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THE FRACTIONAL DIFFER-INTEGRAL OPERATORS AND SOME OF THEIR APPLICATIONS TO CERTAIN MULTIVALENT FUNCTIONS

HÜSEYIN IRMAK

ABSTRACT. With this study, several useful results consisting of certain applications of the fractional differ-integral operators to certain multivalent functions are first determined and a few useful consequences of them between certain equations and inequalities relating to multivalently functions which are analytic in the unit open disk are then pointed out.

1. Introduction, Definitions, Notations And Motivation

As it is well known, ordinary differential and also fractional differ-integral equations are used in many different fields of sciences and, occasionally, the solutions of them will be very important for each one of their fields. But, in this investigation, without getting or searching to find any solutions of the certain types of the fractional differ-integral equations in the complex plane, some useful relationships between those equations and certain multivalent functions are only revealed and their special consequences are also presented. For this goal, there is a need to introduce or recall certain information and definitions connecting with complex equations, fractional calculus and some of their connections, which are below.

Firstly, $\mathbb{N}$ be the set of positive integers, $\mathbb{C}$ be the set of complex numbers, $\mathbb{U}$ be the the open unit disk in the complex plane, that is, the set $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{P}$ be the family of functions $f(z)$ expressed by the the following Taylor-Maclaurin's series:

$$f(z) = 1 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \quad \big(a_n \in \mathbb{C}; n \in \mathbb{N}\big) \tag{1}$$

which are analytic and univalent in $\mathbb{U}$, and also let $\mathcal{A}(p)$ be the family of functions $g(z)$ represented by the the following Taylor-Maclaurin's series:

$$g(z) = z^p + c_{1+p} z^{1+p} + c_{2+p} z^{2+p} + \cdots \quad \big(c_{n+p} \in \mathbb{C}; n, p \in \mathbb{N}\big) \tag{2}$$

which are analytic and multivalent in $\mathbb{U}$.

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For both the definitions of certain fractional differ-integral operators and our main results, there is also a need to recall certain definitions in relation with fractional calculus. (cf., e.g., [1], [4], [9], [13] and [14].)

The first is the *fractional integral of order* λ $(\lambda > 0)$ of a function $f(z)$ analytic in a simply-connected of the complex plane containing the origin, which is denoted by $D_z^{-\lambda}\{f(z)\}$ $\big(or\ D_z^{-\lambda}\{f\}\big)$ and also defined by

$$D_z^{-\lambda}\{f(z)\} = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}d\zeta,$$

where the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $log(z-\zeta)$ to be real when $z-\zeta > 0$.

The second is, under the hypothesis of the definition above, the *fractional derivative* of a function $f(z)$ of order λ, which is denoted by $D_z^{\lambda}\{f(z)\}$ $\big(or\ D_z^{\lambda}\{f\}\big)$ and also defined by

$$D_z^{\lambda}\{f(z)\} = \begin{cases} \dfrac{1}{\Gamma(1-\lambda)}\displaystyle\int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}}d\zeta & \big(0 \le \lambda < 1\big) \\ \dfrac{d^m}{dz^m}\Big(D_z^{\lambda-m}\big(f(z)\big)\Big) & \big(m \le \lambda < m+1; m \in \mathbb{N}\big). \end{cases}$$

Recently, for a function $f(z) \in \mathcal{A}(p)$, the extended *fractional differ-integral operator*:

$$\Omega_z^{(\lambda,p)} \ : \ \mathcal{A}(p) \ \to \ \mathcal{A}(p)$$

was defined by the following form:

$$\Omega_z^{(\lambda,p)}[f] := \Omega_z^{(\lambda,p)}[f(z)] := \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda}\{f(z)\}, \tag{3}$$

where $D_z^{\lambda}\{f(z)\}$ is the fractional integral of order λ $(0 \le \lambda < p+1)$. From just above, for a function $f(z)$ belonging to the class $\mathcal{A}(p)$, we obvious arrive at the following result:

$$\Omega_z^{(\lambda,p)}[f] = z^p + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+p+1-\lambda)} a_k z^k, \tag{4}$$

where $p \in \mathbb{N}$, $\lambda < p+1$ and $z \in \mathbb{U}$. From this result, of course, after simple calculation, the following identity can be easily derived that

$$z\Big(\Omega_z^{(\lambda,p)}[f]\Big)' = (p-\lambda)\Omega_z^{(1+\lambda,p)}[f] + \lambda\Omega_z^{(\lambda,p)}[f], \tag{5}$$

where $p \in \mathbb{N}$, $\lambda < p+1$ and $z \in \mathbb{U}$. For these and also some of their applications, see [1], [12] and [16].

Very recently, for a function $f(z) \in \mathcal{A}(p)$, the *differential operator* $\mathcal{D}_z^q\{f\}$ $\big(or$ $\mathcal{D}_z^q\{f(z)\}\big)$, which is the q th-*order ordinary differential operator*, was defined by the following form:

$$\mathcal{D}_z^q[f] := \mathcal{D}_z^q[f(z)] := \left(\frac{p!}{(p-q)!}z^p + \sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}a_k z^k\right) z^{-q}, \tag{6}$$

where $p \in \mathbb{N}$, $q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $p > q$ and $z \in \mathbb{U}$. For example, one may check it and also its results in [4], [5] and [8].

By taking into account the operators given by (4) and (6), for a function $f(z) \in \mathcal{A}(p)$, we may next define a fractional differential operator represented by $W^{\lambda,p}_{z,q}[f]$ $\big(or\ W^{\lambda,p}_{z,q}[f(z)]\big)$, which is in the following form:

$$\begin{aligned} W^{\lambda,p}_{z,q}[f] &:= \mathcal{D}^q_z\Big\{\Omega^{(\lambda,p)}_z[f]\Big\} \\ &:= \Bigg(\frac{p!}{(p-q)!}z^p \\ &\quad + \sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\frac{\Gamma(k+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+p+1-\lambda)}a_k z^k\Bigg)\, z^{-q}, \end{aligned} \tag{7}$$

where $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, $p > q$, $\lambda < p+1$ and $z \in \mathbb{U}$.

In view of the information (4)-(7), it is clear that there can expose several relationships between the related operators and the functions class $\mathcal{A}(p)$. A number of them can be also presented by the following forms:

$$W^{0,p}_{z,q}[f] \equiv \mathcal{D}^q_z\Big\{\Omega^{(0,p)}_z[f]\Big\} \equiv \mathcal{D}^q_z[f] \equiv f^{(q)}(z) \quad \big(p > q\big), \tag{8}$$

$$z\Big(W^{\lambda,p}_{z,q}[f]\Big)' \equiv zW^{\lambda,p}_{z,1+q}[f] \quad \big(p > q\big), \tag{9}$$

$$W^{\lambda,p}_{z,1}[f] \equiv \mathcal{D}^1_z\Big\{\Omega^{(\lambda,p)}_z[f]\Big\} \equiv \big(\Omega^{(\lambda,p)}_z[f]\big)', \tag{10}$$

$$W^{\lambda,p}_{z,0}[f] \equiv \mathcal{D}^0_z\Big\{\Omega^{(\lambda,p)}_z[f]\Big\} \equiv \Omega^{(\lambda,p)}_z[f], \tag{11}$$

$$W^{0,p}_{z,0}[f] \equiv \Omega^{(0,p)}_z[f] \equiv f(z) \tag{12}$$

and

$$z\Big(W^{0,p}_{z,0}[f]\Big)' \equiv z\Big(\Omega^{(0,p)}_z[f]\Big)' \equiv zf'(z), \tag{13}$$

where $f(z) \in \mathcal{A}(p)$ and $p \in \mathbb{N}$, $\lambda < p+1$ and $z \in \mathbb{U}$.

As certain applications of the comprehensive relationships like (8)-(13) can be obtained by the related operators and also their special cases, from time to time, we encounter several similar results in the literature. More particularly, the results associated by the (ordinary or fractional type) equations and also their consequences in the complex plane have important roles for nearly all natural sciences and also engineering. For some of those, one can look over the papers in the references given by [1], [5], [12] and [15].

In order to prove the main results related to both certain order (ordinary *or* fractional differential) equations in the complex plane and certain multivalently analytic functions, the following well-known lemma, which is a general form of the assertion obtained by [10] (and see also [11]), will be required.

Lemma 1. Let $w(z)$ be a regular function in $\mathbb{U}$ and also satisfy the following conditions:

$$w(0) = w'(0) = w''(0) = \cdots = w^{(m)}(0) = 0 \tag{14}$$

for all $m = 0, 1, 2, \cdots, n$ when $n \in \mathbb{N}$. If $\big|w(z)\big|$ attains its maximum value on the circle:

$$\Big\{z \in \mathbb{C} : |z| = r \quad (0 < r < 1)\Big\}$$

at a point z_0, then we have

$$zw'(z)\big|_{z=z_0} = \kappa\, w(z)\big|_{z=z_0}, \tag{15}$$

where $\kappa \geq n+1$ and $n \in \mathbb{N}$.

2. A Set Of The Main Results

In this section, some general results will be given as a set of our main results. The first consisting of an application of the operator $W^{\lambda,p}_{z,q}[\cdot]$ to the functions in $\mathcal{A}(p)$ is in the following form (Theorem 1 below).

Theorem 1. Let a complex function $\Psi(z)$ satisfy the following inequality:

$$(p-q)!\,\big|\Psi(z)\big| < (p-q)\,p! \quad \big(p > q; p \in \mathbb{N}; q \in \mathbb{N}_0; z \in \mathbb{U}\big). \tag{16}$$

If an analytic function $w := w(z)$ in the class $\mathcal{A}(p)$ is a solution for the (complex) equation given by

$$z\,W^{\lambda,p}_{z,1+q}[\omega] + W^{\lambda,p}_{z,q}[\omega] - \Psi(z) \,=\, 0\,, \tag{17}$$

then the inequality:

$$(p-q)!\,\Big|W^{\lambda,p}_{z,q}[\omega]\Big| < (p-q)\,p! \tag{18}$$

is satisfied, where $p > q$, $\lambda < p+1$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

Proof. Under the conditions $\lambda\ \big(\lambda < p+1\big)$, $p > q$, $p \in \mathbb{N}$ and $q \in \mathbb{N}_0$, we assume that the function $w := w(z)$ belonging to the class $\mathcal{A}(p)$ is in the form (2) and also is a solution for the equation given by (17). In the light of the operator $W^{\lambda,p}_{z,q}[\cdot]$ defined by (7), if define a function $q(z)$ as

$$W^{\lambda,p}_{z,q}[\omega] = \frac{p!}{(p-q)!}q(z) \quad \big(z \in \mathbb{U}\big), \tag{19}$$

then, clearly, the function $q(z)$ is both an analytic function in $\mathbb{U}$ and has the form given by the following series:

$$q(z) = z^{p-q} + c_{1+p}z^{p-q+1} + c_{2+p}z^{p-q+2} + \cdots \ \big(p > q; p \in \mathbb{N}; q \in \mathbb{N}_0\big).$$

Indeed, the conditions in (14) are satisfied for the function $q^{(m)}(z)$ for all positive integer m $\big(0 \leq m \leq p-q-1\big)$. By differentiating of the both sides of (19) with respect to the complex variable z,

$$z\Big(W^{\lambda,p}_{z,q}[\omega]\Big)' = \frac{p!}{(p-q)!}zq'(z) \quad \big(z \in \mathbb{U}\big),$$

or, equivalently,

$$z\,W^{\lambda,p}_{z,1+q}[\omega] = \frac{p!}{(p-q)!}zq'(z) \quad \big(z \in \mathbb{U}\big) \tag{20}$$

is then obtained. By combining (19) and (20),

$$\begin{aligned} z\,W^{\lambda,p}_{z,1+q}[w] + W^{\lambda,p}_{z,q}[\omega]\ \Big(:= \Psi(z)\Big) \qquad (say.) \\ = \frac{p!}{(p-q)!}\big[zq'(z) + q(z)\big] \quad \big(z \in \mathbb{U}\big) \end{aligned} \tag{21}$$

is also derived.

Now, assume that there exists a point $z_0 \in \mathbb{U}$ such that

$$|w(z_0)| = \max\Big\{|w(z)|\ :\ |z| \leq |z_0|\ \ \big(z \in \mathbb{U}\big)\Big\} = 1.$$

By means of (15) of Lemma 1, from (21), it follows:

$$\begin{aligned}\left|\Psi\big(z_0\big)\right| &= \frac{p!}{(p-q)!}\left|zq'(z)+\kappa q(z)\big|_{z=z_0}\right| \\ &= \frac{p!}{(p-q)!}(\kappa+1)\big|q(z_0)\big| \\ &\geq \frac{p!(p-q)}{(p-q)!}\ \ \big(since\ \kappa \geq p-q-1\ \ and\ \big|q(z_0)\big|=1\big),\end{aligned}$$

which is a contradiction with the assumption (16) of Theorem 1. Hence, $|w(z)|<1$ for all $z \in \mathbb{U}$. Therefore, it immediately follows from (19) the inequality (18). Thus, the desired proof is completed.

The second general result including an application of the operator $W_{z,q}^{\lambda,p}[\cdot]$ to the functions in $\mathcal{A}(n)$ is also in the following form (Theorem 2 below).

Theorem 2. Let the function $\Psi(z)$ satisfy the inequality:

$$2\,\Re e\big\{\Psi(z)\big\} < 1 \qquad \big(z \in \mathbb{U}\big). \tag{22}$$

If an analytic function $\omega := \omega(z) \in \mathcal{A}(p)$ is a solution for the (complex) equation given by

$$\begin{aligned}&\Big(W_{z,q}^{\lambda,p}[\omega]\Big)\cdot\Big(z^2\,W_{z,2+q}^{\lambda,p}[\omega]\Big)-\Big(z\,W_{z,1+q}^{\lambda,p}[\omega]\Big)^2 \\ &\qquad + \big[1-\Psi(z)\big]\cdot\Big(W_{z,q}^{\lambda,p}[\omega]\Big)\cdot\Big(z\,W_{z,1+q}^{\lambda,p}[\omega]\Big) \;=\; 0\,,\end{aligned} \tag{23}$$

the inequality:

$$\left|z\,W_{z,1+q}^{\lambda,p}[\omega]-\big(p-q\big)W_{z,q}^{\lambda,p}[\omega]\right| < \big(p-q\big)\left|W_{z,q}^{\lambda,p}[\omega]\right| \tag{24}$$

is satisfied, where $p>q$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

Proof. Under the conditions λ $\big(\lambda < p+1\big)$, $p>q$, $p \in \mathbb{N}$ and $q \in \mathbb{N}_0$, we suppose that the function $\omega := \omega(z) \in \mathcal{A}(p)$ is a solution of the equation given by (23). Then, with the help of (7), define a function $q(z)$ in the implicit form:

$$\frac{z\Big(W_{z,q}^{\lambda,p}[\omega]\Big)'}{W_{z,q}^{\lambda,p}[\omega]} = (p-q)\big[1+q(z)\big] \quad \big(z \in \mathbb{U}\big),$$

or, in the equivalent form:

$$\frac{z\Big(W_{z,q}^{\lambda,p}[\omega]\Big)'}{W_{z,q}^{\lambda,p}[\omega]} \equiv \frac{zW_{z,1+q}^{\lambda,p}[\omega]}{W_{z,q}^{\lambda,p}[\omega]} = (p-q)\big[1+q(z)\big] \quad \big(z \in \mathbb{U}\big), \tag{25}$$

where, of course, $W_{z,q}^{\lambda,p}[\omega] \neq 0$ for all $q \in \mathbb{N}$. Clearly, the function $1+q(z)$ belongs to $\mathcal{P}$ given by (1), i.e., $q(z) \in \mathcal{A}(1)$, and it is analytic in $\mathbb{U}$ with $q(0)=0$ and also noting that the point $z=0$ is a removable singular point for the expression which is the left side of (25). Then, (25) gives us

$$\frac{z\left(\dfrac{z\Big(W_{z,q}^{\lambda,p}[\omega]\Big)'}{W_{z,q}^{\lambda,p}[\omega]}\right)'}{\dfrac{z\Big(W_{z,q}^{\lambda,p}[\omega]\Big)'}{W_{z,q}^{\lambda,p}[\omega]}} = \frac{zq'(z)}{1+q(z)} \quad \big(z \in \mathbb{U}\big),$$

or, equivalently,

$$\frac{z\left(\frac{zW_{z,1+q}^{\lambda,p}[\omega]}{W_{z,q}^{\lambda,p}[\omega]}\right)'}{\frac{zW_{z,1+q}^{\lambda,p}[\omega]}{W_{z,q}^{\lambda,p}[\omega]}} = \Big(\Psi(z) :=\Big)\ \frac{zq'(z)}{1+q(z)}\ . \quad (say.) \tag{26}$$

With the help of $\Psi(z)$ and after simple calculation, it is easy to see that the identity in (26) is equal to the complex equation given by (23) and, accordingly, the related equation is satisfied by that function $\Psi(z)$ as also given by (23).

Assuming now that there exists a point $z_0 \in \mathbb{U}$ such that

$$|w(z_0)| = \max\Big\{|w(z)| \ : \ |z| \leq |z_0| \ \ (z \in \mathbb{U})\Big\} = 1.$$

With the help of (15) of Lemma 1, from (26), it is easily seen that

$$\Re e\Big(\Psi\big(z_0\big)\Big) = \Re e\left(\frac{zq'(z)}{1+q(z)}\bigg|_{z=z_0}\right) = \frac{\kappa}{2} \geq \frac{1}{2} \quad (since\ \kappa \geq 1),$$

which is a contradiction with the assumption (22) of the Theorem 2. Hence, $|w(z)| < 1$ for all $z \in \mathbb{U}$. Thus, the equality (25) immediately yields that the inequality (24). Therefore, the proof of the related theorem is completed.

3. A Number Of Consequences Of The Main Results

In this section, some special results of our main results will be also presented. As we indicated before, clearly, there are many results consisting of the connections between the operators, given by (4), (6) and (7), and functions in $\mathcal{A}(p)$. For both those and other possible relations like (8)-(13), it is enough to choose the suitable values of the parameters λ, q and/or p in the theorems. Most especially, some of them appertaining to Geometric Function Theory, which is the study of the relations between the analytic properties of a function $f(z)$ and the geometric properties of the image domain $f\big(\mathbb{U}\big)$, are important results for literature (see their details in [2], [3] and [17]) and also some of those special consequences, which are here omitted, are comparable with the earlier several results obtained by [5], [6], [7], [8] and [17]. All right, it is not possible to reveal and to indicate all of the consequences of the main results but we want to present some of them, which are known or new.

The following corollary (Corollary 1 below) is the first special consequence consisting of one of the applications of the fractional differ-integral operator to the functions in $\mathcal{A}(p)$ can be given by taking $q := 0$ in the Theorem 1.

Corollary 1. Let the complex function $\Psi(z)$ satisfy the following inequality:

$$\big|\Psi(z)\big| < p \quad \big(p \in \mathbb{N}; z \in \mathbb{U}\big). \tag{27}$$

If an analytic function $\omega := \omega(z)$ in the class $\mathcal{A}(p)$ is a solution for the following equation:

$$(p-\lambda)\Omega_z^{(1+\lambda,p)}[\omega] + (1+\lambda)\Omega_z^{(\lambda,p)}[\omega] - \Psi(z) \,=\, 0\,,$$

then

$$\Big|\Omega_z^{(\lambda,p)}[\omega]\Big| < p \quad \big(\lambda < p+1; p \in \mathbb{N}; z \in \mathbb{U}\big).$$

The following corollary (Corollary 2 below) is the second special consequence containing one of the applications of the ordinary differential operator to the functions in $\mathcal{A}(p)$ can be next given by setting $\lambda := 0$ in the Theorem 1.

Corollary 2. Let the complex function $\Psi(z)$ satisfy the inequality given by (16). If an analytic function $w := w(z)$ in the class $\mathcal{A}(p)$ is a solution for the following equation:

$$z\,\mathcal{D}_z^{1+q}\{w\} + \mathcal{D}_z^{q}\{w\} - \Psi(z) \,=\, 0\,,$$

then

$$\Big|\mathcal{D}_z^{q}\{w\}\Big| < \frac{(p-q)p!}{(p-q)!} \quad \big(p > q; p \in \mathbb{N}; q \in \mathbb{N}_0; z \in \mathbb{U}\big).$$

The following special result (Corollary 3 below) is the next special consequence appertaining to multivalently analytic functions in $\mathcal{A}(p)$ can be also obtained by putting $q := 0$ in the Corollary 2 (or $q := \lambda := 0$ in the Theorem 1).

Corollary 3. Let the function $\Psi(z)$ satisfy the inequality in (27). If the function $\omega(z) \in \mathcal{A}(p)$ is a solution of the complex differential equation: $z\omega'(z) - \omega(z) - \Psi(z) = 0$, then $\big|\omega(z)\big| < p$, where $p \in \mathbb{N}$ and $z \in \mathbb{U}$.

The following corollary (Corollary 4 below) is the third special consequence involving one of the applications of the fractional differ-integral operator to the functions in the class $\mathcal{A}(p)$ can be then presented by receiving $q := 0$ in the Theorem 2.

Corollary 4. Let the function $\Psi(z)$ satisfy the inequality given by (22). If an analytic function $\omega := \omega(z) \in \mathcal{A}(p)$ is a solution of the following equation:

$$\begin{aligned} z \cdot \left(\Omega_z^{(\lambda,p)}[\omega]\right) \cdot \left(\Omega_z^{(\lambda,p)}[\omega]\right)'' - z \cdot \left[\left(\Omega_z^{(\lambda,p)}[\omega]\right)'\right]^2 & \\ +\big[1 - \Psi(z)\big] \cdot \left(\Omega_z^{(\lambda,p)}[\omega]\right) \cdot \left(\Omega_z^{(\lambda,p)}[\omega]\right)' \,=\, 0\,, & \end{aligned}$$

then

$$\left|z\left(\Omega_z^{(\lambda,p)}[\omega]\right)' - p\,\Omega_z^{(\lambda,p)}[\omega]\right| < p\left|\Omega_z^{(\lambda,p)}[\omega]\right|,$$

where $\lambda < p+1$, $p \in \mathbb{N}$ and $z \in \mathbb{U}$.

The following result (Corollary 5 below) is the next special consequence relating to multivalently starlikeness of a function $\omega(z) \in \mathcal{A}(p)$ can be also stated by choosing $\lambda := 0$ in the Corollary 4.

Corollary 5. Let the function $\Psi(z)$ satisfy the inequality in (16) and also let the function $\omega := \omega(z) \in \mathcal{A}(p)$ be the solution for the complex differential equation:

$$z\,\omega\,\omega'' \,-\, z\left[\,\omega'\,\right]^2 \,+\, \big(1 - \Psi(z)\big)\,\omega\,\omega' \,=\, 0\,.$$

Then,

$$\big|z\,\omega' - p\,\omega\big| < p\,\big|\omega\big| \quad \big(p \in \mathbb{N}; z \in \mathbb{U}\big),$$

that is, that $\omega(z)$ is a multivalently starlike function (w.r.t. origin) in $\mathbb{U}$.

The following remark is the next consequence concerning one of the applications of the ordinary differential operator to the analytic functions in the class $\mathcal{A}(p)$ can be given by taking $\lambda := 0$ in the Theorem 2.

Remark 1. [[5], Theorem 1] Let the function $\Psi(z)$ satisfy the inequality given by (22). If an analytic function $\omega := \omega(z) \in \mathcal{A}(p)$ is a solution of the complex equation:

$$\Big(\mathcal{D}_z^q\{\omega\}\Big) \cdot \Big(z^2\, \mathcal{D}_z^{2+q}\{\omega\}\Big) - \Big(z\, \mathcal{D}_z^{1+q}\{\omega\}\Big)^2 + \big[1 - \Psi(z)\big] \cdot \Big(\mathcal{D}_z^q\{\omega\}\Big) \cdot \Big(z\, \mathcal{D}_z^{1+q}\{\omega\}\Big) \;=\; 0\,,$$

then

$$\Big|z\, \mathcal{D}_z^{1+q}\{\omega\} - (p-q)\, \mathcal{D}_z^q\{\omega\}\Big| < (p-q)\Big|\mathcal{D}_z^q\{\omega\}\Big|\,,$$

where $p > q$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ and $z \in \mathbb{U}$.

The next remark (below) is the special consequence relating to multivalently convexity of a function in $\mathcal{A}(p)$ can be lastly given by setting $q := 1$ in the Remark 1.

Remark 2. [[5], Corollary 2] Let the function $\Psi(z)$ satisfy the inequality in (22) and also let the function $\omega := \omega(z) \in \mathcal{A}(p)$ be the solution for the complex differential equation:

$$z\,\omega'\,\omega''' \,-\, z\,\big[\omega''\big]^2 \,+\, \big(1 - \Psi(z)\big)\,\omega'\,\omega'' \,=\, 0\,.$$

Then,

$$\big|z\,\omega'' \,-\, (p-1)\,\omega'\big| < (p-1)\,\big|\omega'\big| \quad \big(p \in \mathbb{N} - \{1\}; z \in \mathbb{U}\big)\,,$$

that is, that $\omega(z)$ is a multivalently convex function (w.r.t. origin) in $\mathbb{U}$.

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Hüseyin Irmak
Department of Mathematics, Faculty of Science, Çankiri Karatekin University, TR-18100, Uluyazi Campus, Çankiri, TURKEY
E-mail address: hisimya@yahoo.com , hirmak@karatekin.edu.tr