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A SURVEY ON EXISTENCE RESULTS OF SOME DIFFERENTIAL AND INTEGRAL EQUATIONS IN ABSTRACT SPACES

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Abstract. The study for weak solutions of the Cauchy differential equation in reflexive Banach spaces was initiated by, among others, *Szep* [47], *Chow* and *Schur*[10]. However, if E is nonreflexive Banach space the situation is quite different.
Here, a review of Cauchy problems in nonreflexive Banach spaces will be given in this survey paper.

1. Introduction

In recent years the study of ordinary differential equations in a Banach space has been developed extensively. However almost all of the work was done using the strong topology see for example, *Deimling* [21], *Szulfa* [48].

The study of first order ordinary differential equations in Banach spaces (reflexive or not) equipped with the weak topology was initiated in the 1950s. Let E be a Banach space and let $f(.,.) : [a, b] \times E \to E$ be continuous. It is well known that if E is finite dimensional, then for each $(t_0, x_0) \in [a, b) \times E$, there exists a continuous differentiable function $x(.)$ which is a solution of the Cauchy problem

$$x'(t) = f(t, x(t)), \ x(t_0) = x_0 \tag{1}$$

on some open interval which contains t_0. In 1950, *Dieudonné* [24] showed that when $E = c_0$ (the space of all real-valued sequences $x = (x_n)$ with $x_n \to 0$, $||x||_{c_0} = sup_n|x_n|$) the Cauchy problem (1) has no solutions for some continuous function $f(.,.)$. *Szep* [47] first established the existence of weak solutions of (1), i.e. weakly differential functions x for which satisfies (1) with its weak derivative if $f(.,.) : [a, b] \times E \to E$ is weakly continuous and E is a reflexive Banach space.

Definition 1. [6] *By a solution of (1) they meant a strongly continuous, once weakly differentiable function* $x : [t_0, t_0 + a] \to E$ *satisfying (1) in* $[t_0, t_0 + a]$, *with* x' *denoting the weak derivative.*

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In 1971 *Chow* and *Schur* [10] found that, existence result may be obtained where E is separable and reflexive and $f(.,.)$ is a weakly continuous function with bounded range. *Kato* in [33] showed that if $f(.,.) : [a,b] \times B_E[x_0,r] \to E$ is weakly continuous, then all that is needed to assure the existence of solutions to (1) is the relatively weak compactness of $f([a,b] \times B_E[x_0,r])$. *Pianigiani* [41] showed that in every nonreflexive retractive Banach space there exists a weakly continuous function $f(.,.)$ such that (1) does not have a weak solution, and *Perri* [40] showed that this property is true in every nonreflexive Banach space.
The notion of the measure of noncompactness was introduced by *Kuratowski* [34] in 1930. *Ambrosetti* [1] used the Kuratowski noncompactness measure and Darbo's fixed point theorem to prove an existence result for (1) in infinite dimensional Banach spaces.
The measure of weak noncompactness was introduced by *De Blasi* [20], and it was used by *Cramer et al.* [12] to obtain an existence result for weak solutions of (1) in nonreflexive Banach spaces. Using the measure of weak noncompactness, *Cichoń* [14], *Cichoń* and *Kubiaczyk* [16], *Dutkiewicz* and *Szulfa* [27], *O'Regan* [39], [38] improved and generalized (for more general notions of solutions) previous results in the literature. For a review of this topic we refer the reader to *Cichoń* [15], *Deimling* [21], *Hashem* [32] and *Teixeira* [51].
Ordinary differential equations in reflexive Banach spaces in the weak topology was examined by *Szep* [47], *Kato* [33], *Salem et al.*[49] and [44]. Also, to study the elliptic functional equations in nonreflexive Banach spaces see *Browder* [5].

Here, A similar review of Cauchy problems in nonreflexive Banach spaces will be given in this survey paper.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this survey paper.
First, *De Blasi* [20] defined the weak measure of noncompactness by that play in important role in the existence of weak solution of ordinary differential equations, and it is defined by

$$\beta(X) = \inf\{t > 0;\ \exists Y \in P_{wk}(E) \textit{ such that } X \subset Y + tB_1\}$$

for any bounded subset $X \subset E$, where $P_{wk}(E)$ denoted the family of all weakly compact subsets of E.

Lemma 1. [7]*Let A, B be bounded subsets of E and $\{x_n\}$, $\{y_n\}$ be bounded sequences in E. Then*

(1) $A \subseteq B$ *then* $\beta(A) \leq \beta(B)$,
(2) $\beta(A) = \beta\overline{A^w}$ *where* $\overline{A^w}$ *denotes the weak closure of* A,
(3) $\beta(A) = 0$ ***if and only if*** $\overline{A^w}$ ***is weakly compact,***
(4) $\beta(A \bigcup B) = \max\{\beta(A), \beta(B)\}$
(5) $\beta(A) = \beta(Co(A))$,
(6) $\beta(A + B) \leq \beta(A) + \beta(B)$,
(7) $\beta(\{x_n\}) - \beta(\{y_n\}) \leq \beta(\{x_n - y_n\})$,
(8) $\beta(x + A) = \beta(A)$ *where* $x \in E$,
(9) $\beta(tA) = t\beta(A),\ t \geq 0$,

(10) $\beta(A) \leq \partial(A)$ *(the diameter of A).*

Theorem 1. [26]*(Eberlein Šmulian) Let A be a subset of a Banach space E. Then the following statements are equivalent*

(i) *A is weakly sequentially compact,*
(ii) *Every infinite subset of A has a weak limit point in E,*
(iii) *The closure of A in the weak topology is weakly compact.*

Definition 2. [2] *A nondecreasing function $g(t,u) : I \times R^+ \to R^+$ is called Kamke function if it satisfies the following conditions*

(i) *$g(t,u)$ is a Carathéodry function, i.e. measurable with respect to t and continuous with respect to u.*
(ii) *there exist an integrable function $M(t)$ such that $g(t,u) \leq M(t)$.*
(iii) *for all $t \in I; g(t,0) = 0$.*
(iv) *$u(t) = 0$ is the only absolutely continuous function which satisfies $u' \leq g(t,u(t))$ a.e. on I and such that $u(0) = 0$.*
The condition (iv) can be replaced by the following condition.
$(iv')u(t) = 0$ is the unique solution of the integral equation $u(t) \leq \int_0^t g(t,u(s))ds$ on I with $u(0) = 0$.

Definition 3. *(Pettis [43]). Let $F : [a,b] \to E$ and $A \subset [a,b]$. The function $f : A \to E$ is a pseudo-derivative of F on A if for each ϕ in E^* the real-valued function ϕF is differentiable almost everywhere on A and $(\phi F)' = \phi f$ almost everywhere on A.*

Clearly, if F is weakly differentiable, then it is also pseudo-differentiable, but the converse implication is not true.

Definition 4. *(Pettis [43]). The function $f : I \to E$ is Pettis integrable ((P) integrable for short) if*

(i) *$\forall_{\phi \in E^*} \phi f$ is Lebesgue integrable on I,*
(ii) *$\forall_{A \ measurable} A \subset I \ \exists g \in E \ \forall_{\phi \in E^*} \phi g = (L) \int_A \phi f(s) \ ds$.*

Definition 5. [44] *Let $x : I \to E$. The (left-sided) fractional Pettis integral (shortly LS-FPI) of x of order $\alpha > 0$ is defined by*

$$I_+^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) \ ds, \ t > 0.$$

In the above definition the sign " $\int$ " denotes the Pettis integral, also we define the right-sided fractional Pettis-integral (shortly RS-FPI)by

$$I_-^\alpha x(t) = \int_t^1 \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} x(s) \ ds, \ t < 1.$$

We will call a function fractionally Pettis integrable provided this integral exists as an element of E (for arbitrary $t < 1$).
Here restrict ourselves to the case of left-sided fractional Pettis-integrals (shortly denote it by I^α). We will consider fractional Pettis integrability for $0 < \alpha < 1$.

Salem and *Cichoń* [44] observed that such an integral $I_+^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) \, ds$ is a convolution of a function $h(\tau) = \tau^{\alpha-1}/\Gamma(\alpha)$ for $\tau > 0, \ h(\tau) = 0$ for $\tau \leq 0$, and

the function $(\widetilde{x})(t) = x(t)$ for $t \in I$, where $(\widetilde{x})(t) = 0$ outside the interval I.
We start with an obvious observation that for $\phi \in E^*$

$$\phi(I_+^\alpha x(t)) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi x(s)\ ds. \tag{2}$$

As a consequence of some properties of a convolution for the Pettis integral ([23], Proposition 9), for arbitrary α, we have the following (see [44]).

Theorem 2. *If $x : I \to E$ is Pettis integrable, then*
(a) *$I_+^\alpha x$ is defined almost a.e. on I,*
(b) *x is fractionally Pettis integrable on I,*
(c) *If x is Pettis integrable and strongly measurable, then $I_+^\alpha x : I \to E$ is bounded, weakly continuous and*

$$\sup_{\|\phi\|\leq 1} \int_0^1 \phi I_+^\alpha x(t)\ dt \leq \sup_{\|\phi\|\leq 1} \int_0^1 \phi x(t)\ dt. \parallel h \parallel_1.$$

By E_ω we will denote the space E equipped with its weak topology.

Lemma 2. [44] *For any $\alpha > 0$ the operator $I_\pm^\alpha$ takes $C[I, E_\omega]$ into $C[I, E_\omega]$ and is well defined.*

For the properties of the fractional Pettis-integral in Banach spaces (see [49], [44] and [3]). Now, we give the definition of the weak derivative of fractional order.

Definition 6. *Let $x : I \to E$ be a weakly differentiable function and x' is weakly continuous, then the weak derivative of x of order $\beta \in (0, 1]$ by*

$$D^\beta x(t) = I_+^{1-\beta} Dx(t),$$

where D the weakly differential operator.

Recall that a function $h : E \to E$ is said to be weakly sequentially continuous if h takes each weakly convergent sequence in E to weakly convergent sequence in E. In reflexive Banach space, both Pettis-integrable and weakly continuous functions are weakly measurable (see [25], [28], [29] and [31]). Moreover, in a reflexive Banach space a weakly measurable function $x(.)$ from I to E is Pettis integrable on I if and only if $\phi(x(.))$ is Lebesgue-integrable on I, for every $\phi \in E^*$ (see [25], [28] and [31]).

3. Existence results of Cauchy problems in abstract spaces

Now we state some existence theorems of the weak solutions of differential equations in nonreflexive Banach spaces.

Let E be nonreflexive Banach space with norm $\parallel . \parallel$ with its dual E^*, and we will denote by $E_\omega = (E, \omega) = (E, \sigma(E, E^*))$ the space E with its weak topology. Let $L^1(I)$ be the space of Lebesgue integrable functions on the interval $I = [0, 1]$. Denote by $C[I, E_\omega]$ the Banach space of weakly continuous functions from I to E_ω endowed with the topology of weak uniform convergence.

In 1978, *E. Cramer, V. Lakshmikantham and A. R. Mitchell* [12], discussed the abstract Cauchy problem (1) where E is nonreflexive Banach space, (H1):$f : [t_0, t_0 + a] \times E \to E$ is weakly weakly continuous on R_0 and $\|f(t, x)\| \leq M$ on R_0 where $R_0 = \{(t, x);\ t_0 \leq t \leq t_0 + a,\ \|x - x_0\| \leq b\}$ and they proved an existence theorem under the following condition (weak compactness condition)

$$\beta(f(I \times A)) \leq g(\beta(A))$$

where $I = [t_0, t_0 + \alpha], A \subset E$ *and bounded* and $g \in C(\Re_+, \Re_+)$, and assume that $u(t) = 0$ is the unique solution of $u' = g(u),\ u(t_0) = 0$ on $[t_0, t_0 + \alpha]$ where $\alpha = min(a, \frac{b}{M})$.
The proof is based on the following:

Lemma 3. *Let the assumption (H1) be satisfied and let $\{\epsilon_n\}$ be given such that $\epsilon_n >$ and $\epsilon_n \to 0$ as $n \to \infty$ be given. Then there exists a sequence of approximate solutions $\{x_n(t)\}$ satisfying*

(i) $x_n(t_0) = x_0$;
(ii) *is strongly equicontinuous and uniformly bounded on* $[t_0, t_0 + \alpha]$
(iii) $x'_n(t) = f(t, x_n(t - \epsilon_n)),\ t \in [t_0, t_0 + \alpha]$, *where* $\alpha = min(a, \frac{b}{M})$.

Lemma 4. *Let the assumption (H1) be satisfied. Suppose that $\{x_n(t)\}$ are constructed as in Lemma 31 and converge weakly to $x(t)$ on $[t_0, t_0 + \alpha]$. Then $x(t)$ is a solution of the Cauchy problem.*

Theorem 3. *Let F be a weakly equicontinuous family of functions from $[t_0, t_0+\alpha] \subset \Re$ to E. Let $\{x_n\}$ be sequence in F such that for each $t \in [t_0, t_0 + \alpha]$, $\{x_n(t)\}$ is weakly pre-compact. Then there exists a subsequence $\{x_{nk}\}$ which converges weakly uniformly on $[t_0, t_0 + \alpha]$ to a weakly continuous function $x(t)$.*

The proof of this theorem was given in three steps, first a sequence of approximate solutions was constructed , second it was shown that the sequence converges (in the weak sense), third, it was proved that the limit function is a solution. *E. Cramer, V. Lakshmikantham and A. R. Mitchell* [12] applied Theorem 3 and Theorem 1.

And *A.R. Mitchell and C.K.L. Smith* [37] established the existence theorem of the abstract Cauchy problem (1) under the following assumptions:

(i) f is weakly-weakly continuous on R_0,
(ii) $\|f(t, x)\| \leq M$, for every $(t, x) \in R_0$,
(iii) there exists $k \geq 0$ such that for any $H \subset S_L(x_0)$,

$$\beta(f(I \times H)) \leq k\beta(H),$$

$ak > 1$ and $Ma \leq L$.

Then the abstract Cauchy problem (1) has a weakly solution,

where $R_0 = I \times S_L(x_0)$ such that $S_L(x_0) = \{x \in E,\ \|x - x_0\| \leq L\},\ E$ is Banach space.
The proof of this theorem is based into the following fixed point theorem.

Theorem 4. *Let $C \subset E$ be nonvoid, closed, convex and bounded if $F : C \to C$ is weakly continuous and what it means $\beta-$ condensing, then F has a fixed point.*

To apply this theorem, C is defined by

$$C = Lip_M(I, E) = \{x(.) \in C(I, E),\ x(.) \ is\ M - Lipschitz\ \}$$

(this set is nonvoid, closed, bounded and convex) and the operator T is defined by

$$(Tx)(t) = x_0 + \int_{t_0}^{t} f(s, x(x))ds$$

Also, the operator T satisfied the conditions of Theorem 4.

In **1980**, *Moses A. Boudourides* [6] proved a local existence of solution by assuming f to be a weakly continuous and what it means β-Lipschitzian where β is a measure of weak noncompactness in the weak topology. Of course E is still a nonreflexive Banach space.

Definition 7. [6] *By a solution of (1) they meant a strongly continuous, once weakly differentiable function* $x : [t_0, t_0 + a] \to E$ *satisfying (1) in* $[t_0, t_0 + a]$*, with* x' *denoting the weak derivative.*

Theorem 5. [6] *Let* $f : I \times D \to E$ *be weakly continuous, (strongly) bounded with* $M = \sup\{\|f(t, x)\|;\ (t, x) \in I \times D\}$ *and* β*-Lipschitzian (that is, there exists* $k \geq 0$ *such that*

$$\beta(f(I \times B)) \leq k\beta(B),\ B \subset D.$$

Then the Cauchy problem (1) has a weak solution on $J = [0, h]$*, where* $h \leq min(a, \frac{b}{M}),\ hk < 1,\ I = [0, a]$ *and*

$$D = \{x \in E;\ \|x - x_0\| \leq b\}$$

By a solution *Boudourides* [6] meant a strongly continuous, once weakly differentiable, but the proof of this theorem has a mistake. Specifically, when the author interprets the notion of weak uniform continuity, he claims that the corresponding inequality holds for all elements of the dual space simultaneously (see [6] page 460), which is not true.(for more detail see *E Cramer*) [12].

In **1982**, *I. Kubiaczyk, S. Szulfa* [36] discussed the Cauchy problem (1) which is equivalent to the integral equation

$$x(t) = x_0 + \int_0^t f(s, x(s))ds,\ t \in I = [0, a]$$

where $\int$ denotes the Pettis integral. *Kubiaczyk, Szulfa* [36] proved, under conditions similar to that imposed by [12], that the set of all weak solutions defined on $[0, 1]$ is nonempty, compact and connected in $C_\omega([0, 1], E)$ (the set of all weakly continuous function from $[0, 1]$ into a Banach space E endowed with the topology of uniform weak convergence) and *Kubiczyk and Szulfa* conditions were given in terms of the Kamke function.

Theorem 6. *Let* $f : I \times B \to E$ *be weakly weakly continuous and* $\|f(t, x)\| \leq M$ *on* $I \times B$*, moreover assume that* E_w *the space* E*, is sequentially weakly complete, and*

$$\beta(f(J \times X)) \leq h(\beta(X)),\ \forall X \subset B,$$

the the set S *of all weak solutions of the Cauchy problem (1) defined on* J *is nonempty, compact and connected in* $C_w(J, E)$*,*
where $B = \{x \in E;\ \|x - x_0\| \leq b\},\ J = [0, d],\ d = \min(a, \frac{b}{M})$ *and* h *is nonnegative*

and nondecreasing real-valued function on $\Re_+$ *and* $u(t) = 0$ *is the unique solution of the integral equation*

$$z(t) = \int_0^t h(z(s))ds, \ t \in J.$$

In the proof of this theorem the measure of weak noncompactness was used. For any $n \in N$ we choose $u_n \in S_{\frac{1}{n}}$ and let $H = \{u_n; \ n \in N\}$, where

$$S_n = \{u : J \to E; \ u(0) = x_0, \ \|u(t) - u(s)\| \leq M|t - s| \ for \ t, s \in J$$
$$and \ \sup_{t \in J} \|u(t) - x_0 - \int_0^t f(s, u(s))ds\| < n\}$$

and it is showed that $\beta(H(t)) = 0$ for each $t \in J$, consequently H is relatively weakly compact, therefore the sequence u_n has a limit point u, since $lim(u_n - F(u_n)) = 0$ and F is continuous, then $u = F(u)$ i.e. $u \in S$, where $F(u)(t) = x_0 + \int_0^t f(s, u(s))ds$.

In 1986. *Papageorgiou* [42] discussed the Cauchy problem (1) in nonreflexive Banach space X and for $f : T \times X \to X$ a weakly continuous vector field. Using a compactness hypothesis involving a measure of weak noncompactness and he prove an existence result that generalizes earlier theorem by *Chow-Shur, Kato* [10, 33] and *Cramer-Lakshmikantham-Mitchell* [12].

Theorem 7. [42] *If* $f : T \times X \to X$ *is a vector field such that*

1. $f(.,.)$ *is continuous from* $T \times X_w$ *into* X_w *(that is* $f(.,.)$ *is weakly-weakly continuous),*
2. *for all* $(t, x) \in T \times X$, $\|f(t, x)\| \leq N$,
3. *for all* $A \subset X$ *nonempty and bounded we have*

$$\lim_{r \downarrow 0} \beta(f(T_{t,r} \times A)) \leq w(t, \beta(A)).$$

The problem (1) admits a solution. Here, by a solution we mean a strongly continuous, once weakly differentiable.

The proof is based into the following.
Firstly he considered the nonlinear integral operator $\phi : Lip_N \to Lip_N$ where $Lip_N = \{x(.) \in C(T, X); \ x(.) \ is \ N - Lipschitz\}$, defined by

$$(\phi x)(t) = x_0 + \int_0^t f(s, x(s))ds,$$

and he proved that is weakly-weakly sequentially continuous, also he considered the Caratheodory approximations

$$x_n(t) = \begin{cases} x_0 & \text{if } 0 \leq t \leq \frac{1}{n}; \\ x_0 + \int_0^{t - \frac{1}{n}} f(s, x_n(s))ds & \text{if } \frac{1}{n} \leq t \leq b. \end{cases}$$

and proved that the weak closure of $K = \{x_n(.)\}_n$ is weakly compact, and using the *Eberlein-Smulian theorem* and he proved that $\|x - \phi x\|_\infty = 0$ i.e, $x(t) = x_0 + \int_0^t f(s, x(x))ds$.

The result by *Cramer-Lakshmikantham-Mitchell* is obtained as corollary when he take $w(t, x) = w(x) = g(x)$ i.e, w is a time independent Kamke function.

Corollary 1. [12] *Let the function* $f : [0, a] \times E \to E$ *be a weak continuous and* $||f(t, x)|| \leq M$ *for all* t, x. *Assume further that, for all bounded set* A *of* E,

$$\omega(f(I \times A)) \leq g(\omega(A)),$$

where g *is Kamke function. Then (1) admits a solution in* $t \in [0, \alpha]$.

Also we notice that, if E is reflexive, by compactness, every bounded set is relatively weakly compact, and hence Szep and Chow-Shur results can be deduced.

Corollary 2. [47] *Let the function* $f : [0, a] \times E \to E$ *be a weakly-weakly continuous and* $||f(t, x)|| \leq M$ *for all* t, x. *Then (1) admits a weak solution on* $[0, \alpha]$.

Also, if E is a Banach space and f is a compact function, then Kato result can be deduced.

Corollary 3. [33]. *Let the function* $f : [0, a] \times E \to E$ *be a weakly-weakly continuous and* $||f(t, x)||k \leq M$ *for all* t, x. *Assume further that* $f(t, x)$ *is a compact function. Then (1) admits a weak solution in* $t \in [0, \alpha]$.

Counterexamples are given in c_0 by *Dieudonné* [24] as well as that given in l_2 by *Yorke* [52] show that, in infinite-dimensional spaces, Peano's existence theorem need not necessarily be true. A natural question then is that of asking whether there could exist infinite dimensional Banach spaces on which Peano's theorem holds, or otherwise, whether the truth of Peano's theorem is a characterization of the finite dimensionality. *Cellina* in [13] offered only a partial answer to this question. He proved that there exist no nonreflexive spaces on which Peano's theorem holds.

Theorem 8. [13] *If* E *is nonreflexive Banach space. Then there exists a continuous function* $f : [0, a] \times E \to E$ *such that the Cauchy problem (1)* $(x(0) = 0)$ *admits no solution on any nonempty interval* I *containing* 0.

In **1999.** *O'Regan* [38], give the sufficient conditions of the existence results for the operator equation

$$x(t) = Fx(t) \text{ on } [0, T] \tag{3}$$

and the proof is based into the following fixed point theorem.

Theorem 9. *Let* Q *be a nonempty, bounded, convex, closed set in a Banach space* E. *Assume* $F : Q \to Q$ *is weakly sequentially continuous and* α $\beta-$*contractive (here* $0 \leq \alpha < 1$*). Then* F *has at least one fixed point in* Q.

And he gave two existence theorems for the operator equation (3).

Theorem 10. *Let* E *be a Banach space with* Q *a nonempty, bounded, closed, convex, equicontinuous subset of* $C([0, T], E)$, *suppose that* $F : Q \to Q$ *is wk-sequentially continuous and there exists* α, $0 \leq \alpha < 1$ *with* $\beta(F(X)) \leq \alpha\beta(X)$ *for all subsets* $X \subset Q$. *Then the operator equation (3) has a solution in* Q.

Theorem 11. *Let* E *be a Banach space with* Q *a nonempty, bounded, closed, convex, equi-continuous subset of* $C([0, T], E)$, *suppose that* $F : Q \to Q$ *is wk-sequentially continuous and assume*

$$FQ(t) \text{ is weakly relatively compact in } E \text{ for each } t \in [0, T]$$

holds. Then (3) has a solution in Q.

Also, he discussed the existence of weak solutions of a particular case of (3), namely

$$y(t) = Fy(t) = x_0 + \int_0^t f(s, y(s))ds, \ for \ t \in [0, T]$$

(which equivalent to Cauchy problem $y'(t) = f(t, y(t)), \ y(0) = x_0 \ on \ [0, T]$).
Under the assumptions

(C1) for each $t \in [0, T], \ f_t = f(t, .)$ is weakly sequentially continuous,
(C2) for each continuous function $y : [0, T] \to E, \ f(., y(.))$ is Pettis integralable for $t \in [0, T]$,
(C3) for any $r > 0$, there exists $h_r \in L^1$ with $|f(t, y)| \leq h_r(t)$ for a.e.$t \in [0, T]$ and all $y \in E$ with $\|y\| \leq r$,
(C4) $FQ(t)$ is weakly relatively compact in E for each $t \in [0, T]$, where Q is nonempty, bounded, closed, convex and equi-continuous subset of $C([0, T], E)$.

Remark: To prove the existence theorem of weak solutions of the Cauchy problem

$$x'(t) = f(t, x(t)), \ x(0) = x_0 \ on \ [0, T]$$

which is equivalent to integral equation

$$x(t) = Fx(t) = x_0 + \int_0^t f(s, x(s))ds, \ t \in [0, T]$$

it sufficient to pose the conditions over f that guaranties that the Pettis integral is well-define and construction subset of $C([0, T], E)$ which is nonempty, bounded, closed, convex and equi-continuous. Moreover, F is ws-sequentially continuous and $FQ(t)$ is relatively weakly compact in E for each $t \in [0, T]$.

In 2001, *J. Banaś , M. Lecko*[8] establish sufficient conditions for the solvability of infinite systems of ordinary differential equations in some Banach sequence spaces, They adopt the technique of measures of noncompactness to the theory of infinite systems of differential equations. Particularly, they present a few existence results for infinite systems of differential equations formulated with the help of convenient and handy conditions..The results presented in the paper create mainly the concrete realizations of sufficient conditions for the solvability of ordinary differential equations in Banach spaces formulated with help of the technique of measures of noncompactness.
The results of this paper extend several ones obtained up to now and create realizations of existence results obtained for ordinary differential equations in Banach spaces with the help of measures of noncompactness.

In 2005, *Cichoń*[17] proved some existence theorems for the Cauchy problem (1) on $I = [0, \alpha]$ using different types of integrals and its properties (he recalled some necessary definitions of integrals and presented some examples of integrable functions under different senses). The requirements on the function f are depended on types of solutions, and as possible as, close to necessary conditions. Also, he presented a brief survey of classes of solutions and he gave some comparison results for them.

We are forced to adjust the definition of solutions to possible assumptions on f. We propose to take two steps: find an appropriate solution for hypotheses on f

and then thanks to some (possibly) additional assumptions ensure that founded solution is from a "better" class (see the last section of this paper [17]).

For each solution problem (1) is equivalent to integral problem

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds,\ t \in I \tag{4}$$

with the integral depending on the type of solution, we have the following theorem:

Theorem 12. [17] *Each solution x of problem (1) is equivalent to the solution y of the integral equation (4) in the following cases:*

(a) *x-classical solution: the Riemann integral,*
(b) *x-weak solution: the weak Riemann integral,*
(c) *x-Carathéodory solution: the Bochner (Lebesgue) integral,*
(d) *x-pseudo-solution: the Pettis integral,*
(e) *x-K-solution: the HenstockKurzweil integral,*
(f) *x-pseudo-K-solution: the HenstockKurzweilPettis integral,*
(g) *x-Denjoy-solution: the Denjoy integral,*
(h) *x-pseudo-D-solution: the DenjoyPettis integral.*

Note that, the definitions of these types of integrals are recalled and some selected implications and examples for different classes of integrals are presented in [17]. As a consequence of the above theorem the following theorem can be obtained:

Theorem 13. [17] *Consider the following classes of solutions for problem (3):*

(a) *classical solutions,*
(b) *Carathéodory solutions,*
(c) *weak solutions,*
(d) *pseudo-solutions,*
(e) *Kurzweil solutions,*
(f) *pseudo-K-solutions,*
(g) *Denjoy solutions,*
(h) *pseudo-D-solutions.*

Then $(a) \subset (c) \subset (b) \subset (d) \subset (f) \subset (h),\ (b) \subset (e) \subset (f) \subset (h) and (b) \subset (e) \subset (g) \subset (h)$. *Moreover, all these inclusions are proper.*

The following lemma explained how different kind of assumptions is covered by "superpositional" ones.

Lemma 5. [17] *Assume that x is absolutely continuous and $f : I \times E \to E$. Thus $f(,x())$ is Pettis-integrable if at least one of the following cases holds:*

(a) *f satisfies Carathéodory conditions,*
(b) *f is weakly-weakly continuous and E is a weakly sequentially complete space,*
(c) *$f(.,x)$ is weakly measurable, $f(t,.)$ is weakly-weakly continuous and E is a WCG-space (weakly compactly generated space),*
(d) *f is strongly measurable and there exists a Young function φ such that $\lim_{x\to\infty} \varphi(x)/x = +\infty$ and $x^* f \in L^{\varphi}(I)$,*
(e) *f is strongly measurable and there exists $p > 1$ such that $x^* f \in L_p$ for each $x^* \in E^*$,*

(f) $f(.,x)$ *is strongly measurable* $f(t,.)$ *is weaklyweakly sequentially continuous and* f *is bounded,*
(g) $f(.,x(.))$ *is strongly measurable,* E *contains no copy of* c_0 *and* f *is bounded.*

Now, an existence theorem for pseudo-solutions (see [14]Theorem 1):

Theorem 14. *(Cichoń* [14]*, see also Knight* [35]*). Let us denote by* $J \in I$ *the interval* $[0,d]$*, where* $d = min(\alpha, r/M)$ *and* $||f(t,x)|| \leq M$ *for* $(t,x) \in J \times B_r$*. Assume that* $f : I \times B_r \to E$ *satisfies*

(a) $f(t,.)$ *is weakly-weakly sequentially continuous,*
(b) *for each Lipschitz-continuous function* $x : I \to E$ *with constant* M *such that* $x(0) = x_0$ *a function* $f(.,x(.))$ *is Pettis-integrable*
(c) $\omega(f(J \times X)) \leq h(\omega(X))$, $X \subset B_r$,

where h *is a non-decreasing* Kamke *function. Then there exists at least one pseudo-solution of the Cauchy problem (1) on* J*.*

Cichoń et al. presented two theorems deal with most general classes of "pseudo" solutions. One is announced in [19], the second is new and most general, versions [17]. Some of these are known"super-positional" assumptions are the only difference.

He presented a result about the existence of pseudo-D-solutions (see [17], Theorem 22).

4. Existence results of some fractional differential in abstract spaces

In recent years fractional differential equations in Banach spaces were studied. The general literature on fractional differential equations in finite or infinite dimensional Banach spaces is extensive and different topics on the existence and qualitative properties of solutions are considered. Only a few papers consider fractional differential equations in reflexive Banach spaces equipped with the weak topology. *Salem and El-Sayed* [49] were the first authors to discuss the existence of weak solutions for fractional differential equation(see also [44] *Salem & Cichoń* investigated the existence of a class of solutions for some boundary value problems of fractional order with integral boundary conditions. The considered problems are very interesting and important from an application point of view. They include two, three, multipoint, and nonlocal boundary value problems as special cases.
Salem & Cichoń[44] stress on single and multivalued problems for which the nonlinear term is assumed only to be Pettis integrable and depends on the fractional derivative of an unknown function. Some investigations on fractional Pettis integrability for functions and multifunctions are also presented. An example illustrating the main result is given).

In **2015.** *Ravi P. Agarwal, Vasile Lupulescu, Donal O'Regan, Ghaus ur Rahman* [3] develop fractional calculus for functions with values in a nonreflexive Banach space equipped with the weak topology. Using the Pettis integral, we introduce the notions of fractional Pettis integrals and pseudo-fractional derivatives. Then they present a very general theory for fractional calculus and fractional differential equations in a nonreflexive Banach spaces equipped with the weak topology and establish an existence result for the fractional differential equation

$$\begin{cases} D_p^\alpha\ y(t) = f(t, y(t)), \\ \\ y(0) = y_0, \end{cases}$$

where D_p^α is a fractional pseudo-derivative of a weakly absolutely continuous and pseudo-differentiable function $y(.) : T \to E$, the function $f(t, .) : T \times E \to E$ is weakly-weakly sequentially continuous for every $t \in T$ and $f(., y(.))$ is Pettis integrable for every weakly absolutely continuous function $y(.) : T \to E$, T is a bounded interval of real numbers and E is a nonreflexive Banach space and develop fractional calculus for functions with values in a nonreflexive Banach space equipped with the weak topology. Using the Pettis integral, *Ravi P. Agarwal et al.*. [3] introduce the notions of fractional Pettis integrals and pseudo-fractional derivatives. Then we present a very general theory for fractional calculus and fractional differential equations in a nonreflexive Banach spaces equipped with the weak topology.

By x'_p we will denote a pseudo-derivative of x.

Remark [3]

(a) Clearly, if $x(.) : T \to E$ is a function a.e. weakly differentiable on T, then $x(.)$ is pseudo-differentiable on T and $x'_p(.) = x'_w(.)$ a.e. on T.
(b) A pseudo-derivative $x'_p(.)$ of a pseudo-differentiable function $x(.) : T \to E$ need not be strongly measurable [45]. However, in [46] it was shown that $x'_p(.)$ is weakly measurable on T.
(c) In general, a pseudo-derivative of a pseudo-differentiable function $x(.)$: $T \to E$ is not unique. Moreover, two pseudo-derivatives of $x(.)$ need not be a.e. equal [45]. However, if E has a countable determining set, that is, a countable set $W^* \subset E^*$ such that $||x|| = \sup_{x^* \in W^*} |\langle x^*, x\rangle|$ for every $x \in E$, then any two pseudo-derivative of $x(.)$ are a.e. equal [45].
(d) Even if E^* is separable and $x(.) : T \to E$ is a Lipschitz function, we cannot guarantee that $x'_p(.)$ exists on T; in fact, $x'_p(.)$ need not exist on any subset of T of positive Lebesgue measure [46].

Ravi P. Agarwal et al.. [3] gave the existence theorem under the following assumptions:

(h1) $f(t, .)$ is weakly-weakly sequentially continuous for every $t \in T$,
(h2) $f(., y(.))$ is Pettis integrable and $\|f(., y(.))\|$ is Lebesgue integrable for every WAC function $y(.) : T \to E$,
(h3) $\|f(t, y)\| \leq M$ for all $(t, y) \in T \times B_r$, where $B_r = \{y \in E;\ \|y - y_0\| \leq r\}$,
(h4) for all $A \subseteq B_r$, we have
$$\beta(f(T \times A)) \leq g(\beta(A)),$$
where g is a Kamke function.

In the proof the *Ravi P. Agarwal et al.*. [3] defined the set B_r of all weakly absolutely continuous function (WAC, for short) $y(.) : T_0 \to B_r$,
where $T_0 = [0, a]$, $a = min(b, (\frac{r\Gamma(\alpha+1)}{M})^{\frac{1}{\alpha}}$ and nonlinear operator $Q(.)$ defined by

$$(Qy)(t) = y_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, y(s))ds,\ t \in T_0.$$

and we have the same steps that used in the work of *N. S. Papageorgiou* [42], and also they presented the case of ordinary differential equations as a corollary.
They gave sufficient conditions of the existence of weak solutions for differential equation

$$x'(t) = f(t, x(t)),\ x(0) = x_0,\ t \in I = [0, T]\ (CP1)$$

where $f : I \times E \to E$ is function which satisfies some conditions and E is Banach space.

Theorem 15. *Assume that f satisfies the following conditions*

(1:) *For each $x \in E$, $f(., x)$ is weakly measurable on I,*
(2:) *For a.e. $t \in I$, $f_t(.) = f(t, .)$ is continuous with respect to the weak topology (i.e, $f(t, .)$ is weakly continuous,*
(3:) *For any bounded set $\Omega \subset B$, we assume that $f(I \times \Omega)$ is weakly relatively compact.*

Then (CP1) has a solution.

By a solution we mean a strongly continuous, once weakly differentiable. Moreover: The following steps can summarize their proof:

Defining the Operator $F : C \to C$ by

$$Fx(t) = x_0 + \int_0^t f(s, x(s))ds\ (IE)$$

where $C = \{x(.) \in C(I, E),\ x(.)\ is\ M - Lipschitz\}$ and the integral is a Pettis integral, we now show that if the assumptions (1:)- (3:) are satisfied then, for any continuous function $y : [0, T] \to E$, then

$$g(t) = f(t, y(t))$$

is weakly measurable see [4](Lemma1. p. 275) and $\|g(t)\| \leq M,\ M > 0$ (g is contained in a weakly compact subset of E) and hence the Pettis integral well define.
Now, they show that F is weakly sequentially continuous, to see this:
Let $\{x_.\}$ be a sequence such that $x_n(.) \rightharpoonup x(.)$, from *Dinculeanu*[22](p. 380) we now that

$$(C(I, E))^\star = M(I, E^\star) = \{bounded,\ regular,\ vector\ measure\ from\ I\ into\ E^\star$$
$$which\ are\ of\ bounded\ variation\}$$

thus for all $m(.) \in M(I, E^\star)$ we have

$$(m, x_n(.) - x(.)) \to 0\ as\ n \to \infty.$$

Let $m = x^\star\delta_t$, where $x^\star \in E^\star,\ t \in I$ and δ_t is the Dirac measure concentrated on t. Then we get

$$(x^\star, x_n(t) - x(t)) \to 0\ as\ n \to \infty,$$

and so $x_n(t) \rightharpoonup x(t)$ as $n \to \infty$ for all $t \in I$.
Then using the weak continuity of $f(t, .)$ and the Lebesgue dominated convergence theorem for the Pettis integral see [50] we get that

$$\int_0^t f(s, x_n(s))ds \rightharpoonup \int_0^t f(s, x(s))ds\ as\ n \to \infty$$

for all $t \in I$. Now for every $m(.) \in M(I, E^\star)$ we have that

$$(m, Fx_n - Fx) = \int_I [\int_0^t (f(s, x_n) - f(s, x(s)))ds] dm(t),$$

using the fact that for all $t \in I$,

$$\int_0^t f(s, x_n(s))ds \rightharpoonup \int_0^t f(s, x(s))ds \ as \ n \to \infty,$$

and by approximating $m(.)$, uniformly on C, by linear combinations of Dirac measure, we finally get that

$$(m, Fx_n - Fx) \to 0 \ as \ n \to \infty.$$

hence F is weakly sequentially continuous.

Now, Let $H \subset C$ be bounded subset of C. Then

$$\begin{aligned}\beta(F(H)) = \sup_{t \in I} \beta(F(H(t))) = \sup_{t \in I} \beta(\int_0^t f(s, x(s))ds, \ x \in H)\\ \leq \sup_{t \in I} \{\beta[(t-0)\overline{co} f([0,t] \times H([0,t]))]\}\\ \leq \sup_{t \in I} t\beta(f(I \times H(I)))\\ \leq Tk\beta(H(I)) \ for \ some \ k > 0\\ = Tk\beta(H),\end{aligned}$$

choose $k > 0$ such that $Tk < 1$ (k exists since $f(I \times H)$ is relatively weakly compact), hence F is $\beta-$condensing therefore the operator F has a fixed point which is solution of (IE) and hence is solution of the problem (CP1).

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