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ON A BOUNDARY VALUE PROBLEM AT RESONANCE ON THE HALF LINE

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Abstract. This paper concerns the solvability of a nonlinear fractional boundary value problem at resonance on the half line. The basic idea is to prove, by using fixed point theorems, the existence and uniqueness of solution for the corresponding perturbed problem, then we establish the existence results for the given problem. An example is given to illustrate the obtained results.

1. Introduction

Fractional differential equations still attracting much attention, this is due to their ability to describe of memory and hereditary properties of various phenomena, for some recent results on fractional differential equations and their applications we refer to the monograph of Kilbas et al [13], Diethelm et al [6], Kiryakova [14, 15], Samko et al [21] Podlubny [20] and Tenreiro Machado et al [23].

The main goal of this paper is to prove the existence of solutions for the following fractional boundary value problem (P)

$$D_{0^+}^{q} u(t) = -f\left(t, u(t)\right), \quad t > 0, \tag{1.1}$$

$$D_{0^+}^{q-2} u(0) = u''\left(0\right) = 0, \quad D_{0^+}^{q-1} u\left(+\infty\right) = \Gamma\left(q\right) u\left(1\right), \tag{1.2}$$

where $f \in C\left(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}\right)$, $2 < q < 3$, $D_{0^+}^{q}$ denotes the Riemann-Liouville fractional derivative. The problem (P) is called at resonance in the sense that the associated linear homogeneous boundary value problem $Lu = D_{0^+}^{q} u(t) = 0$ with conditions (1.2), has $u(t) = ct^{q-1}$, $c \in \mathbb{R}$ as nontrivial solutions. In this case we will apply fixed point theorems together with some ideas from analysis.

Recently many boundary value problems at resonance have been extensively studied and many results have been obtained see [2, 7, 8, 9, 10, 11, 12, 17, 18, 24, 26]. For the resonance case, the boundary value problem is approached in several ways. But the classical method is to decompose the space in the form of a direct sum of subspaces, one of that is $KerL$, and then to work with the corresponding projections on these spaces. In a recent study [19],Nieto investigated a second order boundary

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value problem at resonance by applying fixed point theorem and some analysis tools which motived the present study.

Boundary value problems on the half line have often appeared in applied mathematics and physics, in fact, they may model some physical phenomena, such as the models of gas pressure in a semi-infinite porous medium, see [25]. Many methods are used to investigate these problems such as fixed point theorems, lower and upper method solutions, Mawhin coincidence degree theory...see [1, 2, 3, 7, 8, 9, 10, 11, 12, 16, 17, 18, 19, 22, 24, 25, 26, 27].

In [16] the authors investigated the following m-point boundary value problem on infinite line by using a fixed point theorem on a cone

$$D_{0^+}^q u(t) + a(t) f(t, u(t)) = 0, \quad t > 0, 2 < q < 3$$

$$u(0) = u'(0) = 0, \quad D_{0^+}^{q-1} u(+\infty) = \sum_{i=1}^{m-2} b_i u(\xi_i).$$

By application of Leggett-Williams norm type theorem in [3], the authors established the existence of positive solutions of the following boundary value problem at resonance on infinite interval

$$D_{0^+}^q u(t) + a(t) f(t, u(t)) = 0, \quad t > 0, 3 < q < 4$$

$$u(0) = u'(0) = u''(0) = 0, \quad D_{0^+}^{q-1} u(+\infty) = D_{0^+}^{q-1} u(+\infty).$$

The organization of this work is as follows. In Section 2, we introduce some notations, definitions and lemmas that will be used later. Section 3 treats the existence and uniqueness of solution for the perturbed problem by using respectively Schaefer fixed point theorem and Banach contraction principal. Then by some analysis ideas, we prove that the problem (P) has a solution. Finally, we illustrate the obtained results by an example.

2. Preliminaries

The following lemmas and definitions can be found in [20].

Definition 1. Let $\alpha > 0$,then the Riemann-Liouville fractional integral of a function g is defined by

$$I_{a^+}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} ds.$$

Definition 2. The Riemann fractional derivative of order q of g is defined by

$$D_{a^+}^q g(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{g(s)}{(t-s)^{q-n+1}} ds,$$

where $n = [q] + 1$.($[q]$ is the integer part of q).

Lemma 1. The homogenous fractional differential equation $D_{a^+}^q g(t) = 0$ has a solution

$$g(t) = c_1 t^{q-1} + c_2 t^{q-2} + ... + c_n t^{q-n},$$

where, $c_i \in \mathbb{R}$, $i = 1, ..., n$ and $n = [q] + 1$.

Lemma 2. Let $p, q \geq 0$, $f \in L_1[a, b]$. Then:

1- $I_{0^+}^p I_{0^+}^q f(t) = I_{0^+}^{p+q} f(t) = I_{0^+}^q I_{0^+}^p f(t)$ and ${}^c D_{0^+}^q I_{0^+}^q f(t) = f(t)$, for all $t \in [a, b]$.

2- If $p > q > 0$, then the formula ${}^{c}D_{0^+}^{q} I_{0^+}^{p} f(t) = I_{0^+}^{p-q} f(t)$, holds almost everywhere on $t \in [a,b]$, for $f \in L_1[a,b]$ and it is valid at any point $t \in [a,b]$ if $f \in C[a,b]$.

3- If $q \geq 0$ and $p > 0$ then $D_{a^+}^{q} (t-a)^{p-1}(x) = \frac{\Gamma(p)}{\Gamma(p-q)} (x-a)^{p-q-1}$, $D_{a^+}^{q} (t-a)^{q-j}(x) = 0$, $j = 1, 2, ..., n$.

To prove the main results of this paper, we need the following Lemma.

Lemma 3. Let $2 < q < 3$ and $y \in C(\mathbb{R}_+, \mathbb{R})$. The linear fractional boundary value problem

$$\begin{cases} D_{0^+}^{q} u(t) = -y(t), \\ u''(0) = D_{0^+}^{q-2} u(0) = 0, \quad D_{0^+}^{q-1} u(\infty) = \Gamma(q) u(1), \end{cases} \tag{2.1}$$

has a solution if and only if $\int_0^\infty y(t)\,dt - \int_0^1 (1-t) y(t)\,dt = 0$. In this case the solution can be written as

$$u(t) - t^{q-1} u(1) = \frac{1}{\Gamma(q)} \int_0^\infty H(t,s) y(s) ds, \tag{2.2}$$

where

$$H(t,s) = \begin{cases} -(t-s)^{q-1} + t^{q-1}, s \leq t, \\ t^{q-1}, \quad t \leq s. \end{cases} \tag{2.3}$$

Proof. Applying Lemma 1 to the differential equation in (2.1) we get

$$u(t) = -I_{0^+}^{q} y(t) + c_1 t^{q-1} + c_2 t^{q-2} + c_3 t^{q-3}. \tag{2.4}$$

Differentiating both sides of (2.4) and using Lemma 2, it yields

$$u'(t) = -I_{0^+}^{q-1} y(t) + c_1 (q-1) t^{q-2} + c_2 (q-2) t^{q-3} + c_3 (q-3) t^{q-4}, \tag{2.5}$$

$$\begin{aligned} u''(t) &= -I_{0^+}^{q-2} y(t) + c_1 (q-1)(q-2) t^{q-3} \\ &\quad + c_2 (q-2)(q-3) t^{q-4} + c_3 (q-3)(q-4) t^{q-5}, \\ D_{0^+}^{q-2} u(t) &= -I_{0^+}^{2} y(t) + c_1 \Gamma(q) t + c_2 \Gamma(q-1). \end{aligned}$$

The two first conditions in (2.1) give $c_3 = c_2 = 0$, the third one implies that $\int_0^\infty y(t)\,dt = \Gamma(q) I_{0^+}^{q} y(1)$, hence (2.1) has solution if and only if $\int_0^\infty y(t)\,dt - \int_0^1 (1-t) y(t)\,dt = 0$, then the problem (2.1) has an infinity of solutions given by

$$u(t) = -I_{0^+}^{q} y(t) + c t^{q-1}. \tag{2.6}$$

Now we try to rewrite the function u. We have

$$u(1) = -I_{0^+}^{q} y(1) + c = \frac{-1}{\Gamma(q)} \int_0^\infty y(s)\,ds + c,$$

then

$$c = \frac{1}{\Gamma(q)} \int_0^\infty y(s)\,ds + u(1),$$

substituting c by its value in (2.6) we obtain

$$\begin{aligned} u(t) &= -I_{0^+}^{q} y(t) + \frac{t^{q-1}}{\Gamma(q)} \int_0^\infty y(s)\,ds + t^{q-1} u(1) \\ &= \frac{1}{\Gamma(q)} \int_0^\infty H(t,s) y(s) ds + t^{q-1} u(1). \end{aligned}$$

Hence the linear problem can be written as

$$u(t)-t^{q-1}u(1)=\frac{1}{\Gamma(q)}\int_0^\infty H(t,s)y(s)ds,$$

where $H(t,s)=\begin{cases} -(t-s)^{q-1}+t^{q-1}, & s\le t, \\ t^{q-1}, & t\le s. \end{cases}$ The kernel $H(t,s)$ is continuous according to both variables s,t on $\mathbb{R}_+$ and is positive. The proof is completed.

The nonlinear problem (P) can be transformed to the integral equation

$$u(t)-t^{q-1}u(1)=\frac{1}{\Gamma(q)}\int_0^\infty H(t,s)f(s,u(s))ds. \tag{2.7}$$

Define a new function $v(t)=u(t)-t^{q-1}u(1)$. To find a solution u we have to find v and $u(1)$. Note $v_c(t)=u(t)-t^{q-1}c$, we try to solve for every v_c the problem

$$v_c(t)=\frac{1}{\Gamma(q)}\int_0^\infty H(t,s)f(s,v_c(s)+cs^{q-1})ds, \tag{2.8}$$

if v_c is a solution of (2.8) with $c=u(1)$ then u is a solution of (P).

3. Existence and uniqueness results

We will use the Banach space E defined by

$$E=\left\{u\in C(\mathbb{R}_+,\mathbb{R}), \lim_{t\to+\infty}\frac{|u(t)|}{1+t^{q-1}}<\infty\right\},$$

equipped with the norm $\|u\|=\sup_{t\ge 0}\frac{|u(t)|}{1+t^{q-1}}$. Denote by $L^1(\mathbb{R}_+,\mathbb{R})$ the Banach space of Lebesgue integrable functions from $\mathbb{R}_+$ into $\mathbb{R}$ with the norm $\|y\|_{L^1}=\int_0^\infty|y(t)|\,dt$. Define the integral operator $T:E\to E$ by

$$Tu(t)=t^{q-1}u(1)+\frac{1}{\Gamma(q)}\int_0^{+\infty}H(t,s)f(s,u(s))\,ds, \tag{3.1}$$

and the corresponding perturbed operator $T_c:E\to E$ by

$$T_cv(t)=\frac{1}{\Gamma(q)}\int_0^{+\infty}H(t,s)f\left(s,v(s)+cs^{q-1}\right)ds. \tag{3.2}$$

Theorem 1. Assume that there exist nonnegative functions $g,h,\in L^1(\mathbb{R}_+,\mathbb{R}_+)$ and $0\le\alpha<1$ such that

$$\left|f(t,\left(1+t^{q-1}\right)x)\right|\le h(t)\,|x|^\alpha+g(t),(t,x)\in\mathbb{R}_+\times\mathbb{R}, \tag{3.3}$$

$$\|h\|_{L^1}<\Gamma(q). \tag{3.4}$$

Then the map T_c has at least one fixed point $v^*\in E$.

We apply Schaefer fixed point theorem to prove Theorem 1.

Theorem 2. Let A be a completely continuous mapping of a Banach space X into it self, such that the set $\{x\in X: x=\lambda Ax, 0<\lambda<1\}$ is bounded, then A has a fixed point.

We recall that a continuous mapping T from a subset M of a normed space X into an other normed space Y is called completely continuous, iff T maps bounded subset of M into relatively compact subset of Y. To prove that T_c is completely continuous, we need the following compactness criterion due to Corduneanu [4]:

Lemma 4. Let

$$V = \{u \in C(\mathbb{R}_+, \mathbb{R}), \|u\| < l, \text{ where } l > 0\}, V(t) = \left\{ \frac{u(t)}{1 + t^{q-1}}, u \in V \right\}.$$

V is relatively compact in E, if $V(t)$ is equicontinuous on any finite subinterval of $\mathbb{R}_+$ and equiconvergent at ∞, that is for any $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) > 0$ such that $\left| \frac{u(t_1)}{1+t_1^{q-1}} - \frac{u(t_2)}{1+t_2^{q-1}} \right| < \varepsilon$, $\forall u \in V, t_1, t_2 \geq \eta$, (uniformly according to u).

Proof of Theorem 1. Let us prove that T_c is a completely continuous mapping, indeed:

1- T_c is continuous: Let $(v_n)_{n \in N} \in E$ be a convergent sequence to v in E. Let $r_1 > \max(\|v\|_\infty, \sup \|v_n\|_\infty)$ and remarking that $H(t, s)$ is continuous according to both variables s, t on $\mathbb{R}_+$, nonnegative and $0 \leq \frac{H(t,s)}{1+t^{q-1}} \leq 1$, then we get

$$\begin{aligned}
&\left| \frac{T_c v(t) - T_c v_n(t)}{1 + t^{q-1}} \right| \\
&\leq \frac{1}{\Gamma(q)} \int_0^\infty \frac{H(t,s)}{1 + t^{q-1}} \left| f\left(s, v(s) + cs^{q-1}\right) - f\left(s, v_n(s) + cs^{q-1}\right) \right| ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(q)} \int_0^\infty \left| f\left(s, \frac{\left(1 + s^{q-1}\right)\left(v(s) + cs^{q-1}\right)}{1 + s^{q-1}}\right) \right| \\
&\quad + \left| f\left(s, \frac{\left(1 + s^{q-1}\right)\left(v_n(s) + cs^{q-1}\right)}{1 + s^{q-1}}\right) \right| ds \\
&\leq \frac{1}{\Gamma(q)} \left[\|h\|_{L^1} \left(\|v\| + |c|\right)^\alpha + \left(\|v_n\| + |c|\right)^\alpha + 2\|g\|_{L^1} \right] < \infty.
\end{aligned}$$

Using Lebesgue dominated convergence Theorem and the fact that f is continuous we get when $n \to \infty$

$$\begin{aligned}
&\|T_c v_n - T_c v\| \\
&\leq \frac{1}{\Gamma(q)} \sup_{t \geq 0} \int_0^\infty \frac{H(t,s)}{1 + t^{q-1}} \left| f\left(s, v(s) + cs^{q-1}\right) - f\left(s, v_n(s) + cs^{q-1}\right) \right| ds \to 0.
\end{aligned}$$

2- T is relatively compact: Let $B_r = \{v \in E, \|v\| \leq r\}$, first let us show that $T_c B_r$ is uniformly bounded. Let $v \in B_r$ then

$$\begin{aligned}
\frac{|T_c v|}{1 + t^{q-1}} &\leq \frac{1}{\Gamma(q)} \int_0^\infty \left| f\left(s, v(s) + cs^{q-1}\right) \right| ds \\
&= \frac{1}{\Gamma(q)} \int_0^\infty \left| f\left(s, \frac{\left(v(s) + cs^{q-1}\right)}{1 + s^{q-1}} \left(1 + s^{q-1}\right)\right) \right| ds \\
&\leq \frac{1}{\Gamma(q)} \left(\|h\|_{L^1} \left(\|v\| + |c|\right)^\alpha + \|g\|_{L^1} \right).
\end{aligned}$$

Consequently $\|T_c v\| \leq \frac{1}{\Gamma(q)} \left(\|h\|_{L^1} (r + |c|)^\alpha + \|g\|_{L^1} \right) < \infty$, thus $T_c B_r$ is uniformly bounded.

3- Now we show that T_cB_r is equicontinuous on any compact interval of $\mathbb{R}_+$. Let $v \in B_r$, $\forall t_1, t_2 \in [a,b]$, $0 \le a < b < \infty$, $t_1 \le t_2$, then

$$\begin{aligned}
&\left|\frac{T_c v(t_2)}{1+t_2^{q-1}} - \frac{T_c v(t_1)}{1+t_1^{q-1}}\right| \\
&\le \frac{1}{\Gamma(q)} \int_0^{\infty} \left|\frac{H(t_2,s)}{1+t_2^{q-1}} - \frac{H(t_1,s)}{1+t_1^{q-1}}\right| \left|f\left(s, v(s) + cs^{q-1}\right)\right| ds \\
&\le \frac{1}{\Gamma(q)} \int_0^{\infty} \frac{|H(t_2,s) - H(t_1,s)|}{1+t_2^{q-1}} \left|f\left(s, v(s) + cs^{q-1}\right)\right| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_0^{\infty} \frac{H(t_1,s)\left(t_2^{q-1} - t_1^{q-1}\right)}{\left(1+t_1^{q-1}\right)\left(1+t_2^{q-1}\right)} \left|f\left(s, v(s) + cs^{q-1}\right)\right| ds \\
&\le \frac{2\left(t_2^{q-1} - t_1^{q-1}\right)}{\Gamma(q)\left(1+a^{q-1}\right)} \left(\|h\|_{L^1} (\|v\| + |c|)^{\alpha} + \|g\|_{L^1}\right),
\end{aligned}$$

that converges uniformly to zero as $t_1 \to t_2$. Thus T_c is equicontinuous on the compact $[a,b]$.

4- Let us show hat T_c is equiconvergent at ∞. Since

$$\int_0^{\infty} \left|f\left(s, v(s) + cs^{q-1}\right)\right| ds \le \|h\|_{L^1} (\|v\| + |c|)^{\alpha} + \|g\|_{L^1} < \infty,$$

then $\lim_{t\to+\infty} \left|\frac{T_c v(t)}{1+t^{q-1}}\right| = 0$, consequently T is equiconvergent at ∞. Thus T_c is completely continuous.

Now, let us prove that the set $\{v \in E : v = T_c v, 0 < \lambda < 1\}$ is bounded. Indeed for $\lambda \in (0,1)$ such that $v = \lambda T_c(v)$, we have $v(t) = \frac{\lambda}{\Gamma(q)} \int_0^{\infty} H(t,s) f(s, v(s) + cs^{q-1}) ds$. Using assumptions (3.3), we get

$$\begin{aligned}
\left|\frac{v(t)}{1+t^{q-1}}\right| &= \frac{\lambda}{\Gamma(q)} \int_0^{+\infty} \frac{H(t,s)}{1+t^{q-1}} f(s, \frac{v(s) + cs^{q-1}}{1+s^{q-1}} \left(1+s^{q-1}\right)) ds \\
&\le \frac{1}{\Gamma(q)} \left[\|h\|_{L^1} (\|v\| + |c|)^{\alpha} + \|g\|_{L^1}\right],
\end{aligned}$$

thus,

$$\|v\| \le \frac{1}{\Gamma(q)} \left[\|h\|_{L^1} (\|v\| + |c|)^{\alpha} + \|g\|_{L^1}\right], \tag{3.5}$$

using some ideas from analysis and in view of (3.4), we get

$$\|v\| \le \frac{1}{\Gamma(q) - \|h\|_{L^1}} \left[\|h\|_{L^1} (|c| + 1) + \|g\|_{L^1}\right]. \tag{3.6}$$

Hence v is bounded independently of λ. Schaefer fixed point theorem implies T_c has at least a fixed point. Thus equation

$$v(t) = \frac{1}{\Gamma(q)} \int_0^{+\infty} H(t,s) f\left(s, v(s) + cs^{q-1}\right) ds \tag{3.7}$$

has at least one solution in E. The proof is completed.

The uniqueness result is given by the following theorem:

Theorem 3. Assume there exists a nonnegative function $k \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ such that for all $x, y \in \mathbb{R}, t \in \mathbb{R}_+$ one has

$$\left|f(t, \left(1+t^{q-1}\right)x) - f(t, \left(1+t^{q-1}\right)y)\right| \leq k(t)\left|x-y\right|, \tag{3.8}$$

$$\|k\|_{L^1} < \Gamma(q). \tag{3.9}$$

Then T_c has a unique fixed point v_c^*in E.

Proof. Let v and $w \in E$, then by (3.8) we get

$$\left|\frac{T_cv(t) - T_cw(t)}{(1+t^{q-1})}\right| \leq \frac{1}{\Gamma(q)}\int_0^{+\infty} \frac{H(t,s)}{(1+t^{q-1})} \times$$

$$\left|f(s, \frac{v(s)+cs^2}{(1+s^{q-1})}\left(1+s^{q-1}\right)) - f(s, \frac{w(s)+cs^2}{(1+s^{q-1})}\left(1+s^{q-1}\right))\right| ds$$

$$\leq \frac{1}{\Gamma(q)}\int_0^{+\infty} k(s)\left|\frac{v(s)-w(s)}{(1+s^{q-1})}\right| ds \leq \frac{\|v-w\|\,\|k\|_{L^1}}{\Gamma(q)},$$

thus $\|T_cv - T_cw\| \leq \frac{\|v-w\|\|k\|_{L^1}}{\Gamma(q)} = l\,\|v-w\|$, where $l = \frac{\|k\|_{L^1}}{\Gamma(q)}$. The assumption (3.9) implies that $l < 1$, so the Banach contraction principle ensure the uniqueness of the fixed point. The proof is completed.

Let us remark that under the assumptions of Theorem 3, the map $\Psi : \mathbb{R} \to E$, $\Psi(c) = v_c^*$ is continuous. Moreover the map $\Lambda : \mathbb{R} \to \mathbb{R}$, $\Lambda = \Phi \circ \Psi, \Lambda(c) = v_c^*(1)$ is also continuous, where $\Phi : E \to \mathbb{R}$, $\Phi(v) = v(1)$ and v_c^* is the unique fixed point of T_c. One can write $\Lambda(c)$ as

$$\Lambda(c) = \frac{1}{\Gamma(q)}\int_0^1 \left[1-(1-s)^{q-1}\right] f(s, v_c^*(s) + cs^{q-1})ds \tag{3.10}$$

$$+\frac{1}{\Gamma(q)}\int_1^{\infty} f(s, v_c^*(s) + cs^{q-1})ds.$$

Let us show that the problem (1.1)-(1.2) is solvable.

Theorem 4. Under the assumptions of Theorems 1 and 3 and if

$$lim_{u\to\pm\infty} f\left(t, \left(1+t^{q-1}\right)u\right) = \pm\infty$$

uniformly on $\mathbb{R}_+$ then the problem (1.1)-(1.2) has at least one solution in E.

Proof.

The condition $lim_{u\to\pm\infty} G(u) = \pm\infty$, is assumed to avoid the case $f(t, \left(1+t^{q-1}\right)u(t)) = y(t)$ where the problem may have no solution (in the case $\int_0^{+\infty} y(t)\,dt - \int_0^1 (1-t)\,y(t)\,dt \neq 0$). From (3.5) we obtain $lim_{c\to+\infty} \frac{\|v_c^*\|}{c} = 0$, then $\left(v_c^*(t) + ct^{q-1}\right)$ growths asymptotically as c and uniformly in t. Taking into account that $lim_{u\to\pm\infty} f\left(t, \left(1+t^{q-1}\right)u\right) = \pm\infty$, then by passing to the limit in (3.10) it yields $lim_{c\to\pm\infty}\Lambda(c) = \pm\infty$, consequently there exists $c^* \in \mathbb{R}$ such that $\Lambda(c^*) = 0$, thus $c^* = u_{c^*}(1)$ and hence $u_{c^*}(t) = v_{c^*}^*(t) + t^{q-1}c^*$ is a solution of the nonlinear problem (1.1)-(1.2). The proof is completed.

Corollary 1. Under the assumptions of Theorems 1 and 3 and if there exist two functions $G \in C(\mathbb{R}, \mathbb{R})$ and $H \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ such that $f\left(t, \left(1+t^{q-1}\right)u\right) = F(t)G(u)$, $lim_{u\to\pm\infty} G(u) = \pm\infty$,$\int_0^{+\infty} H(1,s)F(s)ds < \infty$, then the problem (1.1)-(1.2) has at least one solution in E.

Example 1. The following fractional boundary value problem

$$\begin{cases} D_{0^+}^{\frac{5}{2}} u(t) = \frac{(\exp(-t))u^{\frac{7}{3}}}{\left(1+t^{\frac{3}{2}}\right)^{\frac{1}{3}}\left(\left(1+t^{\frac{3}{2}}\right)^2+u^2\right)}, \\ u''(0) = D_{0^+}^{q-2} u(0) = 0, \quad D_{0^+}^{q-1} u(\infty) = \Gamma\left(\frac{5}{2}\right) u(1), \end{cases} \tag{3.11}$$

is solvable in E.

Proof. We have $q = \frac{5}{2}$ and $f(t,u) = \frac{(\exp(-t))u^{\frac{7}{3}}}{\left(1+t^{\frac{3}{2}}\right)^{\frac{1}{3}}\left(\left(1+t^{\frac{3}{2}}\right)^2+u^2\right)}$

$$\left|f(t, \left(1+t^{\frac{5}{2}}\right)u)\right| \leq (\exp(-t))\, |u|^{\frac{1}{3}} \leq h(t)\, |u|^{\frac{1}{3}} + g(t),$$

where $h(t) = \exp(-t)$ and $g(t) = 0$, some calculus give $\|h\|_{L^1} = 1 < \Gamma(q) = 1.3293$. Applying Theorem 1 we conclude that the map T_c has at least one fixed point $v^* \in E$. Now we have

$$\left|f(t, \left(1+t^{\frac{3}{2}}\right)x) - f(t, \left(1+t^{\frac{3}{2}}\right)y)\right| \leq \exp(-t) \left|\frac{x^{\frac{7}{3}}}{1+x^2} - \frac{y^{\frac{7}{3}}}{1+y^2}\right|,$$
$$\leq 0.8 \exp(-t)\, |x-y| = k(t)\, |x-y|,$$

where $k(t) = 0.8 \exp(-t)$, we have $\|k\|_{L^1} = 0.8 < \Gamma(q) = 1.3293$. In view of Theorem 3, T_c has a unique fixed point v_c^* in E. One can check that $f\left(t, \left(1+t^{\frac{3}{2}}\right)u\right) = (\exp(-t)) \frac{u^{\frac{7}{3}}}{(1+u^2)} = F(t)G(u)$ and $\lim_{u\to\pm\infty} G(u) = \lim_{u\to\pm\infty} \frac{u^{\frac{7}{3}}}{(1+u^2)} = \pm\infty$. From Corollary 1 we conclude that the problem (3.11) is solvable in E.

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