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EXISTENCE OF POSITIVE SOLUTIONS TO A PERIODIC BOUNDARY VALUE PROBLEMS FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS AT RESONANCE

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Abstract. In this paper, we study a class of differential equation of fractional order with periodic boundary conditions at resonance. By Leggett-Williams norm-type theorem for coincidences due to O'Regan and Zima, we present a new result on the existence of positive solutions. At last, an example is presented to illustrate our main results.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order. Fractional differential equations have recently proved to be valuable tools in many fields, such as viscoelasticity, engineering, physics and economics, see [1]-[5].

During the last ten years, fractional boundary value conditions have attracted many authors. The researchers studied the existence and multiplicity of solutions or positive solutions for fractional boundary value problems and obtained many interesting results by using some fixed point theorems, such as the Schauder fixed-point theorem, the Leggett-Williams fixed-point theorem, the Guo-Krasnosel??skii fixed-point theorem, etc. For some recent works on the topic, see [6]-[12] and references therein.

Periodic boundary value problems occur in the mathematical modeling of a variety of physical processes and have recently received considerable attention, see [13]-[16]. In recent years, periodic boundary value conditions of fractional order or integral order at resonance have been studied by some authors. Most of the references focused on the existence of solutions (see [17]-[21]). Meanwhile, some researchers have given attention to the existence of positive solution of the boundary value problems at resonance , such as [23]-[25].

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In [17], Zima and Drygaś gained the existence of positive solutions for the following second-order differential equation subject to periodic boundary conditions:

$$\begin{cases} x''(t) + h(t)x'(t) + f(t, x(t), x'(t)) = 0, \quad t \in [0, T], \\ x(0) = x(T), \quad x'(0) = x'(T), \end{cases}$$

where $f : [0, T] \times [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ and $h : [0, T] \to (0, +\infty)$ are continuous.

In [18], the authors applied the continuation theorem to the study of periodic boundary value problem for fractional p-Laplacian equation:

$$\begin{cases} D^{\beta}_{0+}\phi_p(D^{\alpha}_{0+}x(t)) = f(t, x(t), D^{\alpha}_{0+}x(t)), t \in [0, T], \\ x(0) = x(T), D^{\alpha}_{0+}x(0) = D^{\alpha}_{0+}x(T), \end{cases}$$

where $0 < \alpha, \beta \leq 1$, D^{α}_{0+} is a Caputo fractional derivative, and $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

In [19], Chen and Liu investigated the existence of solutions for the following periodic boundary value problem:

$$\begin{cases} x''(t) = f(t, x(t), D^{\alpha}_{0+}x(t)), \quad t \in [0, 1], \\ x(0) = x(1), \quad D^{\alpha}_{0+}x(0) = D^{\alpha}_{0+}x(1), \end{cases}$$

where $0 < \alpha < 2$ is a real number, D^{α}_{0+} is a Caputo fractional derivative, and $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

From the above works, we can see a fact: the study of periodic boundary value problems of fractional equations mainly focuses on the nonresonant periodic boundary conditions and the acquisition of the existence of solution. To the best of our knowledge, the existence of positive solutions to fractional differential equations with periodic boundary value conditions at resonance is at its infancy and much of the work on this topic need to be done. To fill this gap, we investigate the existence of positive solutions of fractional differential equation with periodic boundary value conditions of the form:

$$\begin{cases} D^{\alpha}_{0+}u(t) = f\big(t, u(t)\big), \quad 0 < t < 1, \\ u(0) = u(1), \ u'(0) = u'(1), \ u''(0) = u''(1), \end{cases} \tag{1.1}$$

where $2 < \alpha < 3$, D^{α}_{0+} denotes the Caputo fractional derivative, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

The rest of this paper is organized as follows. Section 2, we give some necessary notations, definitions and lemmas. In Section 3, we obtain the existence of positive solutions of (1.1) by Theorem 2.1. Finally, an example is given to illustrate our results in Section 4.

2. Preliminaries

First of all, we present the necessary definitions and lemmas from fractional calculus theory. For more details see [5].

Definition 2.1 ([5])**.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s)ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 ([5])**.** The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds,$$

where $n-1 < \alpha \le n$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.3 ([5])**.** The fractional differential equation

$$D_{0+}^{\alpha} y(t) = 0$$

has solution $y(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}, c_i \in \mathbb{R}$, $i = 0, 1, \cdots, n-1$, $n = [\alpha] + 1$. Furthermore, for $y \in AC^n[0,1]$,

$$\left(I_{0+}^{\alpha} D_{0+}^{\alpha} y\right)(t) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k$$

and

$$\left(D_{0+}^{\alpha} I_{0+}^{\alpha} y\right)(t) = y(t).$$

Lemma 2.4 ([5])**.** The relation

$$I_{a+}^{\alpha} I_{a+}^{\beta} f(x) = I_{a+}^{\alpha+\beta} f(x),$$

is valid in following case $\beta > 0,\ \alpha + \beta > 0,\ f(x) \in L_1(a,b)$.

In the following, we provide the necessary background definitions on Fredholm operators and cones in Banach space (see [22]).

Let X, Y be real Banach spaces. Consider a linear mapping $L : \operatorname{dom} L \subset X \to Y$ and a nonlinear operator $N : X \to Y$. Assume that

(A1) L is a Fredholm operator of index zero; that is, $\operatorname{Im} L$ is closed and $\dim \ker L = \operatorname{codim} \operatorname{Im} L < \infty$.

This assumption implies that there exist continuous projections $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \ker L$ and $\ker Q = \operatorname{Im} L$. Moreover, since $\dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \ker L$. Denote by L_p the restriction of L to $\ker P \cap \operatorname{dom} L$. Clearly, L_p is an isomorphism from $\ker P \cap \operatorname{dom} L$ to $\operatorname{Im} L$, we denote its inverse by $K_p : \operatorname{Im} L \to \ker P \cap \operatorname{dom} L$. It is known that the coincidence equation $Lx = Nx$ is equivalent to

$$x = (P + JQN)x + K_P(I - Q)Nx.$$

Let C be a cone in X such that

(i) $\mu x \in C$ for all $x \in C$ and $\mu \ge 0$,
(ii) $x, -x \in C$ implies $x = \theta$.

It is well known that C induces a partial order in X by

$$x \preceq y \quad \text{if and only if} \quad y - x \in C.$$

The following property is valid for every cone in a Banach space X.

Lemma 2.5 ([26])**.** *Let C be a cone in X. Then for every $u \in C \setminus \{0\}$ there exists a positive number $\sigma(u)$ such that*

$$\|x + u\| \ge \sigma(u)\|u\| \quad \text{for all } x \in C.$$

Let $\gamma : X \to C$ be a retraction; that is, a continuous mapping such that $\gamma(x) = x$ for all $x \in C$. Set

$$\Psi := P + JQN + K_p(I-Q)N \quad \text{and} \quad \Psi_\gamma := \Psi \circ \gamma.$$

We use the following result due to O'Regan and Zima.

Theorem 2.6 ([26]). Let C be a cone in X and let Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume (A1) and the following assumptions hold:

- (A2) $QN : X \to Y$ is continuous and bounded and $K_p(I-Q)N : X \to X$ be compact on every bounded subset of X,
- (A3) $Lx \neq \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap ImL$ and $\lambda \in (0,1)$,
- (A4) γ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C,
- (A5) $\deg\{[I-(P+JQN)\gamma]|_{\ker L}, \ker L \cap \Omega_2, 0\} \neq 0$,
- (A6) there exists $u_0 \in C \setminus \{0\}$ such that $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for $x \in C(u_0) \cap \partial\Omega_1$, where $C(u_0) = \{x \in C : \mu u_0 \preceq x \ for\ some\ \mu > 0\}$ and $\sigma(u_0)$ such that $\|x + u_0\| \geq \sigma(u_0)\|x\|$ for every $x \in C$,
- (A7) $(P + JQN)\gamma(\partial\Omega_2) \subset C$,
- (A8) $\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$.

Then the equation $Lx = Nx$ has a solution in the set $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Existence and uniqueness

In this section, we prove the existence results for (1.1). We use the Banach space $X = Y = C[0,1]$ with the supremum norm $\|x\| = \max_{t\in[0,1]} |x(t)|$.

Define the operator $L : \mathrm{dom}L \to X$ by

$$Lu = D^\alpha_{0+}u$$

where

$$\mathrm{dom}L = \big\{x \in X : D^\alpha_{0+}u(t) \in Y, u(0) = u(1),\ u'(0) = u'(1),\ u''(0) = u''(1)\big\}.$$

Define the operator

$$N : X \to Y$$

by

$$Nu(t) = f\big(t, u(t)\big).$$

Then the problem (1.1) can be written by $Lu = Nu, u \in \mathrm{dom}L$.

For simplicity of notation, we set

$$G(t,s) = \begin{cases} 1 + \frac{(\alpha-1)(2t+\alpha t^2-2t^\alpha-\alpha t)+\alpha-4}{2(\alpha-1)\Gamma(\alpha+1)} + \frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-1)} - \frac{\Gamma(\alpha-1)}{(\alpha-2)\Gamma(2\alpha-2)}(1-s)^\alpha \\ \quad +\big[\frac{2-\alpha}{2\alpha(\alpha-1)} + \frac{t-t^2}{2}\big]\frac{1-s}{(\alpha-2)\Gamma(\alpha-1)} + \frac{1+t-t\alpha}{(\alpha-1)(\alpha-2)\Gamma(\alpha)}(1-s)^2 \\ \quad +\frac{(1-s)^{3-\alpha}(t-s)^{\alpha-1}}{(\alpha-2)\Gamma(\alpha)},\ 0 \leq s < t \leq 1, \\ 1 + \frac{(\alpha-1)(2t+\alpha t^2-2t^\alpha-\alpha t)+\alpha-4}{2(\alpha-1)\Gamma(\alpha+1)} + \frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-1)} - \frac{\Gamma(\alpha-1)}{(\alpha-2)\Gamma(2\alpha-2)}(1-s)^\alpha \\ \quad +\big[\frac{2-\alpha}{2\alpha(\alpha-1)} + \frac{t-t^2}{2}\big]\frac{1-s}{(\alpha-2)\Gamma(\alpha-1)} \\ \quad +\frac{1+t-t\alpha}{(\alpha-1)(\alpha-2)\Gamma(\alpha)}(1-s)^2,\ 0 \leq t < s \leq 1. \end{cases}$$

We denote a constant $\kappa \in (0,1)$ satisfying

$$\kappa G(t,s) < 1. \tag{3.1}$$

Lemma 3.1. The mapping $L : \text{dom}L \subset X$ is a Fredholm operator of index zero. Furthermore, the operator $K_P : \text{Im}\, L \to \text{dom}\, L \cap \ker P$ can be written by

$$K_P y(t) = \int_0^1 k(t,s)y(s)ds, \ t \in [0,1],$$

where

$$k(t,s) := \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\Gamma(\alpha-1)(1-s)^{2\alpha-3}}{\Gamma(2\alpha-2)} - \frac{(1-s)^{\alpha-2}}{2\Gamma(\alpha)} + \frac{(1-s)^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)} \\ \quad + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha+1)} + \frac{t(1-s)^{\alpha-2}}{2\Gamma(\alpha-1)} - \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^2(1-s)^{\alpha-2}}{2\Gamma(\alpha-1)}, & 0 \le s < t \le 1, \\ -\frac{\Gamma(\alpha-1)(1-s)^{2\alpha-3}}{\Gamma(2\alpha-2)} - \frac{(1-s)^{\alpha-2}}{2\Gamma(\alpha)} + \frac{(1-s)^{\alpha-1}}{(\alpha-1)\Gamma(\alpha)} \\ \quad + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha+1)} + \frac{t(1-s)^{\alpha-2}}{2\Gamma(\alpha-1)} - \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^2(1-s)^{\alpha-2}}{2\Gamma(\alpha-1)} & 0 \le t < s \le 1. \end{cases}$$

Proof. By Lemma 2.3, $D^{\alpha}_{0+}u(t) = 0$ has solution

$$u(t) = c_0 + c_1 t, \ c_0, c_1 \in \mathbb{R}.$$

According to the boundary value conditions of (1.1), we have

$$\ker L = \{c, c \in \mathbb{R}\} \cong \mathbb{R}^1.$$

Let $y \in \text{Im}L$, so there exists a function $u(t) \in \text{dom}L$ satisfying $Lu(t) = y(t)$. By Lemma 2.3, we have

$$u(t) = I^{\alpha}_{0+}y(t) + c_0 + c_1 t + c_2 t^2.$$

By $u''(0) = u''(1)$, we can obtain $\int_0^1 (1-s)^{\alpha-3} y(s)ds = 0$.
On the other hand, suppose $y \in Y$ satisfying $\int_0^1 (1-s)^{\alpha-3} y(s)ds = 0$. Let

$$u(t) = I^{\alpha}_{0+}y(t) + \left[\frac{1}{2} I^{\alpha-1}_{0+}y(1) - I^{\alpha}_{0+}y(1)\right] \cdot t - \frac{1}{2} I^{\alpha-1}_{0+}y(1) \cdot t^2.$$

We can easily prove $u(0) = u(1), u'(0) = u'(1), u''(0) = u''(1)$, that is $u(t) \in \text{dom}L$. Thus, we conclude that

$$\text{Im}L = \left\{ y \in Y : \int_0^1 (1-s)^{\alpha-3} y(s)ds = 0 \right\}.$$

Consider the linear operator $P : X \to X$ defined by

$$Px(t) = (\alpha - 2)\int_0^1 (1-s)^{\alpha-3} x(s)ds, \quad t \in [0,1].$$

Define the operator $Q : Y \to Y$ by

$$Qy(t) = (\alpha - 2)\int_0^1 (1-s)^{\alpha-3} y(s)ds, \quad t \in [0,1].$$

For $u(t) \in X$, we get

$$P(Pu) = P\left[(\alpha-2)\int_0^1 (1-s)^{\alpha-3} u(s)ds\right] = (\alpha-2)\int_0^1 (1-s)^{\alpha-3} u(s)ds = Pu.$$

So we have $P^2 = P$. Obviously, $Q^2 = Q$. Note that $\text{Im}P = \ker L$ and $\ker Q = \text{Im}L$. It follows from $\text{Ind}\, L = \dim \ker L - \text{codim}\, \text{Im}\, L = 0$ that L is a Fredholm mapping of index zero.

Next, we will prove that the operator K_P is the inverse of $L|_{\text{dom}L \cap \text{Ker}P}$.

In fact, for $u(t) \in \mathrm{dom}L \cap \ker P$, we have $D_{0+}^{\alpha}u(t) = y(t)$. By Lemma 2.3, we have $u(t) = I_{0+}^{\alpha}y(t) + c_0 + c_1 t + c_2 t^2$. According to $u(0) = u(1), u'(0) = u'(1)$, we get

$$c_1 = \frac{1}{2} I_{0+}^{\alpha-1} y(1) - I_{0+}^{\alpha} y(1), \ c_2 = -\frac{1}{2} I_{0+}^{\alpha-1} y(1).$$

According to $u(t) \in \ker P$, that is $(\alpha - 2) \int_0^1 (1-s)^{\alpha-3} u(s)ds = 0$, we deduce

$$c_0 = -\Gamma(\alpha - 1) I_{0+}^{2\alpha-2} y(1) - \frac{c_1}{\alpha - 1} - \frac{2c_2}{\alpha(\alpha - 1)}.$$

Define an operator

$$K_P y(t) := I_{0+}^{\alpha} y(t) + c_0 + c_1 t + c_2 t^2.$$

Substituting c_0, c_1, c_2 in above equality, we obtain

$$\begin{aligned}
K_P y(t) &= I_{0+}^{\alpha} y(t) + c_0 + c_1 t + c_2 t^2 \\
&= I_{0+}^{\alpha} y(t) - \Gamma(\alpha-1) I_{0+}^{2\alpha-2} y(1) - \frac{1}{\alpha-1} \left[\frac{1}{2} I_{0+}^{\alpha-1} y(1) - I_{0+}^{\alpha} y(1) \right] \\
&\quad + \frac{1}{\alpha(\alpha-1)} I_{0+}^{\alpha-1} y(1) + \left[\frac{1}{2} I_{0+}^{\alpha-1} y(1) - I_{0+}^{\alpha} y(1) \right] t - \frac{1}{2} I_{0+}^{\alpha-1} y(1) t^2. \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds - \frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-2)} \int_0^1 (1-s)^{2\alpha-3} y(s) ds \\
&\quad - \frac{1}{2\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} y(s) ds + \frac{1}{(\alpha-1)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds \\
&\quad + \frac{1}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha-2} y(s) ds + \frac{t}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s) ds \\
&\quad - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds - \frac{t^2}{2\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s) ds \\
&= \int_0^1 k(t,s) y(s) ds.
\end{aligned}$$

It is easy to see that $LK_p y(t) = y(t)$. Hence, $K_P = \left(L|_{\mathrm{dom}L \cap \mathrm{Ker}P} \right)^{-1}$. This completes the proof Lemma 3.1. □

Lemma 3.2. Assume $\Omega \subset X$ is a open bounded set such that $\mathrm{dom(L)} \cap \overline{\Omega} \neq \emptyset$, then N is L-compact on $\overline{\Omega}$.

Proof. By the continuity of f, we can obtain that $QN(\overline{\Omega})$ and $K_P(I-Q)N(\overline{\Omega})$ are bounded. Hence, for $u(t) \in \overline{\Omega}$, $t \in [0,1]$, there exists a positive constant T such that $|(I-Q)Nu(t)| \leq T$, $|\frac{1}{2} I_{0+}^{\alpha-1}(I-Q)Nu(1) - I_{0+}^{\alpha}(I-Q)Nu(1)| \leq T$ and $|\frac{1}{2} I_{0+}^{\alpha-1}(I-Q)Nu(1)| \leq T$.

Thus, in the view of Arzela-Ascoli theorem, we need only prove that $K_P(I-Q)N(\overline{\Omega})$ is equicontinuous.

For $0 \leq t_1 < t_2 \leq 1$, $u \in \overline{\Omega}$, by the definition of K_P, we have

$$\begin{aligned}
&|K_P(I-Q)Nu(t_2) - K_P(I-Q)Nu(t_1)| \\
&\quad = \Big| \left[I_{0+}^{\alpha}(I-Q)Nu(t) \right]_{t=t_2} + c_0 + c_1 t_2 + c_2 t_2^2
\end{aligned}$$

$$
\begin{aligned}
&\qquad - \big[I_{0+}^{\alpha}(I-Q)Nu(t)\big]_{t=t_1} - c_0 - c_1 t_1 - c_2 t_1^2 \Big| \\
&\le \frac{1}{\Gamma(\alpha)}\Big|\int_0^{t_2}(t_2-s)^{\alpha-1}(I-Q)Nu(s)ds - \int_0^{t_1}(t_1-s)^{\alpha-1}(I-Q)Nu(s)ds\Big| \\
&\qquad + \Big|\frac{1}{2}I_{0+}^{\alpha-1}(I-Q)Nu(1) - I_{0+}^{\alpha}(I-Q)Nu(1)\Big|\cdot|t_2-t_1| \\
&\qquad + \Big|\frac{1}{2}I_{0+}^{\alpha-1}(I-Q)Nu(1)\Big|\cdot|t_2^2-t_1^2| \\
&\le \frac{1}{\Gamma(\alpha)}\Big|\int_0^{t_1}\Big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big](I-Q)Nu(s)ds\Big| \\
&\qquad + \frac{1}{\Gamma(\alpha)}\Big|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(I-Q)Nu(s)ds\Big| + T\big(t_2-t_1+t_2^2-t_1^2\big) \\
&\le \frac{T}{\Gamma(\alpha+1)}\big[t_2^{\alpha}-t_1^{\alpha}+(t_2-t_1)^{\alpha}\big] + T\big(t_2-t_1+t_2^2-t_1^2\big).
\end{aligned}
$$

Notice that t, t^2, t^{α} are uniformly continuous on [0, 1]. Thus, we have $K_P(I-Q)N(\overline{\Omega})$ is equicontinuous on $[0,1]$. The proof is completed. □

Theorem 3.3. Assume that

(H1) for $t\in[0,1]$ and $u(t)\in[0,B]$, one has

$$-\kappa u(t) \le f(t,u(t)) \le -c_1 u(t) + c_2 \text{ and } f(t,u(t)) \le -b_1|f(t,u(t))| + b_2 u(t) + b_3,$$

where $b_1, b_2, b_3, c_1, c_2, B$ are positive constants with

$$B > \frac{c_2}{c_1} + \frac{8b_2c_2}{(\alpha-2)b_1c_1} + \frac{8b_3}{(\alpha-2)b_1}.$$

(H2) there exist $r\in(0,B)$, $t_0\in[0,1]$, $m\in(0,1)$ and $h(u):(0,r]\to[0,+\infty)$ such that $f(t,u)\ge h(u)$ for $t\in[0,1], u\in(0,b]$. Moreover, $\frac{h(u)}{u}$ is non-increasing on $(0,r]$ and

$$(\alpha-2)\frac{h(r)}{r}\int_0^1 G(t_0,s)(1-s)^{\alpha-3}ds \ge \frac{1-m}{m}.$$

Then the problem (1.1) has at least one positive solution on $[0,1]$.

Proof. According to Lemma 3.1 and Lemma 3.2, we have that the conditions (A1) and (A2) of Theorem 2.6 are satisfied.

Consider the cone

$$C = \{x\in X : x(t)\ge 0,\ t\in[0,1]\}.$$

Let

$$\Omega_1 = \big\{x\in X : m\|x\| < |x(t)| < r,\ t\in[0,1]\big\},$$
$$\Omega_2 = \big\{x\in X : \|x(t)\| < B,\ t\in[0,1]\big\}.$$

Obviously, Ω_1 and Ω_2 are bounded and

$$\overline{\Omega}_1 = \big\{x\in X : m\|x\| \le |x(t)| \le r,\ t\in[0,1]\big\} \subset \Omega_2.$$

Moreover, $C\cap(\overline{\Omega}_2\setminus\Omega_1)\ne\emptyset$. Let $J=I$ and $(\gamma x)(t)=|x(t)|$ for $x\in X$, then γ is a retraction and maps subsets of $\overline{\Omega}_2$ into bounded subsets of C, which means that (A4) holds.

Next, we will show (A3) holds. Suppose that there exist $u_0 \in \partial\Omega_2 \cap C \cap \operatorname{dom} L$ and $\lambda_0 \in (0,1)$ such that $Lu_0 = \lambda_0 N u_0$, that is $D_{0+}^{\alpha} u_0(t) = \lambda_0 f(t, u_0(t)), t \in [0,1]$. In view of (H1), we get

$$\begin{aligned} D_{0+}^{\alpha} u_0(t) = \lambda_0 f(t, u_0(t)) &\le -\lambda_0 b_1 |f(t, u_0(t))| + \lambda_0 b_2 u_0(t) + \lambda_0 b_3 \\ &= -b_1 |\lambda_0 f(t, u_0(t))| + \lambda_0 b_2 u_0(t) + \lambda_0 b_3 \\ &= -b_1 |D_{0+}^{\alpha} u_0(t)| + \lambda_0 b_2 u_0(t) + \lambda_0 b_3 \\ &\le -b_1 |D_{0+}^{\alpha} u_0(t)| + b_2 u_0(t) + b_3 \end{aligned} \tag{3.2}$$

and

$$D_{0+}^{\alpha} u_0(t) = \lambda_0 f(t, u_0(t)) \le -\lambda_0 c_1 u_0(t) + \lambda_0 c_2. \tag{3.3}$$

In view of $D_{0+}^{\alpha} u_0(t) = \lambda_0 f(t, u_0(t)) \in \operatorname{Im} L$, from the definition of $\operatorname{Im} L$ and (3.3), we obtain

$$0 = \int_0^1 (1-s)^{\alpha-3} D_{0+}^{\alpha} u_0(s) ds \le \int_0^1 (1-s)^{\alpha-3} \Big(- \lambda_0 c_1 u_0(s) + \lambda_0 c_2 \Big) ds$$

which gives

$$\int_0^1 (1-s)^{\alpha-3} u_0(s) ds \le \frac{c_2}{(\alpha-2)c_1}. \tag{3.4}$$

Furthermore, from (3.2) and (3.4), we have

$$\begin{aligned} 0 &= \int_0^1 (1-s)^{\alpha-3} D_{0+}^{\alpha} u_0(s) ds \\ &\le \int_0^1 (1-s)^{\alpha-3} \big[- b_1 |D_{0+}^{\alpha} u_0(t)| + b_2 u_0(t) + b_3 \big] ds \\ &= -b_1 \int_0^1 (1-s)^{\alpha-3} \left| D_{0+}^{\alpha} u_0(s) \right| ds + b_2 \int_0^1 (1-s)^{\alpha-3} u_0(s) ds + \frac{b_3}{\alpha-2} \end{aligned}$$

which gives

$$\begin{aligned} \int_0^1 (1-s)^{\alpha-3} \left| D_{0+}^{\alpha} u_0(s) \right| ds &\le \frac{b_2}{b_1} \int_0^1 (1-s)^{\alpha-3} u_0(s) ds + \frac{b_3}{(\alpha-2)b_1} \\ &\le \frac{b_2 c_2}{(\alpha-2) b_1 c_1} + \frac{b_3}{(\alpha-2)b_1}. \end{aligned} \tag{3.5}$$

According to the function expression of $k(t,s)$, it is easy to see that

$$|k(t,s)| \le 8(1-s)^{\alpha-3}, s, t \in [0,1]. \tag{3.6}$$

From (3.4), (3.5), (3.6) and the equation $u_0 = (I-P)u_0 + Pu_0 = K_P L(I-P)u_0 + Pu_0 = Pu_0 + K_P L u_0$, we can get

$$\begin{aligned} u_0 &= Pu_0 + K_P L u_0 \\ &= (\alpha-2) \int_0^1 (1-s)^{\alpha-3} u_0(s) ds + \int_0^1 k(t,s) D_{0+}^{\alpha} u_0(s) ds \\ &\le (\alpha-2) \cdot \frac{c_2}{(\alpha-2)c_1} + \int_0^1 |k(t,s)| \cdot \left| D_{0+}^{\alpha} u_0(s) \right| ds \\ &= \frac{c_2}{c_1} + \int_0^1 \frac{|k(t,s)|}{(1-s)^{\alpha-3}} \cdot (1-s)^{\alpha-3} \left| D_{0+}^{\alpha} u_0(s) \right| ds \end{aligned}$$

$$\le \frac{c_2}{c_1} + 8\int_0^1 (1-s)^{\alpha-3}\big|D_{0+}^{\alpha}u_0(s)\big|ds$$
$$\le \frac{c_2}{c_1} + \frac{8b_2c_2}{(\alpha-2)b_1c_1} + \frac{8b_3}{(\alpha-2)b_1}.$$

Then, we have

$$B = \|u_0\| \le \frac{c_2}{c_1} + \frac{8b_2c_2}{(\alpha-2)b_1c_1} + \frac{8b_3}{(\alpha-2)b_1},$$

which contradicts (H1). Hence (A3) holds.

To prove (A5), consider $u(t) \in \text{ker}L \cap \overline{\Omega}_2$, then $u(t) \equiv c$. For $c \in [-B, B]$ and $\lambda \in [0, 1]$, we have

$$\begin{aligned} H(c,\lambda) &= [I - \lambda(P + JQN)\gamma]c \\ &= c - \lambda(\alpha-2)\int_0^1 (1-s)^{\alpha-3}|c|ds - \lambda(\alpha-2)\int_0^1 (1-s)^{\alpha-3} f(s,|c|)ds \\ &= c - \lambda|c| - \lambda(\alpha-2)\int_0^1 (1-s)^{\alpha-3} f(s,|c|)ds. \\ &= c - \lambda(\alpha-2)\int_0^1 (1-s)^{\alpha-3}\big[f(s,|c|) + |c|\big]ds. \end{aligned}$$

By use of proof by contradiction, it is easy to show that $H(c, \lambda) = 0$ implies $c \ge 0$. Suppose $H(B, \lambda) = 0$ for some $\lambda \in (0, 1]$, then we have

$$0 = B - \lambda B - \lambda(\alpha-2)\int_0^1 (1-s)^{\alpha-3} f(s,B)ds.$$

According to (H1), we have

$$0 \le B(1-\lambda) = \lambda(\alpha-2)\int_0^1 (1-s)^{\alpha-3} f(s,B)ds \le \lambda(-c_1B + c_2) < 0,$$

which is a contradiction. In addition, if $\lambda = 0$, then $B = 0$, which is impossible. As a result, for $x \in \text{ker}L \cap \partial\Omega_2$ and $\lambda \in [0, 1]$, we have $H(x, \lambda) \ne 0$. Thus,

$$\begin{aligned} &\deg\{[I - (P{+}JQN)\gamma]_{\text{Ker}L}, \text{Ker}L \cap \Omega_2, 0\} \\ &\qquad = \deg\{H(\cdot,1), \text{Ker}L \cap \Omega_2, 0\} \\ &\qquad = \deg\{H(\cdot,0), \text{Ker}L \cap \Omega_2, 0\} \\ &\qquad = \deg\{I, \text{Ker}L \cap \Omega_2, 0\} \\ &\qquad = 1 \ne 0. \end{aligned}$$

So (A5) holds.

Next, we prove (A6). Let $u_0(t) \equiv 1, t \in [0, 1]$, then $u_0 \in C \setminus \{0\}$, $C(u_0) = \{x \in C : x(t) > 0, t \in [0, 1]\}$. We take $\sigma(u_0) = 1$. Let $u \in C(u_0) \cap \partial\Omega_1$, then $0 < \|u\| \le r$ and $u(t) \ge m\|u\|$ on $[0, 1]$.

By (H2), for $u \in C(u_0) \cap \partial\Omega_1$, we have

$$\begin{aligned} (\Psi)u(t_0) &= \big[\big(P + JQN + K_p(I-Q)N\big)u(t)\big]_{t=t_0} \\ &= \big[Pu(t)\big]_{t=t_0} + \big[\big(JQN + K_p(I-Q)N\big)u(t)\big]_{t=t_0} \\ &= (\alpha-2)\int_0^1 (1-s)^{\alpha-3}u(s)ds + (\alpha-2)\int_0^1 G(t_0,s)(1-s)^{\alpha-3} f\big(s,u(s)\big)ds \end{aligned}$$

$$
\begin{aligned}
&\geq (\alpha-2)\int_0^1 (1-s)^{\alpha-3} m\|x\| ds + (\alpha-2)\int_0^1 G(t_0,s)(1-s)^{\alpha-3} f\big(s,u(s)\big)ds\\
&\geq m\|u\| + (\alpha-2)\int_0^1 G(t_0,s)(1-s)^{\alpha-3} h\big(u(s)\big)ds\\
&= m\|u\| + (\alpha-2)\int_0^1 G(t_0,s)(1-s)^{\alpha-3}\cdot\frac{h\big(u(s)\big)}{u(s)}u(s)ds\\
&\geq m\|u\| + (\alpha-2)\int_0^1 G(t_0,s)(1-s)^{\alpha-3}\cdot\frac{h\big(u(s)\big)}{u(s)}\cdot m\|u\|ds\\
&\geq m\|u\| + m\|u\|\cdot(\alpha-2)\int_0^1 G(t_0,s)(1-s)^{\alpha-3}\frac{h(r)}{r}ds\\
&\geq m\|u\| + m\|u\|\cdot\frac{1-m}{m}\\
&= \|u\|.
\end{aligned}
$$

Thus, for all $x \in C(u_0)\cap\partial\Omega_1$, we have $\|x\| \leq \sigma(u_0)\|\Psi x\|$, i.e. (A6) holds.
For $u \in \partial\Omega_2$, by (H2), we have

$$
\begin{aligned}
&\big[(P+JQN)\circ\gamma\big]u(t) = P(|u(t)|) + JQN(|u(t)|)\\
&= (\alpha-2)\int_0^1 (1-s)^{\alpha-3}|u(s)|ds + (\alpha-2)\int_0^1 (1-s)^{\alpha-3} f(s,|u(s)|)ds\\
&\geq (\alpha-2)\int_0^1 (1-s)^{\alpha-3}(1-\kappa)|u(s)|ds\\
&\geq 0.
\end{aligned}
$$

Thus, for $u \in \partial\Omega_2$, one has $\big[(P+JQN)\circ\gamma\big]x(t) \subset C$. Then (A7) holds.

Next, we prove (A8). For $u(t) \in \overline{\Omega}_2\setminus\Omega_1$, by (H2) and (3.1), we have

$$
\begin{aligned}
&\Psi_\gamma u(t) = \big[\big(P+JQN+K_p(I-Q)N\big)\circ\gamma\big]u(t)\\
&= \big(P+JQN+K_p(I-Q)N\big)|u(t)|\\
&= P\big(|u(t)|\big) + \big[JQN+K_p(I-Q)N\big]|u(t)|\\
&= (\alpha-2)\int_0^1 (1-s)^{\alpha-3}|u(s)|ds + (\alpha-2)\int_0^1 (1-s)^{\alpha-3}G(t,s)f(s,|u(s)|)ds\\
&> (\alpha-2)\int_0^1 (1-s)^{\alpha-3}|u(s)|ds + (\alpha-2)\int_0^1 (1-s)^{\alpha-3}G(t,s)(-\kappa|u(s)|)ds\\
&> (\alpha-2)\int_0^1 (1-s)^{\alpha-3}|u(s)|\big(1-\kappa G(t,s)\big)ds\\
&\geq 0.
\end{aligned}
$$

Hence, $\Psi_\gamma(\overline{\Omega}_2\setminus\Omega_1) \subset C$, that is (A8) holds.

Hence, applying Theorem 2.6, BVP (1.1) has a positive solution $u^*(t)$ on $[0,1]$ with $r \leq \|u^*(t)\| \leq B$. This completes the proof. □

4. An example

In this section, we give an example to illustrate our main results.

Example 4.1. Consider the fractional periodic boundary value problem

$$\begin{cases} D_{0^+}^{2.5}u(t) = f(t,u(t)), \quad 0 < t < 1, \\ u(0) = u(1),\ u'(0) = u'(1),\ u''(0) = u''(1), \end{cases} \tag{4.1}$$

where

$$f(t,u) = \frac{1}{100}(1+t^2)\Big(-\frac{1}{2}u + \frac{1}{2}\Big).$$

Corresponding to BVP (4.1), we have that $\alpha = 2.5$ and

$$G(t,s) = \begin{cases} 1 + \frac{2t+2.5t^2-2t^{2.5}-2.5t-1}{2\Gamma(3.5)} + \frac{\Gamma(1.5)}{6} - \Gamma(1.5)(1-s)^{2.5} \\ \quad +(t-t^2-\frac{2}{15})\frac{1-s}{\Gamma(1.5)} + \frac{4+4t-10t}{3\Gamma(2.5)}(1-s)^2 \\ \quad +\frac{2(1-s)^{0.5}(t-s)^{1.5}}{\Gamma(2.5)}, & 0 \le s < t \le 1, \\ 1 + \frac{2t+2.5t^2-2t^{2.5}-2.5t-1}{2\Gamma(3.5)} + \frac{\Gamma(1.5)}{6} - \Gamma(1.5)(1-s)^{2.5} \\ \quad +(t-t^2-\frac{2}{15})\frac{1-s}{\Gamma(1.5)} + \frac{4+4t-10t}{3\Gamma(2.5)}(1-s)^2, & 0 \le t < s \le 1. \end{cases}$$

By simple calculation, we find that $\frac{1}{15}G(t,s) < 1$ and if $t \in [0,1]$ and $u \in [0,2]$, one has

$$-\frac{1}{15}u(t) \le f(t,u(t)) \le -\frac{1}{100}u(t) + \frac{2}{125},$$
$$f(t,u(t)) \le -\big|f(t,u(t))\big| + \frac{1}{100}u(t) + \frac{11}{1000}.$$

So, we can choose $\kappa = \frac{1}{15}$, $B = 2$, $c_1 = \frac{1}{100}$, $c_2 = \frac{3}{200}$, $b_1 = 1$, $b_2 = \frac{1}{100}$, $b_3 = \frac{11}{1000}$. Furthermore, it is easy to verify that

$$\frac{c_2}{c_1} + \frac{8b_2c_2}{(\alpha-2)b_1c_1} + \frac{8b_3}{(\alpha-2)b_1} = 1.916 < 2 = B.$$

So, (H1) is satisfied.

We take $r = 0.9 \in [0,2]$, $\rho = 1$, $h(u) = \frac{u}{2000}$ and $q(t) = \frac{1}{100}(1+t^2)$. By calculation, we obtain

$$f(t,u(t)) \ge h(u) = \frac{1}{2000}u(t),\ (t,u) \in [0,1] \times (0,0.9]$$

and

$$\frac{h(u)}{u} = \frac{\frac{1}{20}u(t)}{u(t)} = \frac{1}{20},$$

which is non-increasing on $(0,0.9]$.

Let $t_0 = 0$, then we have

$$G(t_0,s) = G(0,s) \approx 0.99 - 0.88(1-s)^{2.5} - 0.5(1-s) + 1.01(1-s)^2 > 0.6 > 0.$$

Using the given data, we have

$$(\alpha-2)\frac{h(r)}{r}\int_0^1 G(0,s)(1-s)^{\alpha-3}ds \approx 0.00044 > \frac{1-m}{m}$$

holds for $m = 0.9995$. One sees that (H2) is satisfied. In consequence, the conclusion of Theorem 3.3 implies that the problem (4.1) has a positive solution on $[0,1]$.

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