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ON A FRACTIONAL ORDER STUDY OF MIDDLE EAST RESPIRATORY SYNDROME CORONA VIRUS (MERS-COV)

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Abstract. MERS-CoV is a dangerous epidemic that exists in Saudi-Arabia and some other countries. Its fatality rate is approximately 35% which is quite high. So the possibility of its transmission between different areas is important. Here we study this possibility using fractional order (FO) model.

1. Introduction

MERS-CoV is a dangerous epidemic that exists in Saudi-Arabia and some other countries. Its fatality rate is approximately 35% which is quite high. Mathematical models for infectious diseases may be helpful in controlling them. Here we present a FO model for MERS-CoV in two regions.

Fractional order (FO) models [14-18] are quite useful in epidemic models to predict the spread of diseases, how to prevent epidemics and so much more. FO models naturally include both memory and nonlocality effects. These effects are quite relevant to epidemic spread.

In sec. 2 FO formalism is presented. In sec. 3 the model [22] will be generalized to FO. The possibility of transmission of the epidemic from an infected to an uninfected region will be studied as a function of the human movement rate between the two regions.

2. Fractional order calculus

Definition 1 The fractional integral of order $\beta \in R^+$ of the function $f(t)$, $t > 0$ is defined by

$$I^\beta f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds \tag{1}$$

and the fractional derivative of order $\alpha \in (n-1, n)$ of $f(t)$, $t > 0$ is defined by

$$D_*^\alpha\ f(t) = I^{n-\alpha} D^n f(t),\ D_* = \frac{d}{dt}. \tag{2}$$

The following properties are some of the main ones of the fractional derivatives and integrals (see [6]-[8], [10], [12], [20], [21]).

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Let $\beta, \gamma \in R^+$ and $\alpha \in (0,1)$. Then
(i) $I_a^\beta : L^1 \to L^1$, and if $f(y) \in L^1$, then $I_a^\gamma \; I_a^\beta \; f(y) = I_a^{\gamma+\beta} \; f(y)$.
(ii) $\lim_{\beta \to n} I_a^\beta \; f(y) = I_a^n \; f(y)$ uniformly on $[a,b]$, $n = 1, \; 2, \; 3, \cdots$,
where $I_a^1 \; f(y) = \int_a^y f(s) \; ds$.
(iii) $\lim_{\beta \to 0} I_a^\beta \; f(y) = f(y)$ weakly.
(iv) If $f(y)$ is absolutely continuous on $[a,b]$, then $\lim_{\alpha \to 1} D_*^\alpha \; f(y) = \frac{df(y)}{dy}$.
(v) If $f(y) = k \neq 0$, k is a constant, then $D_*^\alpha \; k = 0$.
The following lemma can be easily proved (see [10]).
Lemma 1 Let $\beta \in (0,1)$ if $f \in C[0,T]$, then $I^\beta f(t)|_{t=0} = 0$.

2.1. Equilibrium points and their asymptotic stability

Let $\alpha \in (0,1]$ and consider the system ([1]-[3], [11], [13])

$$\begin{aligned} D_*^\alpha y_1(t) &= f_1(y_1, y_2, y_3, y_4), \\ D_*^\alpha y_2(t) &= f_2(y_1, y_2, y_3, y_4), \\ D_*^\alpha y_3(t) &= f_3(y_1, y_2, y_3, y_4), \\ D_*^\alpha y_4(t) &= f_4(y_1, y_2, y_3, y_4), \end{aligned} \tag{3}$$

with the initial values

$$y_1(0) = y_{o1} \text{ and } y_2(0) = y_{o2} \text{ and } y_3(0) = y_{o3} \text{ and } y_4(0) = y_{o4}. \tag{4}$$

To evaluate the equilibrium points, let

$$D_*^\alpha y_i(t) = 0 \Rightarrow f_i(y_1^{eq}, y_2^{eq}, y_3^{eq}, y_4^{eq}) = 0, \; i = 1,2,3,4$$

from which we can get the equilibrium points $y_1^{eq}, \; y_2^{eq}, y_3^{eq}, y_4^{eq}$.
To evaluate the asymptotic stability, let

$$y_i(t) = y_i^{eq} + \varepsilon_i(t),$$

so the the equilibrium point $(y_1^{eq}, y_2^{eq}, y_3^{eq}, y_4^{eq})$ is locally asymptotically stable if the eigenvalues of the Jacobian matrix A

$$\begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial y_3} & \frac{\partial f_1}{\partial y_4} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} & \frac{\partial f_2}{\partial y_4} \\ \frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial y_2} & \frac{\partial f_3}{\partial y_3} & \frac{\partial f_3}{\partial y_4} \\ \frac{\partial f_4}{\partial y_1} & \frac{\partial f_4}{\partial y_2} & \frac{\partial f_4}{\partial y_3} & \frac{\partial f_4}{\partial y_4} \end{bmatrix}$$

evaluated at the equilibrium point satisfiesis ([2], [3], [13], [19])

$$(|\arg(\lambda_1)| > \alpha\pi/2, |\arg(\lambda_2)| > \alpha\pi/2, |\arg(\lambda_3)| > \alpha\pi/2, |\arg(\lambda_4)| > \alpha\pi/2).$$

The stability region of the fractional-order system with order α is illustrated in Fig. 1 (in which σ, ω refer to the real and imaginary parts of the eigenvalues, respectively, and $j = \sqrt{-1}$). From Fig. 1, it is easy to show that the stability region of the fractional-order case is greater than the stability region of the integer-order case.

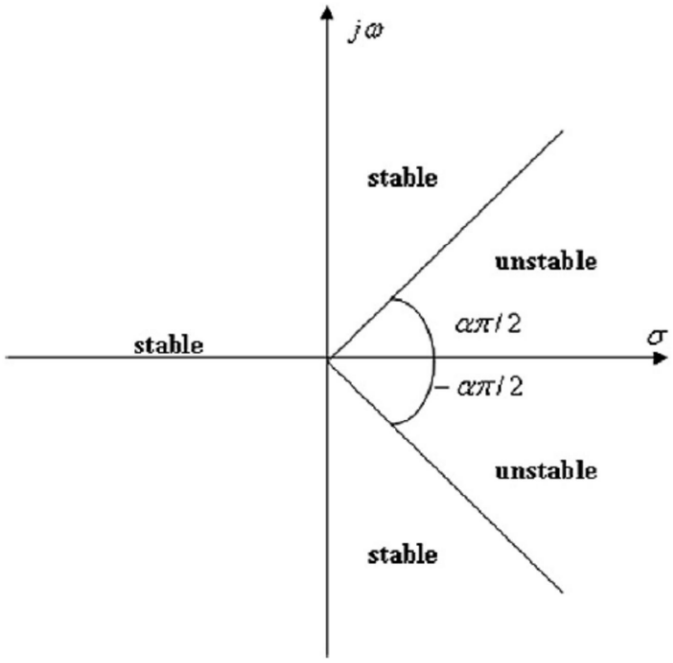

Fig. 1: Stability region of the fractional-order system.

3. The fractional order Middle East Respiratory Syndrome Corona Virus (MERS-CoV) model

We divide the population (N) into two areas, namely area x and y. In each area, we have two sub-populations, according to their disease status; population who are susceptible to infection (S_x and S_y) and population who have the disease (I_x and I_y) [22].

The fractional order Middle East Respiratory Syndrome Corona Virus (MERS-CoV) model is given by

$$\begin{aligned}
D_*^\alpha I_x(t) &= \frac{\beta S_x I_x}{S_x+I_x} - (c+d+\alpha_1)I_x + \alpha_2 I_y + \frac{\omega\alpha_2 S_y I_y}{S_y+I_y},\\
D_*^\alpha I_y(t) &= \frac{\beta S_y I_y}{S_y+I_y} - (c+d+\alpha_2)I_y + \alpha_1 I_x + \frac{\omega\alpha_1 S_x I_x}{S_x+I_x},\\
D_*^\alpha S_x(t) &= a_1 - \frac{\beta S_x I_x}{S_x+I_x} - (b+\alpha_1)S_x + \alpha_2 S_y + d\ I_x - \frac{\omega\alpha_2 S_y I_y}{S_y+I_y},\\
D_*^\alpha S_y(t) &= a_2 - \frac{\beta S_y I_y}{S_y+I_y} - (b+\alpha_2)S_y + \alpha_1 S_x + d\ I_y - \frac{\omega\alpha_1 S_x I_x}{S_x+I_x}, \qquad (5)
\end{aligned}$$

where $0 < \alpha \leq 1$. The variable domain of the model is

$$\Omega = \{(I_x, I_y, S_x, S_y) \in R^4 : I_x, I_y, S_x, S_y \geq 0\}$$

and all the parameters a_1, a_2, b, c, d, $\beta, \alpha_1, \alpha_2$ and ω are positive [22].

Parameter	Description
a_1	Number of newly recruited to the susceptible x population
a_2	Number of newly recruited to the susceptible y population
b	Natural death rate for susceptible individuals
c	MERS-CoV death rate of human population
d	Recovery rate from MERS-CoV
β	Transmission rate within an area
α_1	Movement rate of human population from area x leave to area y
α_2	Movement rate of human population from area y leave to area x
ω	Transmission rate in different area

Table 1. Parameters used in FO model (5) and their description.

To evaluate the equilibrium points, let

$$\begin{aligned} D_*^\alpha I_x &= 0, \\ D_*^\alpha I_y &= 0, \\ D_*^\alpha S_x &= 0, \\ D_*^\alpha S_y &= 0, \end{aligned}$$

then

$$(I_x^{eq}, I_y^{eq}, S_x^{eq}, S_y^{eq}) = (0, 0, \frac{a_2\alpha_2 + a_1(b+\alpha_2)}{b(\alpha_1+\alpha_2+b)}, \frac{a_1\alpha_1 + a_2(b+\alpha_1)}{b(\alpha_1+\alpha_2+b)}), (I_x^*, I_y^*, S_x^*, S_y^*),$$

are the equilibrium points.

For a disease-free equilibrium point

$$(I_x^{eq}, I_y^{eq}, S_x^{eq}, S_y^{eq}) = (0, 0, \frac{a_2\alpha_2 + a_1(b+\alpha_2)}{b(\alpha_1+\alpha_2+b)}, \frac{a_1\alpha_1 + a_2(b+\alpha_1)}{b(\alpha_1+\alpha_2+b)})$$

we find that its eigenvalues are

$$\lambda_1 = -b < 0,$$

$$\lambda_2 = -(b + \alpha_1 + \alpha_2) < 0,$$

$$\lambda_{3,4} = (2\beta - 2c - 2d - \alpha_1 - \alpha_2 \pm \sqrt{\rho})/2,$$

$$\rho = \alpha_1^2 + 2(1 + 4\omega + 2\omega^2)\alpha_1\alpha_2 + \alpha_2^2 > 0.$$

Hence a disease-free equilibrium point is locally asymptotically stable if $\lambda_{3,4} < 0$, if

$$\beta < c + d + \frac{\alpha_1 + \alpha_2 - \sqrt{\rho}}{2}. \tag{6}$$

A sufficient condition for the local asymptotic stability of a unique endemic equilibrium point $(I_x^*, I_y^*, S_x^*, S_y^*)$ is

$$|\arg(\lambda_1)| > \alpha\pi/2, |\arg(\lambda_2)| > \alpha\pi/2, |\arg(\lambda_3)| > \alpha\pi/2, |\arg(\lambda_4)| > \alpha\pi/2. \tag{7}$$

3.1. Numerical methods and results

An Adams-type predictor-corrector method has been introduced and investigated further in ([1]-[3], [4], [5], [9]). In this paper we use an Adams-type predictor-corrector method for the numerical solution of fractional integral equations.

The key to the derivation of the method is to replace the original problem (5) by an equivalent fractional integral equations

$$\begin{aligned}
I_x(t) &= I_x(0) + I^\alpha \left[\frac{\beta S_x I_x}{S_x + I_x} - (c + d + \alpha_1) I_x + \alpha_2 I_y + \frac{\omega \alpha_2 S_y I_y}{S_y + I_y} \right], \\
I_y(t) &= I_y(0) + I^\alpha \left[\frac{\beta S_y I_y}{S_y + I_y} - (c + d + \alpha_2) I_y + \alpha_1 I_x + \frac{\omega \alpha_1 S_x I_x}{S_x + I_x} \right], \qquad (8) \\
S_x(t) &= S_x(0) + I^\alpha \left[a_1 - \frac{\beta S_x I_x}{S_x + I_x} - (b + \alpha_1) S_x + \alpha_2 S_y + d\ I_x - \frac{\omega \alpha_2 S_y I_y}{S_y + I_y} \right], \\
S_y(t) &= S_y(0) + I^\alpha \left[a_2 - \frac{\beta S_y I_y}{S_y + I_y} - (b + \alpha_2) S_y + \alpha_1 S_x + d\ I_y - \frac{\omega \alpha_1 S_x I_x}{S_x + I_x} \right],
\end{aligned}$$

and then apply the **PECE** (Predict, Evaluate, Correct, Evaluate) method.

The approximate solutions displayed in Figs. 2-7 for $a_1 = 4326, a_2 = 13461$, $b = 0.01, c = 0.05, d = 0.1, \beta = 0.1, \omega = 1.0, \alpha_2 = 0$, $0 < \alpha \leq 1$ and different $\alpha_1(0.001, 0.01, 0.1, 0.6)$. we take $I_x(0) = 100$, $I_y(0) = 0$, $S_x(0) = 500$, $S_y(0) = 500$ and found that a disease-free equilibrium point is locally asymptotically stable. Yet there are transient regions where there is transmission for the disease may occur despite eventually going to a disease-free equilibrium point.

Attractive example to our study has been given in [23].

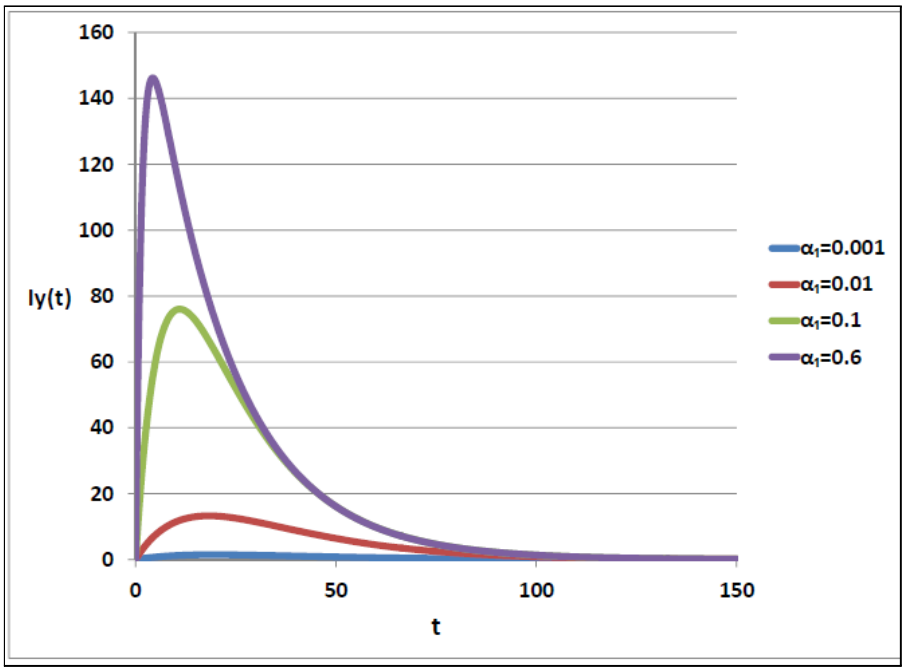

Fig. 2. $\alpha = 1.0$.

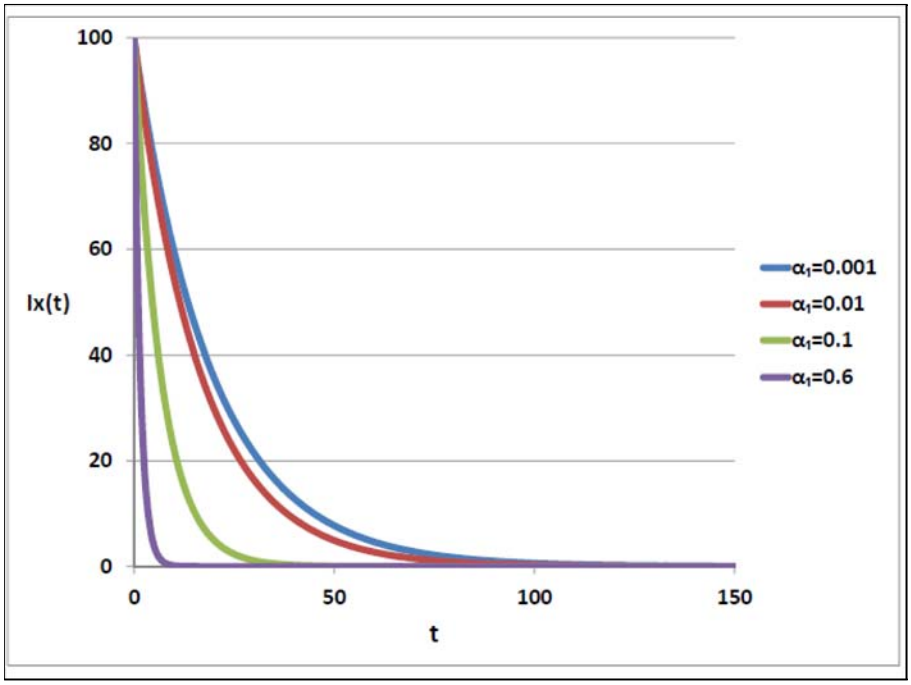

Fig. 3. $\alpha = 1.0$.

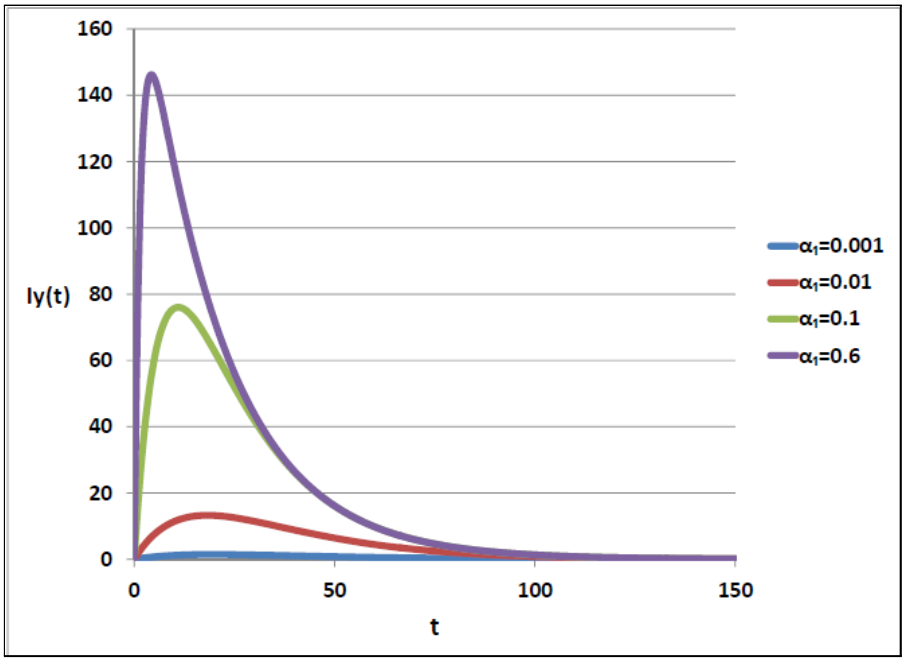

Fig. 4. $\alpha = 0.95$.

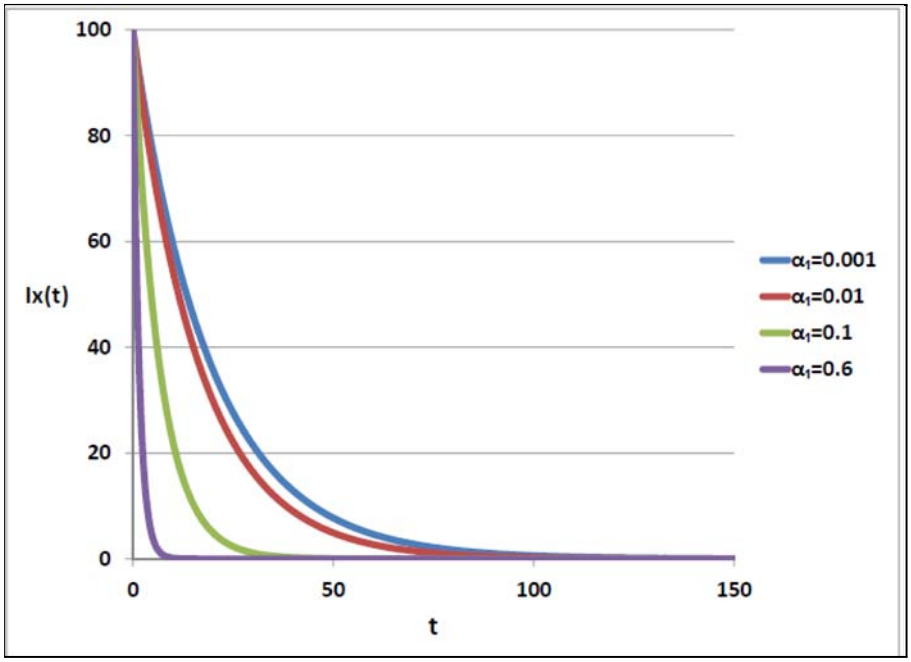

Fig. 5. $\alpha = 0.95$.

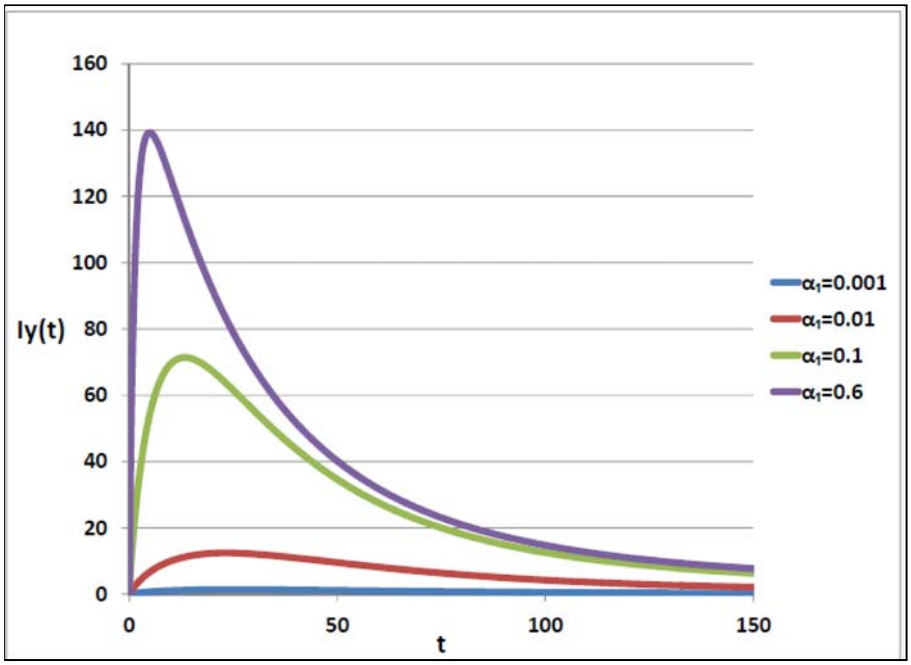

Fig. 6. $\alpha = 0.9$.

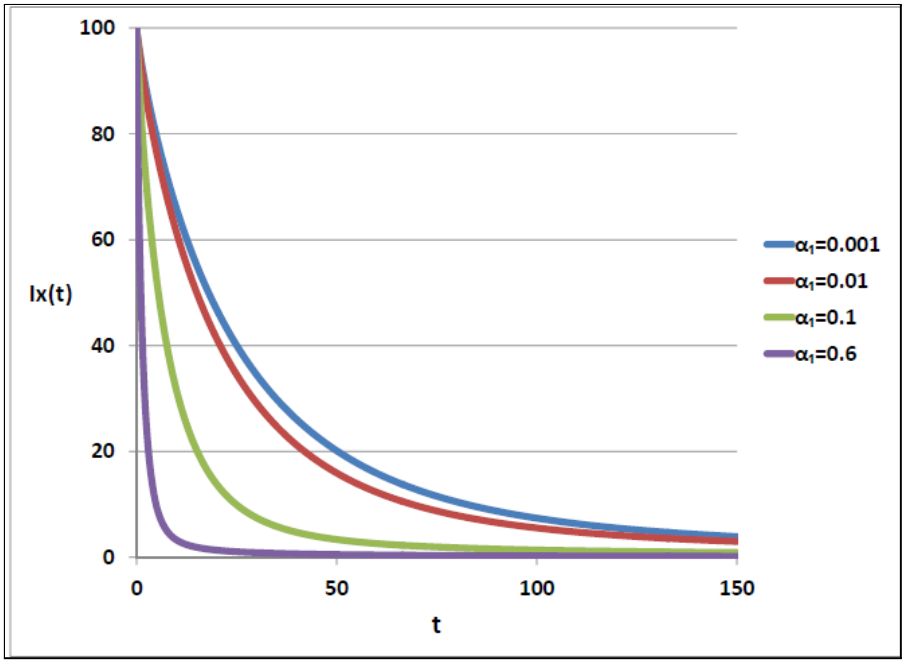

Fig. 7. $\alpha = 0.9$.

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