues are complex; $\lambda = a \pm ib$, and $a \neq 0$, then the equilibrium point is a focus.
V. If the eigenvalues are complex; $\lambda = a \pm ib$, and $a = 0$, then
   - the equilibrium point is either a center or a focus for $\alpha = 1$;
   - the equilibrium point is a focus for $0 < \alpha < 1$.

Proof. This theorem can be proved by using Theorems 10 and 6, so it is omitted here.

Remark 6 Since the type of phase portrait of commensurate and incommensurate fractional order systems is determined by the eigenvalues of Jacobian matrix J, the nature of the equilibrium point in commensurate and incommensurate fractional order systems is same.

Now, we state a generalization of the Hartman-Grobman theorem to show that the dynamics of nonlinear fractional order systems are topologically equivalent to those of linear fractional systems locally.

Theorem 12 Suppose that $\bar{x}$ is a simple equilibrium point of nonlinear fractional order system (9) or (11). Then there is a neighborhood of this equilibrium point on which the phase portrait for the nonlinear system resembles that of the linearized system (10) or (12).

Proof. Since the type of phase portrait of nonlinear commensurate or incommensurate fractional order system is determined by the eigenvalues of Jacobian matrix J, the nature of the equilibrium point of nonlinear fractional order system is the same as that for the linearized system. Therefore, the behavior of a nonlinear fractional dynamical system in a domain near a simple equilibrium point is qualitatively the same as the behavior of its linearization near this equilibrium point.

The above theorem tells us that, at least in a neighborhood of the simple equilibrium point, we can get a qualitative idea of the behaviour of solutions in the nonlinear system. Such qualitative characteristics we can glean include whether solution trajectories approach or move away from the equilibrium point over time, and whether the solutions spiral or if the equilibrium point acts as a node.

5. Numerical symulation

In this section we will apply the results established in the previous sections for the following fractional dynamical systems. In all of them, in the commensurate case we take $\alpha = \alpha_1 = \alpha_2 = 0.98$, and we do $\alpha = \alpha_1 = \alpha_2 = 0.6$ too. In the incommensurate case we set $(\alpha_1, \alpha_2) = (0.98, 0.88)$. Thus, we have $M = 100$ and $\gamma = 0.01$. We use the Adams-type predictor-corrector method for the numerical solution of fractional differential equation [17].

Example 1 Consider the fractional order system

$$\begin{cases} {}^C D^{\alpha_1} x_1(t) = 2x_1 + 2, \\ \\ {}^C D^{\alpha_2} x_2(t) = x_1^2 + x_2 + 2x_1. \end{cases} \tag{14}$$

This system has one equilibrium point at $(-1, 1)$. We obtain

$$J|_{(-1,1)} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have $\lambda_1 = 2$ and $\lambda_2 = 1$. Thus $(-1, 1)$ is a node for commensurate and incommensurate fractional order systems (14). The linearization at $(-1, 1)$ is

$$\begin{cases} {}^C D^{\alpha_1} x_1(t) = 2x_1, \\ \\ {}^C D^{\alpha_2} x_2(t) = x_2. \end{cases} \tag{15}$$

This system has one equilibrium point at the origin. Hence, $(0, 0)$ is a node for commensurate and incommensurate fractional order systems (15). In the commensurate case, since both eigenvalues of Jacobian matrix J are real and positive, the absolute value of the arguments of the eigenvalues are zero which are less than $\frac{\alpha\pi}{2}$, thus $(-1, 1)$ and $(0, 0)$ are unstable node as shown in Figures 1(a)-1(d). All of them are drawn with the same initial conditions. In the incommensurate case, the characteristic equation of the nonlinear system (14) and linearized system (15) evaluated at the equilibrium point $(-1, 1)$ is

$$\Delta(\lambda) = \det(diag(\lambda^{98}, \lambda^{88}) - J\mid_{(-1,1)}) = (\lambda^{98} - 2)(\lambda^{88} - 1) = 0.$$

Because $\min_i(\mid \arg(\lambda_i)\mid) = 0 < \frac{\gamma\pi}{2} = 0.01571$, thus incommensurate fractional order systems (14) and (15) exhibit an unstable node as shown in Figures 1(e) and 1(f), respectively.

Example 2 Consider the fractional order system

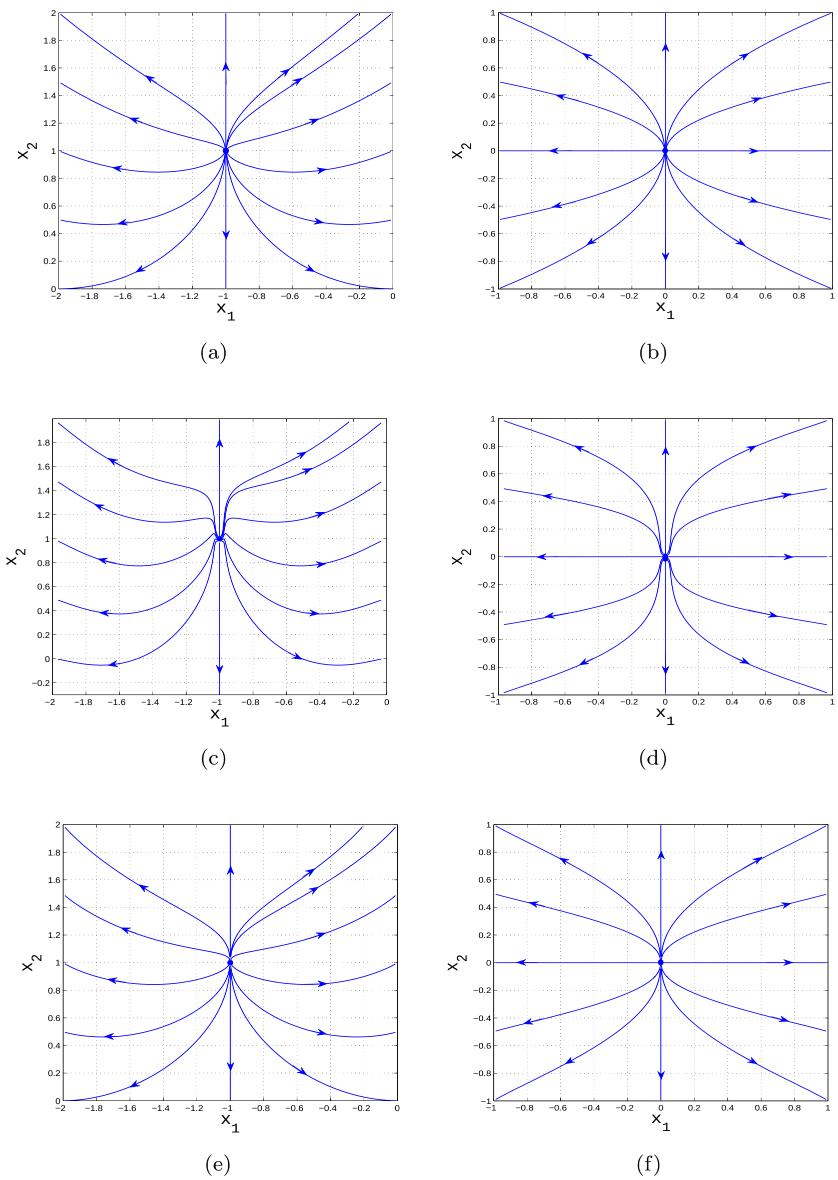


FIGURE 1. Phase portraits for: (a) nonlinear system (14) in commensurate case when $\alpha = 0.98$; (b) linearized system (15) in commensurate case when $\alpha = 0.98$; (c) system (14) in commensurate case when $\alpha = 0.6$; (d) system (15) in commensurate case when $\alpha = 0.6$; (e) system (14) in incommensurate case when $(\alpha_1, \alpha_2) = (0.98, 0.88)$ and (f) system (15) in incommensurate case when $(\alpha_1, \alpha_2) = (0.98, 0.88)$.

$$\begin{cases} {}^{C}D^{\alpha_1}x_1(t) = e^{x_1+x_2} - 1, \\ \\ {}^{C}D^{\alpha_2}x_2(t) = x_2. \end{cases} \tag{16}$$

The origin is the only equilibrium point. Linearize by finding the Jacobian matrix at this point; hence we have

$$\begin{cases} {}^{C}D^{\alpha_1}x_1(t) = x_1 + x_2, \\ \\ {}^{C}D^{\alpha_2}x_2(t) = x_2, \end{cases} \tag{17}$$

and

$$J|_{(0,0)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The eigenvalues are $\lambda_1 = \lambda_2 = 1$. Thus the origin is an improper node. In the commensurate case, since both eigenvalues of Jacobian matrix J are real and positive, the absolute value of the arguments of the eigenvalues are zero which are less than $\frac{\alpha\pi}{2}$, the origin is unstable. Figures 2(a)-2(d) show the unstable improper node for commensurate fractional order systems (16) and (17) with the same initial conditions.

In the incommensurate case, the characteristic equation of the nonlinear system (16) and linearized system (17) evaluated at the equilibrium point $(0,0)$ is

$$\Delta(\lambda) = \det(diag(\lambda^{98}, \lambda^{88}) - J\mid_{(0,0)}) = (\lambda^{98} - 1)(\lambda^{88} - 1) = 0.$$

Because $\min_i(\mid \arg(\lambda_i) \mid) = 0 < \frac{\gamma\pi}{2} = 0.01571$, thus incommensurate fractional order systems (16) and (17) exhibit an unstable improper node which is shown in Figures 2(e) and 2(f), respectively.

Example 3 Consider the fractional order system

$$\begin{cases} {}^{C}D^{\alpha_1}x_1(t) = x_1 + x_1x_2^2, \\ \\ {}^{C}D^{\alpha_2}x_2(t) = x_2 + x_2^5. \end{cases} \tag{18}$$

The origin is the only equilibrium point. The Jacobian matrix evaluated at this point is

$$J|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus the linearized system at origin is

$$\begin{cases} {}^{C}D^{\alpha_1}x_1(t) = x_1, \\ \\ {}^{C}D^{\alpha_2}x_2(t) = x_2. \end{cases} \tag{19}$$

The eigenvalues are $\lambda_1 = \lambda_2 = 1$. Thus the origin is a star node. In the commensurate case, since both eigenvalues of Jacobian matrix J are real and positive, the absolute value of the arguments of the eigenvalues are zero which are less than $\frac{\alpha\pi}{2}$, the origin is unstable. Figures 3(a)-3(d) show the unstable star node for commensurate fractional order systems (18) and (19) with the same initial conditions.

In the incommensurate case, the characteristic equation of the nonlinear system (18) and linearized system (19) evaluated at the equilibrium point $(0,0)$ is

$$\Delta(\lambda) = \det(diag(\lambda^{98}, \lambda^{88}) - J\mid_{(0,0)}) = (\lambda^{98} - 1)(\lambda^{88} - 1) = 0.$$

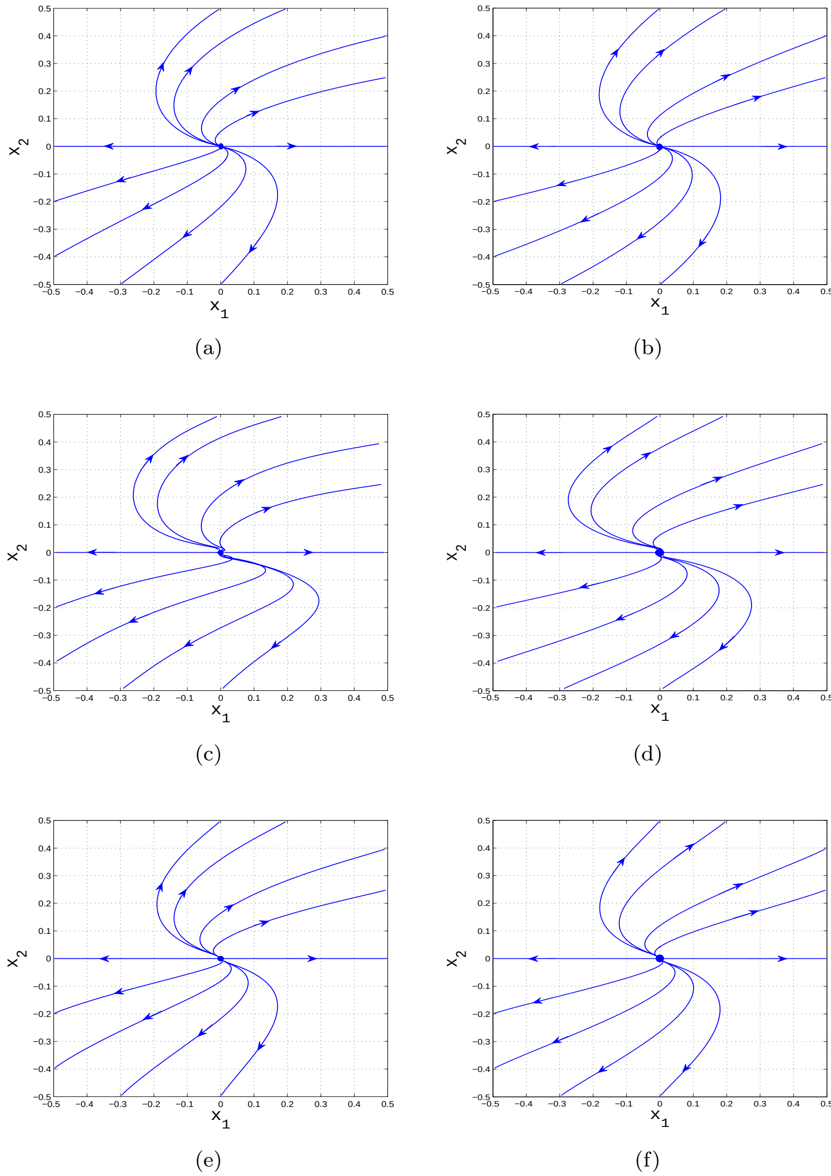


FIGURE 2. Phase portraits for: (a) nonlinear system (16) in commensurate case when $\alpha = 0.98$; (b) linearized system (17) in commensurate case when $\alpha = 0.98$; (c) system (16) in commensurate case when $\alpha = 0.6$; (d) system (17) in commensurate case when $\alpha = 0.6$; (e) system (16) in incommensurate case when $(\alpha_1, \alpha_2) = (0.98, 0.88)$ and (f) system (17) in incommensurate case when $(\alpha_1, \alpha_2) = (0.98, 0.88)$.

Because $\min_i(|\arg(\lambda_i)|) = 0 < \frac{\gamma\pi}{2} = 0.01571$, thus incommensurate fractional order systems (18) and (19) exhibit an unstable star node which is shown in Figures 3(e) and 3(f), respectively.

Example 4 Consider the fractional order system

$$\begin{cases} {}^{C}D^{\alpha_1}x_1(t) = x_1 + 4x_2 + e^{x_1} - 1, \\ \\ {}^{C}D^{\alpha_2}x_2(t) = -x_2 - x_2e^{x_1}. \end{cases} \tag{20}$$

The origin is the only equilibrium point. The Jacobian matrix evaluated at this point is

$$J|_{(0,0)} = \begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix}.$$

The linearized system is then of the form

$$\begin{cases} {}^{C}D^{\alpha_1}x_1(t) = 2x_1 + 4x_2, \\ \\ {}^{C}D^{\alpha_2}x_2(t) = -2x_2. \end{cases} \tag{21}$$

The Jacobian matrix has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -2$. Since one eigenvalues is real and positive and the other is real and negative, the equilibrium point at the origin is a saddle point. Phase portraits for commensurate fractional order systems (20) and (21) are plotted in Figures 4(a)-4(d) when $\alpha = 0.98$ and $\alpha = 0.6$.

In the incommensurate case, the characteristic equation of the systems (20) and (21) evaluated at the equilibrium point $(0,0)$ is

$$\Delta(\lambda) = \det(diag(\lambda^{98}, \lambda^{88}) - J\mid_{(0,0)}) = (\lambda^{98} - 2)(\lambda^{88} + 2) = 0.$$

Because $\min_i(|\arg(\lambda_i)|) = 0 < \frac{\gamma\pi}{2} = 0.01571$, thus incommensurate fractional order systems (20) and (21) exhibit an unstable saddle as shown in Figures 4(e) and 4(f), respectively.

Example 5 Consider the fractional order system

$$\begin{cases} {}^{C}D^{\alpha_1}x_1(t) = x_2, \\ \\ {}^{C}D^{\alpha_2}x_2(t) = -(1 + x_1^2 + x_1^4)x_2 - x_1. \end{cases} \tag{22}$$

The origin is a unique equilibrium point. The Jacobian matrix is given by

$$J|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Therefore the linearized system at $(0,0)$ is

$$\begin{cases} {}^{C}D^{\alpha_1}x_1(t) = x_2, \\ \\ {}^{C}D^{\alpha_2}x_2(t) = -x_1 - x_2. \end{cases} \tag{23}$$

The eigenvalues are $\lambda_{1,2} = \dfrac{-1 \pm i\sqrt{3}}{2}$ and real part of both of them is negative. Thus, equilibrium point at the origin is a asymptotically stable focus in the commensurate and incommensurate cases. Note that the absolute value of the argument λ_1 and λ_2 is equal to $\frac{2\pi}{3}$ which is more than $\frac{\alpha\pi}{2}$ with $\alpha = 0.98$ and $\alpha = 0.6$. Figures 5(a)-5(d) show the asymptotically stable focus for commensurate nonlinear system (22) and commensurate linearized system (23) with the same initial conditions $(x_1(0), x_2(0)) = (1,0)$.

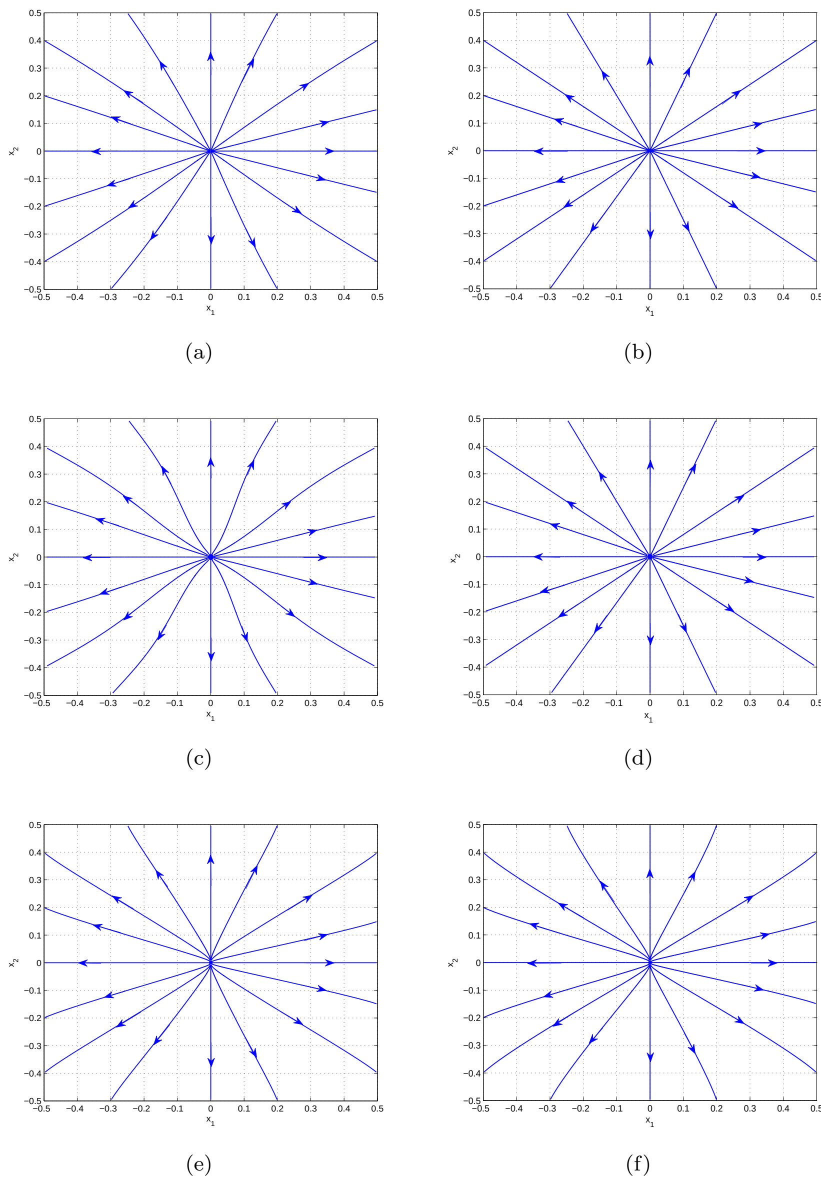


FIGURE 3. Phase portraits for: (a) nonlinear system (18) in commensurate case when $\alpha = 0.98$; (b) linearized system (19) in commensurate case when $\alpha = 0.98$; (c) system (18) in commensurate case when $\alpha = 0.6$; (d) system (19) in commensurate case when $\alpha = 0.6$; (e) system (18) in incommensurate case when $(\alpha_1, \alpha_2) = (0.98, 0.88)$ and (f) system (19) in incommensurate case when $(\alpha_1, \alpha_2) = (0.98, 0.88)$.

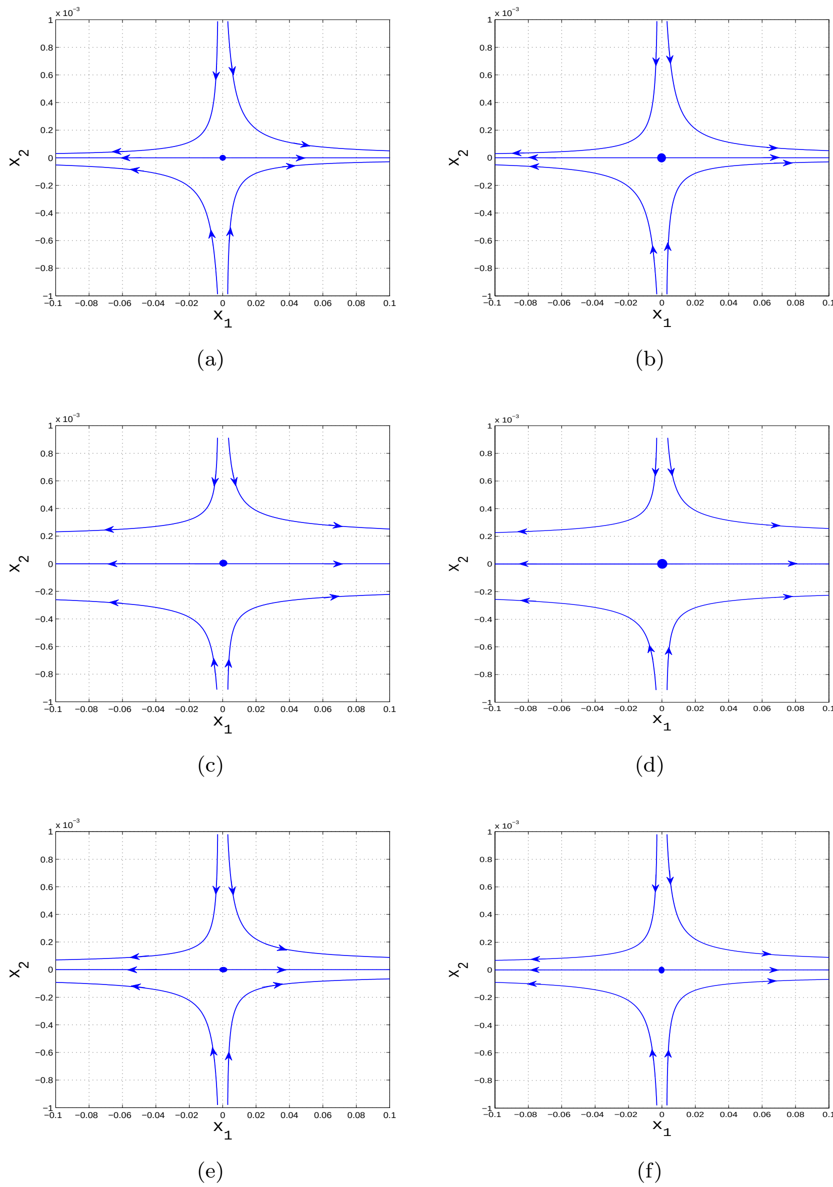


FIGURE 4. Phase portraits for: (a) nonlinear system (20) in commensurate case when $\alpha = 0.98$; (b) linearized system (21) in commensurate case when $\alpha = 0.98$; (c) system (20) in commensurate case when $\alpha = 0.6$; (d) system (21) in commensurate case when $\alpha = 0.6$; (e) system (20) in incommensurate case when $(\alpha_1, \alpha_2) = (0.98, 0.88)$ and (f) system (21) in incommensurate case when $(\alpha_1, \alpha_2) = (0.98, 0.88)$.

In the incommensurate case, the characteristic equation of the systems (22) and (23) evaluated at the equilibrium point $(0,0)$ is

$$\Delta(\lambda) = \det(diag(\lambda^{98}, \lambda^{88}) - J \mid_{(0,0)}) = \lambda^{186} + \lambda^{98} + 1 = 0.$$

Because $\min_i(\mid \arg(\lambda_i) \mid) = 0.02245 > \frac{\gamma\pi}{2} = 0.01571$, thus incommensurate fractional order nonlinear system (22) and incommensurate fractional order linearized system (23) exhibit a asymptotically stable focus as shown in Figures 5(e) and 5(f), respectively, in which the initial states are $(x_1(0), x_2(0)) = (1, 0)$.

Example 6 Consider two systems

$$^{C}D^{\alpha_1} x_1(t) = -x_2 + x_1(x_1^2 + x_2^2), \quad ^{C}D^{\alpha_2} x_2(t) = x_1 + x_2(x_1^2 + x_2^2), \tag{24}$$

and

$$^{C}D^{\alpha_1} x_1(t) = -x_2 - x_1(x_1^2 + x_2^2), \quad ^{C}D^{\alpha_2} x_2(t) = x_1 - x_2(x_1^2 + x_2^2). \tag{25}$$

Both systems have the origin as only equilibrium point and they have the linearization

$$^{C}D^{\alpha_1} x_1(t) = -x_2, \quad ^{C}D^{\alpha_2} x_2(t) = x_1. \tag{26}$$

Therefore, we have

$$J|_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and $\lambda_{1,2} = \pm i$. The real parts of eigenvalues are zero. Thus, equilibrium point at the origin of systems (24), (25) and linearized system (26) is a focus in commensurate and incommensurate cases. Note that the absolute value of the argument λ_1 and λ_2 is equal to $\frac{\pi}{2}$ which is more than $\frac{\alpha\pi}{2}$ with $\alpha = 0.98$ and $\alpha = 0.6$. Therefore, the origin of systems (24), (25) and linearized system (26) is asymptotically stable in commensurate case. Figures 6 and 7 show the asymptotically stable focus for systems (24), (25) and linearized system (26) in commensurate case with the initial states $(x_1(0), x_2(0)) = (0.001, 0)$ when $\alpha = 0.98$ and $\alpha = 0.6$, respectively.

In the incommensurate case, the characteristic equation of the systems (24), (25) and linearized system (26) evaluated at the equilibrium point $(0,0)$ is $\Delta(\lambda) = \lambda^{186} + 1 = 0$. Because

$$\min_i(\mid \arg(\lambda_i) \mid) = 0.01689 > \frac{\gamma\pi}{2} = 0.01571,$$

thus incommensurate fractional order systems (24), (25) and linearized system (26) exhibit a asymptotically stable focus as shown in Figure 8, in which the initial conditions are $(x_1(0), x_2(0)) = (0.001, 0)$.

Now we consider systems (24) and (25) with $\alpha = 1$. Thus, we have

$$\dot{x}_1 = -x_2 + x_1(x_1^2 + x_2^2), \quad \dot{x}_2 = x_1 + x_2(x_1^2 + x_2^2), \tag{27}$$

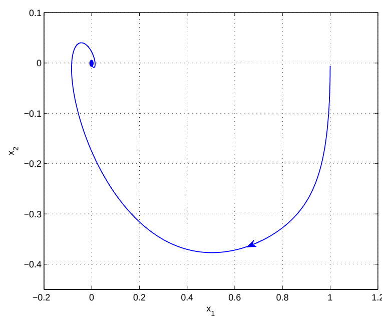
$$\dot{x}_1 = -x_2 - x_1(x_1^2 + x_2^2), \quad \dot{x}_2 = x_1 - x_2(x_1^2 + x_2^2). \tag{28}$$

Both systems (27) and (28) have the linearization

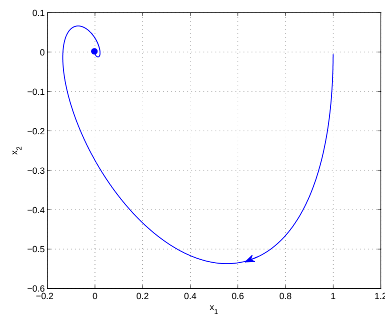
$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1, \tag{29}$$

which has a center at origin as shown in Figure 9(a). In polar coordinates system (27) becomes

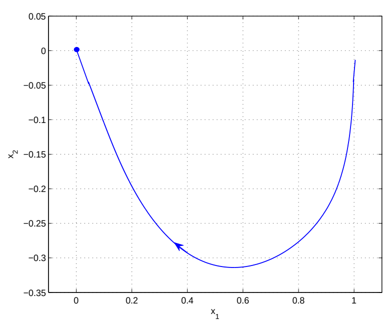
$$\dot{r} = r^3, \quad \dot{\theta} = 1, \tag{30}$$



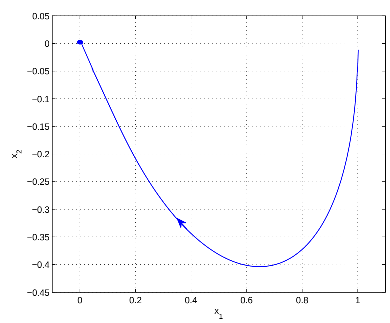
(a)



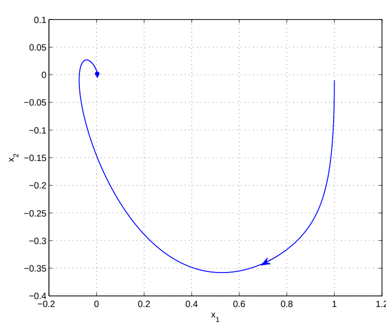
(b)



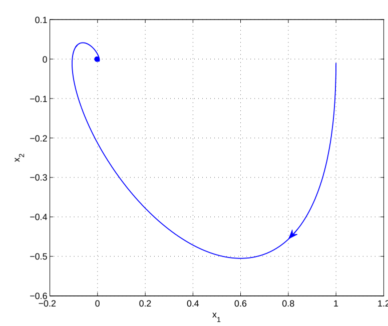
(c)



(d)

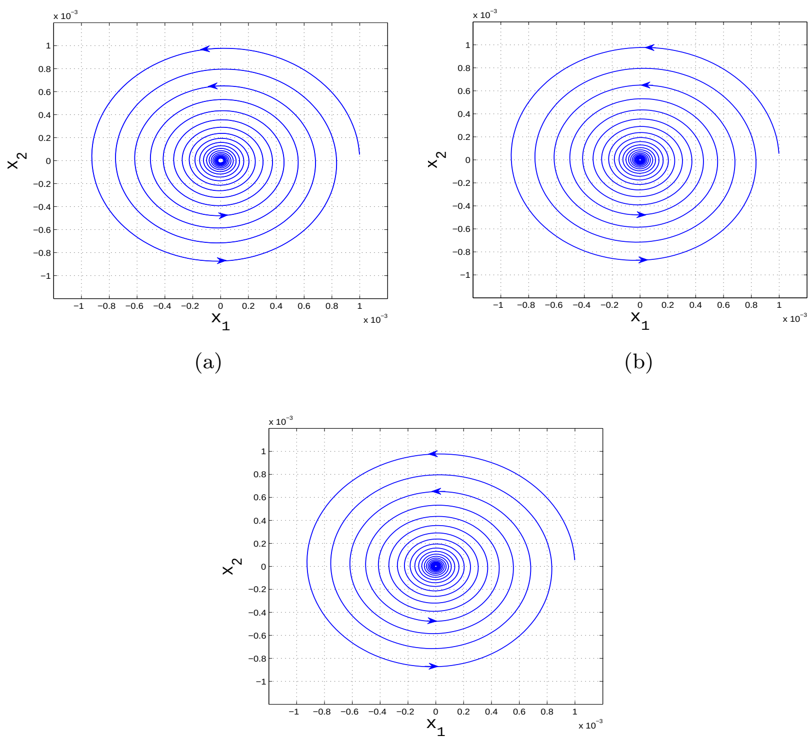


(e)



(f)

FIGURE 5. Phase portraits for: (a) nonlinear system (22) in commensurate case when $\alpha = 0.98$; (b) linearized system (23) in commensurate case when $\alpha = 0.98$; (c) system (22) in commensurate case when $\alpha = 0.6$; (d) system (23) in commensurate case when $\alpha = 0.6$; (e) system (22) in incommensurate case when $(\alpha_1, \alpha_2) = (0.98, 0.88)$ and (f) system (23) in incommensurate case when $(\alpha_1, \alpha_2) = (0.98, 0.88)$.



(a) (b)

(c)

FIGURE 6. Phase portraits for systems (a) (24); (b) (25) and (c) (26) in commensurate case when $\alpha = 0.98$.

while system (28) gives

$$\dot{r} = -r^3, \quad \dot{\theta} = 1. \tag{31}$$

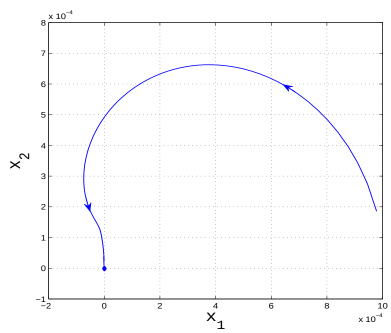
Equation (30) shows that $\dot{r} > 0$ for all $r > 0$ and so the trajectories of system (27) spiral outwards as t increases. On the other hands, equation (31) shows that $\dot{r} < 0$ for all $r > 0$ and so the trajectories of system (28) spiral inwards as shown in Figures 9(b) and 9(c) with the initial states $(x_1(0), x_2(0)) = (0.12, 0)$ and $(x_1(0), x_2(0)) = (0.4, 0)$, respectively. Thus system (27) shows an unstable focus while system (28) is an asymptotically stable focus.

Hence, If we consider systems (24), (25) and linearized system (26) with $\alpha = 1$, then the equilibrium point at the origin of these systems is an unstable focus, an asymptotically stable focus and a centre, respectively.

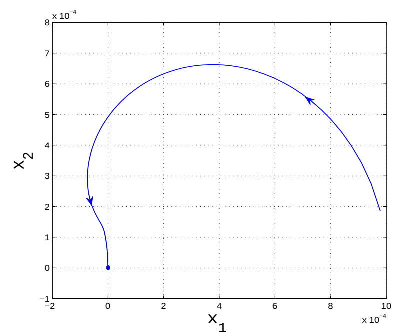
This example shows the difference in qualitative behaviour of fractional order and ordinary differential equations. In the case of fractional order, it is possible to change the nature and stability of equilibrium point.

Example7 Consider the system

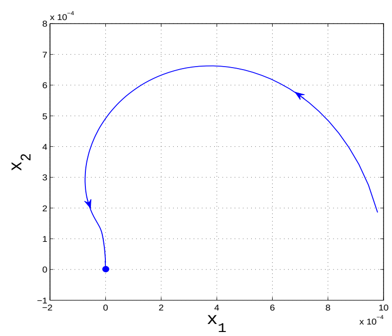
$$\begin{cases} {}^C D^{\alpha_1} x_1(t) = x_2^2, \\ \\ {}^C D^{\alpha_2} x_2(t) = x_1. \end{cases} \tag{32}$$



(a)



(b)



(c)

FIGURE 7. Phase portraits for systems (a) (24); (b) (25) and (c) (26) in commensurate case when $\alpha = 0.6$.

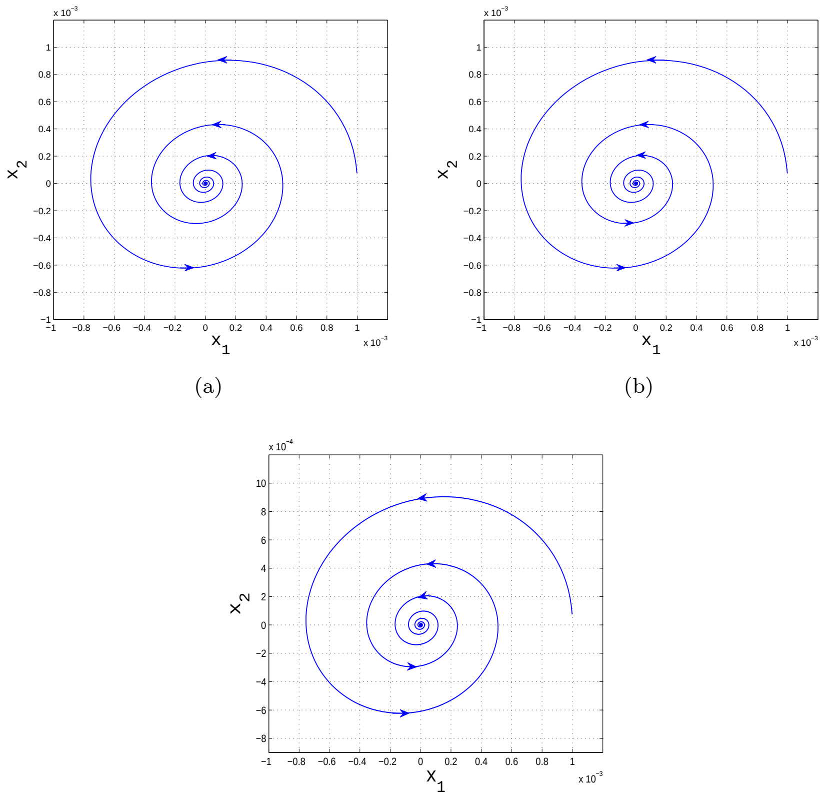
The origin is the only equilibrium point. Linearize at the origin to obtain

$$J|_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Thus, the origin is a nonhyberbolic equilibrium point and the system (32) is non-simple. Consider the linearized system

$$\begin{cases} {}^{C}D^{\alpha_1}x_1(t) = 0, \\ \\ {}^{C}D^{\alpha_2}x_2(t) = x_1. \end{cases} \tag{33}$$

Every point on the x_2-axis is an equilibrium point of linearized system (33). Figures 10 and 11 show the phase potraits for the nonlinear commensurate fractional order system (32) and its linearization, i.e. system (33), at origin when $\alpha = 0.98$ and $\alpha = 0.6$, respectively. A phase portrait for incommensurate fractional order systems (32) and system (33) is plotted in Figure 12. As can be seen from Figures 10, 11 and 12, systems (32) and (33) are unstable.



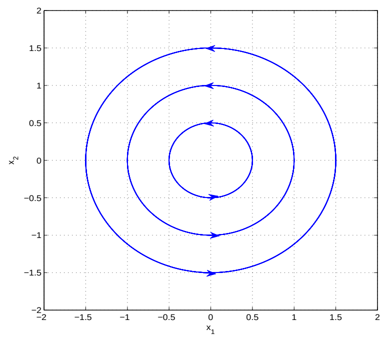
(a) (b)

(c)

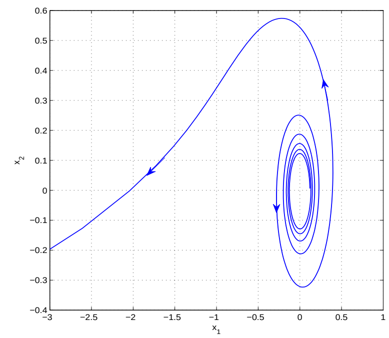
FIGURE 8. Phase portraits for systems (a) (24); (b) (25) and (c) (26) in incommensurate case when $(\alpha_1, \alpha_2) = (0.98, 0.88)$.

6. Conclusions

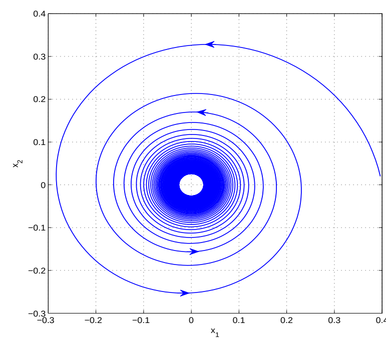
In this paper, we have investigated the equilibrium points classification of linear and nonlinear fractional order differential equations in both cases, commensurate and incommensurate. We have discussed about the qualitative type of phase portrait of equilibrium points. We have showed that equilibrium points classification of fractional order differential equations recovers a standard result from the theory of ordinary differential equations of integer order except in the case the eigenvalues are imaginary, depending on the nature of the eigenvalues of the Jacobian matrix and the value of the fractional order α. Also, we have generalized the Hartman-Grobman theorem to fractional order dynamical systems to show that the dynamics of nonlinear fractional order systems in a neighborhood of the simple equilibrium point are topologically equivalent to those of linear systems locally. All the theoretical results are verified by numerical simulations to demonstrate the effectiveness of the proposed claims.



(a)

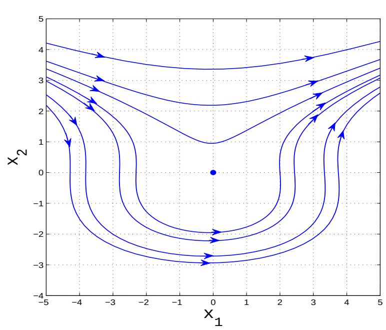


(b)

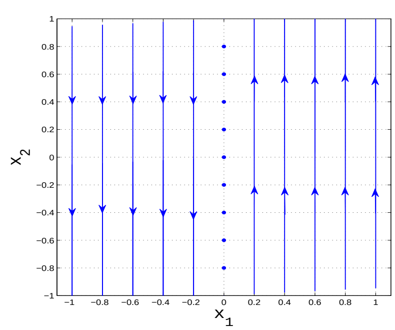


(c)

FIGURE 9. Phase portraits for systems (a) linearized system (26); (b) (24) and (c) (25) when $\alpha = 1$.



(a)



(b)

FIGURE 10. Phase portraits for systems (a) (32); (b) (33) in commensurate case when $\alpha = 0.98$.

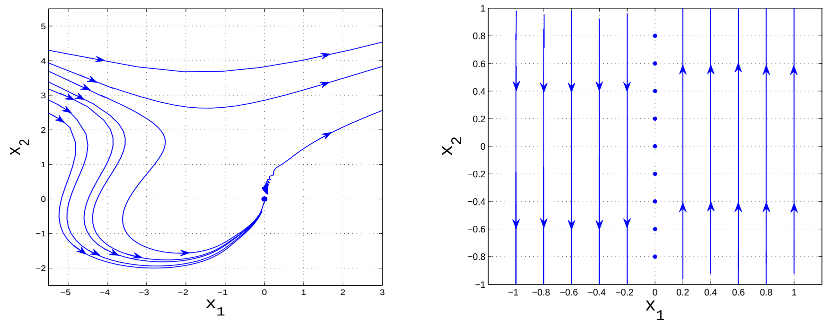


FIGURE 11. Phase portraits for systems (a) (32); (b) (33) in commensurate case when $\alpha = 0.6$.

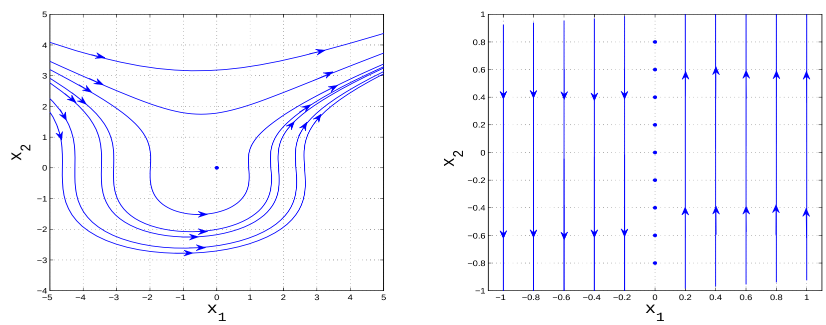


FIGURE 12. Phase portraits for systems (a) (32); (b) (33) in incommensurate case when $(\alpha_1, \alpha_2) = (0.98, 0.88)$.

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