Journal of Fractional Calculus and Applications
Vol. 8(1) Jan. 2017, pp. 142-158.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

APPROXIMATE CONTROLLABILITY OF NONLINEAR DELAY EVOLUTION INTEGRODIFFERENTIAL SYSTEMS

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Abstract. In this paper the approximate controllability of nonlinear evolution delay integrodifferential systems with preassigned responses is studied. These controllability results are for nonlinear systems that are not associated with linear systems and no compactness assumption is imposed.

1. Introduction

Controllability of the nonlinear systems in infinite dimensional spaces has been extensively studied. Several authors [[1], [5], [8]] have studied the concept of exact controllability for systems represented by nonlinear evolutions equations, in which the authors have effectively used fixed point technique. From the mathematical point of view, the problems of exact and approximate controllability are to be distinguished. In infinite-dimensional spaces the concept of exact controllability is usually too strong and, indeed has limited applicability [17]. Approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications [[9], [12]]. Therefore, it is necessary to study the weaker concept of controllability, namely approximate controllability for nonlinear integrodifferential systems.

Kartsatos and Mabry [16] introduced a new type of controllability concept for the following system

$$x'(t) + A(t)x(t) = B(t)u(t),$$

where

$$A(t) : D(A) \subset X \to X, \quad B(t) : D(B) \subset X \to X, \quad t \in [0, a]$$

are nonlinear operators with constant domains $D(A), D(B)$ and where $0 \in D(A)$. In the same work they also discussed the LS-controllability of the functional evolution system

$$x'(t) + A(t, x_t)x(t) = u'(t) + B(t)u(t).$$

Kaplan and Kartsatos [14] have studied the K-controllability of nonlinear evolution systems with preassigned responses, where as Kartsatos and Liang in [15]

2010 *Mathematics Subject Classification.* 34A34; 34G20; 34K30.

Key words and phrases. Approximate controllability, nonlinear delay integrodifferential systems, preassigned responses.

Submitted Aug. 6, 2016.

used more general preassigned responses than in [14]. Subalakshmi and Balachandran [23] discussed about the approximate controllability properties of nonlinear stochastic impulsive integrodifferential and neutral functional stochastic impulsive integrodifferential equations in Hilbert spaces.

Several authors [[2], [3], [7]] have discussed the approximate controllability of nonlinear evolution systems with preassigned responses. Recently, Muthukumar and Balasubramaniam [19] studied the approximate controllability of nonlinear stochastic evolution time-varying delay systems of the form

$$\begin{aligned} d(x(t)) - A(t)x(t)dt &= f(t, x(t), x(\alpha_1(t)), \cdots, x(\alpha_n(t)), u(t))dt \\ &\quad + g(t, x(t), x(\alpha_1(t)), \cdots, x(\alpha_n(t)), u(t))dw(t), \\ & t \in J = [t_0, b], \quad x(t) = x_0, \quad t \le t_0 \end{aligned}$$

with preassigned responses. The necessary conditions for controllability results of nonlinear systems are not associated with linear systems and no compactness assumptions are imposed. Rykaczewski [22] studied the problem of approximate controllability of semilinear differential inclusion by using resolvent of controllability Grammian operator and fixed point theorem, assuming that semigroup generated by the linear part of the inclusion is compact and under the assumption that the corresponding linear system is approximately controllable.

The outline of this paper is as follows: In Section 2, the nonlinear evolution delay integrodifferential systems are described and further it contains basic notations, definitions and some preliminary results. In Section 3, sufficient conditions for the approximate controllability for the nonlinear evolution delay integrodifferential systems are discussed with preassigned responses. Approximate Controllability for the Implicit Delay Systems are discussed in Section 4. Finally, we provide an example to demonstrate the effectiveness of our method.

2. Preliminaries

The approximate controllability problems for linear and nonlinear systems with preassigned responses is considered in few literature . In order to fill this gap, this paper studies the approximate controllability of the following nonlinear integrodifferential equation of the form

$$\begin{aligned} & x'(t) - A(t)x(t) \\ &\quad = f\Big(t, x(\delta_1(t)), \int_{t_0}^{t} g(t, s, x(\delta_2(s)), \int_{t_0}^{s} k(s, \tau, x(\delta_3(\tau)))d\tau)ds, u(t)\Big), \quad t \ge t_0, \\ & x(t) = x_0, \quad t \le t_0, \end{aligned} \tag{1}$$

and the more general delay evolution integrodifferential systems

$$\begin{aligned} & L(x(t), x'(t)) - A(t)x(t) \\ &\quad = f\Big(t, x(\delta_1(t)), \int_{t_0}^{t} g(t, s, x(\delta_2(s)), \int_{t_0}^{s} k(s, \tau, x(\delta_3(\tau)))d\tau)ds, u(t)\Big), \quad t \ge t_0, \\ & x(t) = x_0, \quad t \le t_0, \end{aligned} \tag{2}$$

where $A(t)$ is a linear operator on a Hilbert space H for each t, $k : \Delta \times H \to H$ and $g : \Delta \times H^2 \to H$ are nonlinear functions, and f is a nonlinear function from $[t_0, T] \times H^3$ to H, the delays $\delta_1, \delta_2, \delta_3$ are continuous functions on R, while L is an operator from H^2 to H, and $\Delta = \{(t,s) : t_0 \le s \le t \le T\}$. Motivation for these kind of equation can be found in [18].

Let H be a separable Hilbert space with norm $\|\cdot\|$ and inner product $\prec \cdot,\cdot \succ$, $I = [t_0, T]$, $I_\delta = I \cup R(\delta)$, and $R(\delta)$ the range of $\delta = \max\{\delta_1(t), \delta_2(t), \delta_3(t)\}$. Let $f : I \times H \times H \to H$ be a nonlinear operators, measurable in the first terms and continuous in the last two terms; the delays $\delta_i : I \to R$ are bounded continuous function; $D = D(A(t))$, the domain of $A(t)$, is independent of t and dense in H; $x_0 \in D$ and $A(t) : D \subset H \to H$ is linear. Denote the space of all continuous functions x from I to H with the usual maximum norm $\|\cdot\|_c$ by $C(I,H)$, and the space of all square-integrable functions with the usual L^2 norm $\|\cdot\|_L$ and inner product $\prec \cdot,\cdot \succ_L$ by $L^2(I,H)$. The closure and boundary of any subset Ω are denoted by $\overline{\Omega}$ and $\partial\Omega$, respectively. For system (1) assume that $A(t)$ generates an evolution system $\{E(t,s)\}$ [21], and for system (2) assume only that $A(\cdot)x \in L^2(I,H)$ for each $x \in H$.

Definition 2.1.

: (i) For a given $u \in L^2(I,H)$, a function $x_u \in C(I_\delta, H)$ is said to be a *solution* of (1) (or (2)) on I if it satisfies (1) (or (2)) almost everywhere on I.

: (ii) $x_u \in C(I_\delta, H)$ is said to be a *mild solution* of (1) on I if

$$\begin{aligned} x_u(t) &= E(t,t_0)x(t_0) + \int_{t_0}^{t} E(t,s) f\Big(s, x_u(\delta_1(s)), \int_{t_0}^{s} g(s,\tau,x_u(\delta_2(\tau)), \\ &\qquad \int_{t_0}^{\tau} k(\tau,\theta,x_u(\delta_3(\theta)))d\theta)d\tau, u(t)\Big)ds, \quad (3) \\ x_u(t) &= x_0, \quad t \in I_\delta, \quad t \le t_0 \end{aligned}$$

A mild solution of (2) is defined similarly.

Definition 2.2. The mapping $u \to S(u)$ defined by $S(u) = \{x \in C(I_\delta, H) : x$ is a mild solution of (1) or (2) for some $u \in L^2(I,H)\}$ is said to be the *solution mapping* of (1) (or (2)).

Note that S is generally a set-valued mapping and $S(u)$ may be empty for some u. Several authors [[6], [10], [11], [13], [20]] have assumed S to be a well-defined continuous single-valued operator.

Definition 2.3.

: (i) The set $R_T(x_0) = \{x(T) : x \in S(u)$ for $u \in L^2(I,H)\}$ is called the *reachable set* of (1) from the initial state x_0.

: (ii) If $R_T(x_0) = H$, then system (1) is called *exactly controllable.*

: (iii) If $\overline{R_T(x_0)} = H$, then system (1) is called *approximately controllable.*

Definition 2.4. An operator $N : D(N) \subset H \to H$ is called *strongly monotone* (or *monotone*) if there exists $\beta > 0$ (or $\beta = 0$) such that

$$\prec Nx - Ny, x - y \succ \ \ge \ \|x-y\|^2, \text{ for every } \ x, y \in D(N).$$

The operator N is called *hemicontinuous* if $x \in D(N), h \in H, t > 0, x + th \in D(N)$, and if $t \to 0 \Rightarrow N(x+th) \xrightarrow{w} N(x)$; here $\xrightarrow{w}$ means weak convergence in H.

Let X be a Banach space and let $\Omega \subset X$ be a subset. The Hausdorff measure of noncompactness of Ω is defined by

$$\psi(\Omega) = \inf\{\epsilon > 0 \ : \ \Omega \text{ has a finite } \epsilon\text{-net}\}.$$

Lemma 2.1.[4] Suppose that Ω is a bounded and equicontinuous subset of $C(I, H)$, then

$$\psi(\Omega) = \sup_{t\in I} \psi(\Omega(t)).$$

Lemma 2.2.[4] Let X be a Banach space and $\Omega \subset X$ be an open and bounded subset with $0 \in \Omega$. Suppose that $f : \overline{\Omega} \to X$ is continuous and there exists $k \in [0, 1)$ such that

$$\psi(f(\Omega)) \leq k\psi(\Omega) \ \text{ for all bounded subsets } \ \Omega \subset X.$$

If $x \neq \lambda F x$ for $\lambda \in (0, 1)$ and $x \in \partial\Omega$, then f admits fixed points in $\overline{\Omega}$.

Lemma 2.3.[9] Let 2^H be the set of all subsets of H. Suppose that the multifunction $h : [a, b] \to 2^H$ is measurable and integrably bounded that is, $\sup\{\|y\| : y \in h(t) \leq \beta(t)\}$, with $\beta \in L^1([a, b], H)$. If $\psi(\cdot)$ is the Hausdorff measure of noncompactness on H, then $\psi(h(\cdot)) \in L^1([a, b], H)$ and

$$\psi(\int_D h(s)ds) \leq \int_D \psi(h(s))ds$$

for each measurable subset $D \subset [a, b]$.

Lemma 2.4.[10] Suppose that $N : H \to H$ is hemicontinuous and monotone. If there exists $r > 0$ such that

$$\prec Nx, x \succ \geq 0 \text{ for all } \ x \in \partial B_r, \quad \text{with} \ \ B_r = \{x \in H : \|x\| \leq r\},$$

then $Nx = 0$ has solutions in B_r.

3. Controllability of Integrodifferential Systems

In this section it is assumed that $t \to f(t, x, y, u)$ is almost everywhere continuous and ψ is the Hausdorff measure of noncompactness on H. Further, the following conditions are assumed to hold for f and A:

- (C1) $\psi(f(t, \Lambda_1, \Lambda_2, \Omega)) \leq h_1(t)\psi(\Omega)$, for every $t \in I$, all compact subsets $\Lambda_1, \Lambda_2 \subset H$, and bounded subsets $\Omega \subset H$. Here $h_1 \in L^2(I, R)$ is nonnegative.
- (C2) $k : \Delta \times H \to H$ is continuous and there exists constants $k_1, k_2 > 0$ such that

$$\|k(t, s, x_1) - k(t, s, x_2)\| \leq k_1\|x_1 - x_2\| \quad \text{and}$$

$$k_2 = \max\{\|k(t, s, 0)\| : (t, s) \in \Delta\}$$

- (C3) $g : \Delta \times H \times H \to H$ is continuous and there exists constants $k_3, k_4 > 0$ such that

$$\|g(t, s, x_1, y_1) - g(t, s, x_2, y_2)\| \leq k_3(\|x_1 - x_2\| + \|y_1 - y_2\|) \quad \text{and}$$

$$k_4 = \max\{\|g(t, s, 0, \int_0^s k(s, \tau, 0)ds)\| : (t, s) \in \Delta\}$$

and there exist continuous functions $a, b : \Delta \to I$ such that

$$\|g(t,s,x(\delta_2(s))), \int_{t_0}^{s} k(s,\tau,x(\delta_2(\tau))d\tau))\| \le a(t,s)[\|x(\delta_2(s))\| + \int_{t_0}^{s} b(s,\tau)\|x(\delta_3(\tau))d\tau\|]$$

and

$$\alpha = \sup_{t_0 \le t \le T} \int_{t_0}^{t} a(t,s)ds$$

$$\beta = \sup_{t_0 \le t \le T} \int_{t_0}^{t} \int_{t_0}^{s} a(t,s)b(s,\tau)d\tau ds$$

: (C4) There exist positive constants $a_i, i = 1,2,3,4$ such that for every $(t,x,y,u) \in I \times H^3$

$$\|f(t,x,y,u)\| \le a_1 + a_2\|x\| + a_3\|y\| + a_4\|u\|.$$

: (C5) For each collection of bounded subset $D_i \subset H$, there exists a constant $b_1 > 0$ and measurable functions $b_2, b_3 \in L^2(I,R)$ (each b_i may depend on D) such that

$$\prec f(t,x,y,u), u \succ \ \ge \ b_1\|u\|^2 - b_2(t)\|u\| - b_3(t)$$

for every $t \in I, u \in H, x_i \in D_i,$ for $i = 1,2,3$.

: (C6) There exists $h_2 \in L^2(I,R)$ such that

$$\|f(t,x_1,y_1,u) - f(t,x_2,y_2,u)\| \le h_2(t)\Big(\|x_1 - x_2\| + \|y_1 - y_2\|\Big)$$

for every $t \in I, u, x_1, x_2, y_1, y_2 \in H$.

: (C7) $\delta_i \in C^1(I,R), \delta_i(t) \le t, \delta_i'(t) \ge k_i > 0$, for every $t \in I,\ i = 1,2,3$.

: (C8) There exists a uniformly bounded function $c(t)$ such that

$$\prec -A(t)x, x \succ \ \ge \ c(t)\|x\|^2$$

for every $t \in I, x \in D$.

: (C9) $A(\cdot)x \in L^2(I,H)$ for every $x \in D$.

: (C10) $t \to E(t,s)$ is continuous in the uniform operator topology for each $s \le t$.

Theorem 3.1. Suppose that the conditions (C1),(C4),(C7) and (C10) are satisfied. Then, for given functions $x \in C(I_\delta, H), y \in C(I,H)$ and constant $n > 0$, the integral equation

$$\begin{aligned} u(t) &= y(t) + \int_{t_0}^{t} E(t,s) f\Big(s, x(\delta_1(s)), \int_{t_0}^{s} g(s,\tau,x(\delta_2(\tau)), \\ &\qquad \int_{t_0}^{\tau} k(\tau,\theta,x(\delta_3(\theta)))d\theta)d\tau, u(t)\Big)ds, \end{aligned} \tag{4}$$

admits solutions in $C(I,H)$.

Proof. Since $E(t,s)$ is strongly continuous, it can be assumed that $\|E(t,s)\| \le M$ for some constant $M > 0$. Also assume that

$$M_1 = \Big[\int_{t_0}^{T} h_1^2(s)ds\Big]^{\frac{1}{2}}.$$

Let $a > 0$ be such that

$$\frac{nMM_1}{\sqrt{2a}} < 1.$$

For each $u \in C(I, H)$, let

$$\|u\|_a = \max_{t\in I} \|u(t)\| exp(-at),$$

$$\begin{aligned}(Nu)(t) &= y(t) + \int_{t_0}^{t} E(t,s) f\Big(s, x(\delta_1(s)), \int_{t_0}^{s} g(s,\tau, x(\delta_2(\tau)), \\ &\qquad \int_{t_0}^{\tau} k(\tau,\theta, x(\delta_3(\theta)))d\theta)d\tau, u(t)\Big)ds.\end{aligned}$$

Then $\|\cdot\|_a$ is a norm on $C(I, H)$ and is equivalent to $\|\cdot\|_c$. The space $C(I, H)$ endowed with this new norm is denoted by $C_a(I, H)$. By condition (C4) and the Lebesgue dominated convergence theorem, it follows that N is continuous from $C_a(I, H)$ to $C_a(I, H)$.

Denote the Hausdorff measure of noncompactness on $C_a(I, H)$ by ϕ. Then it is clear that for each $\Omega \in C_a(I, H)$ and $t \in I$,

$$\psi(\Omega(t)) \leq \phi(\Omega) exp(at).$$

Assume that $\Omega \subset C_a(I, H)$ is a bounded subset. Since

$$\begin{aligned}N(\Omega)(t) &= y(t) + n\int_{t_0}^{t} E(t,s) f\Big(s, x(\delta_1(s)), \int_{t_0}^{s} g(s,\tau, x(\delta_2(\tau)), \\ &\qquad \int_{t_0}^{\tau} k(\tau,\theta, x(\delta_3(\theta)))d\theta)d\tau, u(t)\Big)ds, \\ x_u(t) &= x_0, \quad t \in I_{\delta_i}, \quad t \leq t_0,\end{aligned}$$

by the definition of ψ, condition (C1) and Lemma 2.3, it follows that

$$\begin{aligned}\psi(N(\Omega)(t)) &= \psi\Big(y(t) + n\int_{t_0}^{t} E(t,s) f\Big(s, x(\delta_1(s)), \int_{t_0}^{s} g(s,\tau, x(\delta_2(\tau)), \\ &\qquad \int_{t_0}^{\tau} k(\tau,\theta, x(\delta_3(\theta)))d\theta)d\tau, u(t)\Big)ds\Big) \\ &\leq n\int_{t_0}^{t} \psi\Big(E(t,s) f\Big(s, x(\delta_1(s)), \int_{t_0}^{s} g(s,\tau, x(\delta_2(\tau)), \\ &\qquad \int_{t_0}^{\tau} k(\tau,\theta, x(\delta_3(\theta)))d\theta)d\tau, u(t)\Big)\Big)ds \\ &\leq n\int_{t_0}^{t} M h_1(s)\psi(\Omega(s))ds \\ &\leq nMM_1\phi(\Omega)\Big(\int_{t_0}^{t} exp(2as)ds\Big)^{\frac{1}{2}} \\ &\leq nMM_1\phi(\Omega)\frac{1}{\sqrt{2a}} exp(at),\end{aligned}$$

that is,

$$\psi\Big(N(\Omega)(t)\Big) \leq \frac{n}{\sqrt{2a}} MM_1\phi(\Omega) exp(at).$$

Therefore

$$\begin{aligned}\psi\Big(exp(-at)N(\Omega)(t)\Big) &= \psi\Big(N(\Omega)(t)\Big)exp(-at)\\ &\leq \frac{n}{\sqrt{2a}}MM_1\phi(\Omega).\end{aligned} \tag{5}$$

From (C4) and (C10) it follows that $N(\Omega)$ is bounded and equicontinuous in $C(I,H)$. Therefore

$$[N(\Omega)]_a = \{v \in C(I,H) : v(t) = u(t)exp(-at), \text{ for some } u \in N(\Omega)\}$$

is also bounded and equicontinuous in $C(I,H)$. So, by Lemma 2.1

$$\psi\Big([N(\Omega)]_a\Big) = \sup_{t\in I}\psi\Big([N(\Omega)]_a(t)\Big) = \sup_{t\in I}\psi\Big(exp(-at)N(\Omega)(t)\Big).$$

From the definition of ψ and ϕ, it follows that

$$\psi\Big([N(\Omega)]\Big)_a = \phi(N(\Omega)).$$

Since t in (5) is arbitrary,

$$\phi(N(\Omega)) \leq \frac{n}{\sqrt{2a}}MM_1\phi(\Omega).$$

Suppose there exists $u \in C_a(I,H), \lambda \in [0,1]$ such that $u = \lambda Nu$. Then, note that $\alpha_i, i = 1,2,3$ are bounded and so by (C2)-(C4) and (C7)

$$\begin{aligned}\|u(t)\| &\leq \|y(t)\| + n\int_{t_0}^{t}\|E(t,s)\|\\ &\quad\times\|f\Big(s, x(\delta_1(s)), \int_{t_0}^{s} g(s,\tau,x(\delta_2(\tau))), \int_{t_0}^{\tau} k(\tau,\theta,x(\delta_3(\theta)))d\theta)d\tau, u(t)\Big)\|ds\\ &\leq n\|y(t)\| + n\int_{t_0}^{t} M\Big[a_1 + a_2\|x(\delta_1(s))\|\\ &\quad + a_3\int_{t_0}^{s}\|g(s,\tau,x(\delta_2(\tau))), \int_{t_0}^{s} k(\tau,\theta,x(\delta_3(\theta)))d\theta)\|d\tau + a_4\|u(s)\|\Big]ds\\ &\leq n[\|y\|_c + Ma_1T + Ma_2T\|x\| + Ma_3(\alpha+\beta)\|x\|] + nMa_4\int_{t_0}^{t}\|u(s)\|ds\end{aligned}$$

Here, $|I_\delta|$ means the measure of I_δ and $\|x\|_{\alpha_i}$ means the norm in $C(I_\delta, H)$.

From Gronwall's inequality, it follows that

$$\begin{aligned}\|u\|_a &\leq \|u\|_c\\ &\leq n\Big[\|y\|_c + Ma_1T + Ma_2T\|x\| + Ma_3(\alpha+\beta)\|x\|\Big]exp(nMa_4T),\end{aligned}$$

which means that $\{u \in C_a(I,H) : u = \lambda Nu, \lambda \in [0,1]\}$ is bounded. Let

$$K \geq \sup\Big\{\|u\|_a : u = \lambda Nu, \lambda \in [0,1], u \in C_a(I,H)\Big\}$$

be a given number. Then $u \neq \lambda Nu$ for every $\lambda \in [0,1], u \in C_a(I,H)$ with $\|u\|_a = K$. By Lemma 2.2 there exists $u \in C_a(I,H)$ satisfying (4). Hence the proof is completed.

Remark 3.1. If the condition (C1) is replaced by assuming the compactness of $E(t,s)$, then Theorem 3.1 is still true. Indeed, in this case the operator $K : C(I,H) \to C(I,H)$ defined by

$$(Ku)(t) = \int_{t_0}^{t} E(t,s)u(s)ds, u \in C(I,H)$$

is compact. Therefore, the operator N defined in Theorem 3.1 is compact.

Theorem 3.2. If (C1)–(C10) are satisfied, then the system (1) is approximately controllable.

Proof. Without loss of generality take $x_0 = 0$. Since $\overline{D} = H$, it need only be shown that $\overline{R_T(0)} \supset D$. So, let $x_T \in D$ and

$$\begin{aligned} x(t) &= \frac{t-t_0}{T-t_0} x_T, \qquad t \in I, \\ x(t) &= 0, \qquad t \in I_\delta, \quad t \le t_0. \end{aligned}$$

Then $x(T) = x_T$ and $x(t) \in D$ for each t. Consider the approximate equation

$$\begin{aligned} x(t) - \frac{1}{n}u_n(t) &= \int_{t_0}^{t} E(t,s) f\Big(s, x(\delta_1(s)), \int_{t_0}^{s} g(s,\tau,x(\delta_2(\tau)), \\ &\qquad \int_{t_0}^{\tau} k(\tau,\theta,x(\delta_3(\theta)))d\theta)d\tau, u(t)\Big)ds, \quad t \in I. \quad (6) \end{aligned}$$

By Theorem 3.1, for each integer n equation (6) has a solution $u_n \in C(I,H)$. Since f is almost everywhere continuous with respect to the first argument, $E(t,s)$ is strongly continuous and satisfies $(C10)$. By differentiating (6) it follows that

$$\begin{aligned} \frac{1}{T-t_0}x_T - \frac{1}{n}u_n'(t) &= f\Big(t, x(\delta_1(t)), \int_{t_0}^{t} g(t,s,x(\delta_2(s)), \int_{t_0}^{s} k(s,\tau,x(\delta_3(\tau)))d\tau)ds, u(t)\Big) \\ &\quad + A(t)\Big[x(t) - \frac{1}{n}u_n(t)\Big] \\ &= f\Big(t, x(\delta_1(t)), \int_{t_0}^{t} g(t,s,x(\delta_2(s)), \int_{t_0}^{s} k(s,\tau,x(\delta_3(\tau)))d\tau)ds, u(t)\Big) \\ &\quad + \frac{t-t_0}{T-t_0}A(t)x_T - \frac{1}{n}A(t)u_n(t). \end{aligned}$$

Let $w(t) = \frac{t-t_0}{T-t_0}A(t)x_T$. Then, by (C5) and (C8) it follows that

$$\begin{aligned} \frac{1}{T-t_0}x_T - \frac{1}{n}u_n'(t) &= f\Big(t, x(\delta_1(t)), \int_{t_0}^{t} g(t,s,x(\delta_2(s)), \int_{t_0}^{s} k(s,\tau,x(\delta_3(\tau)))d\tau)ds, u(t)\Big) \\ &\quad + w(t) - \frac{1}{n}A(t)u_n(t). \end{aligned}$$

Taking inner product with $u_n(t)$, it follows that

$$\begin{aligned}
&\prec \frac{1}{T-t_0}x_T, u_n(t) \succ - \prec \frac{1}{n}u_n'(t), u_n(t) \succ \\
&= \quad \prec f\Big(t, x(\delta_1(t)), \int_{t_0}^{t} g(t,s,x(\delta_2(s)), \int_{t_0}^{s} k(s,\tau,x(\delta_3(\tau)))d\tau)ds, u(t)\Big), u_n(t) \succ \\
&\quad + \prec w(t), u_n(t) \succ - \prec \frac{1}{n}A(t)u_n(t), u_n(t) \succ \\
&\geq \quad b_1\|u_n(t)\|^2 - b_2(t)\|u_n(t)\| - b_3(t) + \prec w(t), u_n(t) \succ + \frac{c(t)}{n}\|u_n(t)\|^2, \ a.e.
\end{aligned}$$

Here b_1 is a constant and b_2, b_3 are functions in (C5) related to the bounded subsets,

$$\begin{aligned}
D_1 &= \{x(\delta_1(t)) : t \in I\}, \\
D_2 &= \{\int_{t_0}^{t} g(t,s,x(\delta_2(s)), \int_{t_0}^{s} k(s,\tau,x(\delta_3(\tau)))d\tau) : t \in I\}.
\end{aligned}$$

By (C8)

$$c = \max_{t\in I} |c(t)| < \infty.$$

From the above inequality, we have

$$\begin{aligned}
b_1\|u_n(t)\|^2 \quad &\leq \quad \prec \frac{x_T}{T-t_0}, u_n(t) \succ - \frac{1}{n} \prec u_n'(t), u_n(t) \succ + b_2\|u_n(t)\| \\
&\quad + b_3(t) - \prec w(t), u_n(t) \succ - \frac{c(t)}{n}\|u_n(t)\|^2.
\end{aligned}$$

Integrating the above inequality yields

$$\begin{aligned}
\int_{t_0}^{T} b_1\|u_n(t)\|^2 dt \quad &\leq \quad \int_{t_0}^{T} \prec \frac{x_T}{T-t_0} - w(t), u_n(t) \succ dt \\
&\quad - \frac{1}{n}\int_{t_0}^{T} \prec u_n'(t), u_n(t) \succ dt + \int_{t_0}^{T} b_2(t)\|u_n(t)\|dt \\
&\quad + \int_{t_0}^{T} b_3(t)dt - \int_{t_0}^{T} \frac{c(t)}{n}\|u_n(t)\|^2 dt
\end{aligned}$$

and so,

$$\begin{aligned}
b_1\|u_n\|_L^2 \quad &\leq \quad \Big\|\frac{x_T}{T-t_0} - w\Big\|_L \|u_n\|_L - \frac{1}{2n}\|u_n(T)\|^2 \\
&\quad + \|b_2\|\|u_n\|_L + \int_{t_0}^{T} |b_3(t)|dt + \int_{t_0}^{T} \frac{|c(t)|}{n}\|u_n(t)\|^2 dt \\
&\leq \quad \Big[\frac{\|x_T\|_L}{T-t_0} + \|w\|_L + \|b_2\|\Big]\|u_n\|_L + \int_{t_0}^{T} |b_3(t)|dt + \frac{c}{n}\|u_n\|_L^2. \qquad (7)
\end{aligned}$$

This implies that $\{u_n\}$ is bounded in $L^2(I,H)$. Let $N_1 = \sup\|u_n\|_L < \infty$. By (C6), the solution map $S(u)$ of (1) is a single-valued continuous operator on $C(I,H)$. The claim is that $\{S(u_n)(s)\}$ is uniformly bounded and equicontinuous.

In fact, noting that $S(u_n)(t) = 0$ for $t \leq t_0$ and $\delta_i(t) \leq t$ for every $t \in I$,

$$\begin{aligned}
\|S(u_n)(t)\| &\leq \int_{t_0}^{t} \|E(t,s)\| \Big\| f\Big(s, S(u_n)(\delta_1(s)), \int_{t_0}^{s} g(s,\tau,S(u_n)(\delta_2(\tau)), \\
&\qquad \int_{t_0}^{\tau} k(\tau,\theta,S(u_n)(\delta_3(\theta)))d\theta)d\tau, u_n(s)\Big)\Big\| ds \\
&\leq M \int_{t_0}^{t} \Big[a_1 + a_2\|S(u_n)(\delta_1(s))\| \\
&\quad + a_3\|\int_{t_0}^{s} g(s,\tau,S(u_n)(\delta_2(\tau)), \int_{t_0}^{\tau} k(\tau,\theta,S(u_n)(\delta_3(\theta)))d\theta)d\tau\| \\
&\quad + a_4\|u_n(s)\|\Big] ds \\
&\leq M\Big(Ta_1 + \sqrt{T}a_4\|u_n\|_L\Big) + M\Big(a_2 + a_3(\alpha+\beta)\Big)\int_{t_0}^{t} \|S(u_n)(s)\| ds.
\end{aligned}$$

By Gronwall's inequality, it follows that

$$\begin{aligned}
\|S(u_n)(t)\| &\leq M\Big(Ta_1 + \sqrt{T}a_4\|u_n\|_L\Big)\exp\Big(\Big(a_2 + a_3(\alpha+\beta)\Big)MT\Big) \\
&= N_2 < \infty.
\end{aligned}$$

This shows that $\{S(u_n)(t)\}$ is uniformly bounded on I. Therefore, it is uniformly bounded by N_2 on I_δ. Now, let $t_1, t_2 \in I$ with $t_1 > t_2$. Then for every n

$$\begin{aligned}
&\|S(u_n)(t_1) - S(u_n)(t_2)\| \\
&\leq \int_{t_2}^{t_1} \|E(t_1,s)\| \Big\| f\Big(s, S(u_n)(\delta_1(s)), \int_{t_0}^{s} g(s,\tau,S(u_n)(\delta_2(\tau)), \\
&\qquad \int_{t_0}^{\tau} k(\tau,\theta,S(u_n)(\delta_3(\theta)))d\theta)d\tau, u_n(s)\Big)\Big\| ds \\
&\quad + \int_{t_0}^{t_2} \|E(t_1,s) - E(t_2,s)\| \Big\| f\Big(s, S(u_n)(\delta_1(s)), \int_{t_0}^{s} g(s,\tau,S(u_n)(\delta_2(\tau)), \\
&\qquad \int_{t_0}^{\tau} k(\tau,\theta,S(u_n)(\delta_3(\theta)))d\theta)d\tau, u_n(s)\Big)\Big\| ds \\
&\leq M(t_1 - t_2)\Big[a_1 + a_2N_2\Big] + Ma_3N_2[\alpha+\beta][t_1 - t_2] + Ma_4N_1\sqrt{t_1 - t_2} \\
&\quad + [\int_{t_0}^{t_2} \|E(t_1,s) - E(t_2,s)\|^2 ds]^{\frac{1}{2}}\Big(a_1 + a_2N_2\Big)\sqrt{T} \\
&\quad + a_3N_2[\alpha+\beta][\int_{t_0}^{t_2} \|E(t_1,s) - E(t_2,s)\|^2 ds]^{\frac{1}{2}}\sqrt{T} \\
&\quad + a_4N_1[\int_{t_0}^{t_2} \|E(t_1,s) - E(t_2,s)\|^2 ds]^{\frac{1}{2}}\sqrt{T}
\end{aligned}$$

Because of (C10), the equicontinuity of $\{S(u_n)\}$ follows. By (6) and (C6),

$$\begin{aligned}
\|S(u_n)(t)-x(t)\| &= \Big\| \int_{t_0}^{t} E(t,s)\Big[f\Big(s, S(u_n)(\delta_1(s)), \int_{t_0}^{s} g(s,\tau,S(u_n)(\delta_2(\tau)), \\
&\qquad \int_{t_0}^{\tau} k(\tau,\theta,S(u_n)(\delta_3(s)))d\theta)d\tau, u_n(s)\Big) \\
&\quad - f\Big(s, x(\delta_1(s)), \int_{t_0}^{s} g(s,\tau,x(\delta_2(\tau)), \\
&\qquad \int_{t_0}^{\tau} k(\tau,\theta,x(\delta_3(s)))d\theta)d\tau, u_n(s)\Big)\Big]ds\Big\| \\
&\le M\int_{t_0}^{t} h_2(s)\Big[\frac{1}{k}\|S(u_n)(s)-x(s)\| \\
&\quad + \int_{t_0}^{s}\|g(s,\tau,S(u_n)(\delta_2(\tau)), \int_{t_0}^{\tau} k(\tau,\theta,S(u_n)(\delta_3(s)))d\theta)d\tau \\
&\quad - g(s,\tau,x(\delta_2(\tau)), \int_{t_0}^{\tau} k(\tau,\theta,x(\delta_3(s)))d\theta)d\tau\|\Big]ds + \frac{1}{n}\|u_n(t)\| \\
&\le \frac{1}{n}\|u_n(t)\| + \frac{M}{k}\Big(1+k_3T(1+k_1T)\Big)\int_{t_0}^{t} h_2(s)\|S(u_n)(s)-x(s)\|ds.
\end{aligned}$$

Applying the Gronwall inequality, it follows that

$$\begin{aligned}
\|S(u_n)(t)-x(t)\| &\le \frac{1}{n}\|u_n(t)\|exp\Big[\frac{M}{k}\Big(1+k_3T(1+k_1T)\Big)\int_{t_0}^{t} h_2(s)ds\Big] \\
&\le \frac{1}{n}\|u_n(t)\|exp\Big[\frac{M}{k}\Big(1+k_3T(1+k_1T)\Big)\|h_2\|_L\Big].
\end{aligned}$$

Therefore,

$$\|S(u_n)-x\|_L \le \frac{1}{n}exp\Big[\frac{M}{k}\Big(1+k_3T(1+k_1T)\Big)\|h_2\|_L\Big]\|u_n\|_L \to 0 \quad \text{as } n\to\infty. \tag{8}$$

This means that $S(u_n) \to x$ in L_2. Therefore, there exists a subsequence $S(u_n)(t) \to x(t)$ a.e. in H. Let $t_m \in I$ with $t_m \to T$ be such that $S(u_n)(t_m) \to x(t_m)$ for each m. Since $\{S(u_n)\}$ is equicontinuous, for each $\epsilon > 0$ there exists m_0 such that

$$\|x(t_{m_0}) - x_T\| = \Big\|\frac{t_{m_0}-t_0}{T-t_0}x_T - x_T\Big\| = \frac{T-t_{m_0}}{T-t_0}\|x_T\| < \frac{\epsilon}{3},$$

$$\|S(u_n)(T) - S(u_n)(t_{m_0})\| < \frac{\epsilon}{3} \text{ for all } n.$$

For this m_0, equation (8) implies there exists $N > 0$ such that

$$\|S(u_n)(t_{m_0}) - x(t_{m_0})\| < \frac{\epsilon}{3} \text{ for all } n \ge N.$$

Hence,

$$\begin{aligned}
\|S(u_n)(T)-x_T\| &\le \|S(u_n)(T)-S(u_n)(t_{m_0})\| + \|S(u_n)(t_{m_0})-x(t_{m_0})\| \\
&\quad + \|x(t_{m_0})-x_T\| \\
&< \epsilon.
\end{aligned}$$

That is, $S(u_n)(T) \to x_T$ and $x_T \in \overline{R_T(0)}$.

Thus, $x_T \in D$ implies that $x_T \in \overline{R_T(0)}$. Therefore, $D \subset \overline{R_T(0)}$. But $\overline{R_T(0)} \subset D$. Combining these two inclusions yields $D = \overline{R_T(0)}$. Thus, the system (1) is approximately controllable. Hence the proof is completed.

Remark 3.2. The results of Theorem 3.2 remain valid when (C5) is replaced by the following assumption:

(C5′) There exists a constant $c_1 > 0$ and measurable functions $c_2, c_3 \in L^2(I, R)$ for each bounded subsets $D \subset H$ such that for every $t \in I, u \in H, x, y \in D$,

$$\prec f(t, x, y, u), u \succ \leq -c_1\|u\|^2 + c_2(t)\|u\| + c_3(t).$$

Theorem 3.3. If (C1)–(C4),(C5′),(C6)–(C10) are satisfied, then the system (1) is approximately controllable.

Proof. Let $x(t)$ be defined as in the proof of Theorem 3.2. By Theorem 3.1, for each n there exists $u_n \in C(I, H)$ such that

$$x(t) + \frac{1}{n}u_n(t) = \int_{t_0}^{t} E(t,s) f\Big(s, x(\delta_1(s)), \int_{t_0}^{s} g(s, \tau, x(\delta_2(\tau)), \int_{t_0}^{\tau} k(\tau, \theta, x(\delta_3(\theta)))d\theta)d\tau, u(t)\Big)ds.$$

Differentiating this equation yields

$$\begin{aligned}
&\frac{1}{T-t_0}x_T + \frac{1}{n}u_n'(t)\\
&= f\Big(t, x(\delta_1(t)), \int_{t_0}^{t} g(t, s, x(\delta_2(s), \int_{t_0}^{s} k(s, \tau, x(\delta_3(\tau)))d\tau)ds, u(t)\Big),\\
&\quad + A(t)\int_{t_0}^{t} E(t,s) f\Big(s, x(\delta_1(s)), \int_{t_0}^{s} g(s, \tau, x(\delta_2(\tau)),\\
&\qquad \int_{t_0}^{\tau} k(\tau, \theta, x(\delta_3(\theta)))d\theta)d\tau, u(t)\Big)ds,\\
&= f\Big(t, x(\delta_1(t)), \int_{t_0}^{t} g(t, s, x(\delta_2(s), \int_{t_0}^{s} k(s, \tau, x(\delta_3(\tau)))d\tau)ds, u(t)\Big)\\
&\quad + A(t)\Big[\frac{t-t_0}{T-t_0}x_T + \frac{1}{n}u_n(t)\Big].
\end{aligned}$$

Taking the inner product with $u_n(t)$, produces

$$\begin{aligned}
&\prec \frac{x_T}{T-t_0}, u_n(t) \succ + \frac{1}{n} \prec u_n'(t), u_n(t) \succ\\
&= \prec f\Big(t, x(\delta_1(t)), \int_{t_0}^{t} g(t, s, x(\delta_2(s), \int_{t_0}^{s} k(s, \tau, x(\delta_3(\tau)))d\tau)ds, u(t)\Big), u_n(t) \succ\\
&\quad + \frac{t-t_o}{T-t_0} \prec A(t)x_T, u_n(t) \succ - \frac{1}{n} \prec -A(t)u_n(t), u_n(t) \succ\\
&\leq -c_1\|u_n(t)\|^2 + c_2(t)\|u_n(t)\| + c_3(t) + \frac{t-t_0}{T-t_0} \prec A(t)x_T, u_n(t) \succ - \frac{c(t)}{n}\|u_n(t)\|^2.
\end{aligned}$$

By the same method as used to obtain (7), the same inequality is obtained with b_i replaced by c_i, i=1,2,3. Therefore $\|u_n\|_L$ is bounded. The rest of the proof is similar to that of Theorem 3.2 and hence omitted.

Theorem 3.4. Suppose that $E(t,s)$ is compact for every $t_0 \le s < t \le T$, then conditions (C2),(C5) [or (C5′)] and (C6)–(C9) imply the approximate controllability of system (1).

Proof. By Theorem 2.3.2 of [[19]] it follows that (C10) is satisfied. So, by Remark 3.1 the proof can be completed in the same way as in Theorem 3.2.

Remark 3.3. Obviously other preassigned responses can be taken in the above results to obtain the same conclusions. For example, $x(\cdot)$ can be defined as

$$x(t) = \frac{(t-t_0)^2}{(T-t_0)^2}(x_T - x_0) + x_0.$$

4. Controllability of Implicit Delay Systems

In this section the approximate controllability of the implicit system (2) is considered. Here, assume that $L : H \to H$ satisfies the condition

- **:** (C11) If $x : I_{\alpha_i} \to D$ is an affine function, then $L(x(\cdot), x'(\cdot)) \in L^2(I_{\alpha_i}, H)$.

Theorem 4.1. Suppose that conditions (C2),(C7),(C9) and either one of the following conditions are satisfied:

- **:** (C12) $u \to f(t,x,y,u)$ is monotone for each (t,x,y) and (C5) is satisfied, or
- **:** (C12′) $u \to -f(t,x,y,u)$ is monotone for any (t,x,y) and (C5′) is satisfied.

Then $D \subset R_T(x_0)$ for each $x_0 \in H$, and system (2) is approximately (or exactly) controllable provided $\overline{D} = H$ (or $D = H$).

Proof. Let $x_T \in D$ and

$$\begin{aligned} x(t) &= \frac{t-t_0}{T-t_0}(x_T - x_0) + x_0, \quad t \in I, \\ x(t) &= x_0, \quad t \le t_0. \end{aligned}$$

Then $L(x,x') \in L^2(I,H)$, $x(T) = x_T$, and $x(t) \in D$ for every $t \in I$. If (C10) holds, define an operator N on $L^2(I,H)$ by

$$\begin{aligned} Nu(t) &= f\Big(t, x(\delta_1(t)), \int_{t_0}^{t} g(t,s,x(\delta_2(s))), \int_{t_0}^{s} k(s,\tau,x(\delta_3(\tau)))d\tau)ds, u(t)\Big) \\ &\quad - L(x(t),x'(t)) + A(t)x(t) \\ &= f\Big(t, x(\delta_1(t)), \int_{t_0}^{t} g(t,s,x(\delta_2(s))), \int_{t_0}^{s} k(s,\tau,x(\delta_3(\tau)))d\tau)ds, u(t)\Big) \\ &\quad - L(x(t),x'(t)) + \frac{t-t_0}{T-t_0}A(t)(x_T - x_0) + A(t)x_0. \end{aligned}$$

Then conditions (C2)-(C4) and (C9),(C11) imply that N maps $L^2(I,H)$ to $L^2(I,H)$ and is hemicontinuous, since f is continuous in u. By (C12) it can easily be shown that N is monotone.

For each $u \in L^2(I, H)$, (C5) and (C7) imply that

$$
\begin{aligned}
\prec Nu, u \succ_L &= \int_{t_0}^{T} \prec Nu(t), u(t) \succ dt \\
&= \int_{t_0}^{T} \prec f\Big(t, x(\delta_1(t)), \int_{t_0}^{t} g(t, s, x(\delta_2(s)), \int_{t_0}^{s} k(s, \tau, x(\delta_3(\tau)))d\tau)ds, u(t)\Big), u(t) \succ dt \\
&\quad - \int_{t_0}^{T} \prec L(x(t), x'(t)), u(t) \succ dt + \int_{t_0}^{T} \prec \frac{t-t_0}{T-t_0} A(t)(x_T - x_0), u(t) \succ dt \\
&\quad + \int_{t_0}^{T} \prec A(t)x_0, u(t) \succ dt \\
&\geq \int_{t_0}^{T} \Big[c_1 \|u(t)\|^2 - c_2(t)\|u(t)\| - c_3(t)\Big] dt - \|L(x, x')\|_L \|u\|_L \\
&\quad + \|\frac{t-t_0}{T-t_0} A(\cdot)(x_T - x_0)\|_L \|u\|_L + \|A(\cdot)x_0\|_L \|u\|_L \\
&\geq c_1 \int_{t_0}^{T} \|u(t)\|^2 dt - \int_{t_0}^{T} |c_3(t)| dt - \Big[\|c_2\|_L + \|L(x, x')\|_L \\
&\quad - \frac{t-t_0}{T-t_0} \|A(\cdot)(x_T - x_0)\|_L - \|A(\cdot)x_0\|_L\Big] \|u\|_L. \qquad (9)
\end{aligned}
$$

If $(C12')$ holds define the operator N by

$$
\begin{aligned}
Nu(t) \quad = \quad & L(x(t), x'(t)) - A(t)x(t) \\
& - f\Big(t, x(\delta_1(t)), \int_{t_0}^{t} g(t, s, x(\delta_2(s)), \int_{t_0}^{s} k(s, \tau, x(\delta_3(\tau)))d\tau)ds, u(t)\Big),
\end{aligned}
$$

and similarly prove that N is monotone and (9) holds with b_i replaced by $c_i, i = 1, 2, 3$.

So, in each case there exists $r > 0$ such that whenever $\|u\|_L = r$ it follows that

$$\prec Nu, u \succ_L \geq 0.$$

By Lemma 2.4, $Nu = 0$ has solutions in B_r; that is, there exists $u \in L^2(I, H)$ such that $x \in S(u)$. It is obvious that $x(T) = x_T$ and $x(t_0) = x_0$. Hence the proof is completed.

Theorem 4.2. Under the conditions of Theorem 4.1, system (2) is null controllable on $D(A)$.

Proof. Let $x(t) = \frac{t_1 - t}{t_1 - t_0} x_0,\ x_0 \in D(A),\ t_1 \in [t_0, T]$. Using almost the same method as in Theorem 4.1, $u \in L^2(I, H)$ can be found such that $x \in S(u)$ with $x(t_0) = x_0$ and $x(t_1) = 0$, and the proof follows.

5. Example

Consider the following nonlinear distributed-parameter delay control system

$$\begin{aligned}\frac{\partial}{\partial t}z(y,t) &+ \sum_{i=1}^{n}\frac{\partial}{\partial y_i}(p(y,t)(\frac{\partial}{\partial y_i}z(y,t))\\ &= \frac{z(y,t-h)\sin(z(y,t))}{(1+t)(1+t^2)} + \int_0^t \Big[\frac{z(y,s)}{(1+t)(1+t^2)^2(1+s)^2}\\ &\quad + \frac{1}{(1+t)(1+t^2)}\int_0^s \frac{z(y,\tau)}{(1+s)(1+\tau)}e^z d\tau\Big]ds, \qquad (10)\\ z(y,t) &= 0 \text{ on } \partial\Omega\times(I\cup[-h,0]),\\ z(y,0) &= z_0(y) \text{ for } y\in\Omega.\end{aligned}$$

Here $I=[0,T]$ and Ω is a bounded open set in R. Let $\delta(t)=t-h, h>0$ and $p: I\times\Omega\to R$ be such that

: (i) $p(y,t)\geq c>0$ for every $y\in\Omega,\ t\in I$,

: (ii) p is Lipschitz with respect to t, continuously differentiable with respect to y, and $p\in L^\infty$.

Let $H=L^2(\Omega)$ and $D=H^2(\Omega)\cap H_0^2(\Omega)$. Then D is dense in H. Define the linear operators $A(t): D\subset H\to H$, for each $t\in I$, by

$$\prec A(t)z, v\succ = \sum_{i=1}^{n}\int_\Omega -p(y,t)(\frac{\partial z}{\partial y_i})(\frac{\partial v}{\partial y_i})dy, \quad \text{for} \quad z,v\in D.$$

Let

$$\begin{aligned}f\Big(t, x(\delta_1(t)), &\int_{t_0}^t g(t,s,x(\delta_2(s)), \int_{t_0}^s k(s,\tau,x(\delta_3(\tau)))d\tau)ds\\ &= \frac{x(t-h)\sin(x(t))}{(1+t)(1+t^2)} + \int_0^t \Big[\frac{x(s)}{(1+t)(1+t^2)^2(1+s)^2}\\ &\quad + \frac{1}{(1+t)(1+t^2)}\int_0^s \frac{x(\tau)}{(1+s)(1+\tau)}e^x d\tau\Big]ds\end{aligned}$$

then,

$$\begin{aligned}\|f(t,x,j)\| &= \|\frac{1}{(1+t)(1+t^2)}(x\sin x + j)\|\\ &\leq \frac{1}{(1+t^2)}\|x\| + \frac{1}{(1+t)}\|j\|\end{aligned}$$

Then (10) is equivalent to

$$\begin{aligned}z'(t)-A(t)z(t) &= f\Big(t, z(\delta_1(t)), \int_{t_0}^t g(t,s,z(\delta_2(s)), \int_{t_0}^s k(s,\tau,z(\delta_3(\tau)))d\tau)ds, u(t)\Big), \quad t\geq t_0,\\ z(t) &= z_0, \quad t\leq t_0.\end{aligned}$$

By the above assumptions, there exist $\mu\geq 0, k>0$ such that

$$\prec -A(t)z, z\succ = \int_\Omega \sum_{i=1}^{n} p(y,t)\Big\|\frac{\partial z}{\partial y_i}\Big\|^2 dy \geq \mu\|z\|^2, \quad z\in D, \qquad (11)$$

$$\|A(t)z - A(s)z\| \leq k|t-s|\|z\|, \quad z\in D. \qquad (12)$$

Note that if $c > 0$, then (11) and (12) can be obtained from the Poincare inequality; if $c = 0$, then take $\mu = 0$. So, A satisfies conditions (C8), and $A(t)$ generates a strongly continuous compact evolution operator system $E(t,s)$. Since all the conditions of Theorem (3.3) is satisfied and hence (10) is approximately controllable. Similarly, if conditions (C2),(C3) and (C11) hold, then Theorem (4.1) implies that (10) will be approximately controllable.

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