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A SUBCLASS OF MULTIVALENT FUNCTIONS DEFINED BY USING FAMILIAR FRACTIONAL DERIVATIVE OPERATOR

SH. KHOSRAVIANARAB, S.R. KULKARNI AND O.P. AHUJA

ABSTRACT. The authors introduce a new class $\mathcal{T}(n,p,\lambda,\alpha,\xi)$ of functions which are analytic and p–valent in the open unit disk $\mathcal{U}$ by means of fractional derivative operator. Coefficient estimates, radii of starlikeness and convexity, and many other interesting and useful properties and characteristics of this class are obtained.

1. Introduction

In generalized integration and differentiation, it is natural for mathematician to ask question: Can the meaning of derivatives of integer order $\frac{d^n y}{dx^n}$ be extended to have meaning where n is any fractional , irrational, or even complex? Such questions attracted the attention of several mathematicians in the last two centuries.
Fractional calculus is old but studied little. In this paper, we make use of the following well–known fractional calculus operators $D_z^{-\xi}$, D_z^{ξ}, and $D_z^{n+\xi}$. For an analytic function f defined in a simply connected region of the complex z–plane containing its origin, these operators are designed as follows:
Definition 1. The fractional integral of order ξ is defined, for a function f, by

$$D_z^{-\xi} f(z) = \frac{1}{\Gamma(\xi)} \int_0^z \frac{f(t)}{(z-t)^{1-\xi}} dt \quad (\xi > 0)\,,$$

where f is analytic in a simply–connected region of the complex z–plane containing the origin, and the multiplicity of $(z-t)^{\xi-1}$ is removed by requiring $log(z-t)$ to be real when $z-t>0$.
Definition 2. The fractional derivative of order ξ is defined, for a function f, by

$$D_z^{\xi} f(z) = \frac{1}{\Gamma(1-\xi)} \int_0^z \frac{f(t)}{(z-t)^{\xi}} dt \quad (0 \le \xi < 1)\,,$$

where the function f is constrained, and the multiplicity of $(z-t)^{-\xi}$ is removed, as in Definition 1 above.

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Definition 3. The fractional derivative of order $n+\xi$ is defined, for a function f, by

$$D_z^{n+\xi} f(z) = \frac{d^n}{dz^n} D_z^{\xi} f(z) \quad \left(0 \le \xi < 1; \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\right).$$

Applying Definitions 1, 2 and 3, it is easily seen that

$$D_z^{-\xi} z^k = \frac{\Gamma(k+1)}{\Gamma(k+\xi+1)} z^{k+\xi} \quad (k \in \mathbb{N},\ \xi > 0)\,, \tag{1.1}$$

$$D_z^{\xi} z^k = \frac{\Gamma(k+1)}{\Gamma(k-\xi+1)} z^{k-\xi} \quad (k \in \mathbb{N},\ 0 \le \xi < 1)\,, \tag{1.2}$$

and

$$D_z^{q+\xi} z^k = \frac{d^q}{dz^q} D_z^{\xi} z^k = \frac{\Gamma(k+1)}{\Gamma(k-q-\xi+1)} z^{k-(q+\xi)}$$

$$(q \in \mathbb{N}_0,\ k \in \mathbb{N},\ 0 \le \xi < 1;\ q \le k \ for\ \xi = 0)\,.$$

Let $\mathcal{T}(n,p)$ denote the class of functions f of the form

$$f(z) = z^p - \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \qquad (a_{p+k} \ge 0;\ p \in \mathbb{N} = \{1,2,3,...\};\ n \in \mathbb{N})\,, \tag{1.3}$$

which are analytic in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.
A function $f \in \mathcal{T}(n,p)$ is said to be in the class $\mathcal{T}(n,p,\lambda,\alpha,\xi)$ if it satisfies the inequality

$$\Re\left\{\frac{zD_z^{\xi+1} f(z) + \lambda z^2 D_z^{\xi+2} f(z)}{(1-\lambda) D_z^{\xi} f(z) + \lambda z D_z^{\xi+1} f(z)}\right\} > \alpha, \tag{1.4}$$

for some $\alpha\ (0 \le \alpha < p), \lambda\ (0 \le \lambda \le 1), \xi\ (0 \le \xi < 1)$ and for all $z \in \mathcal{U}$.
The class

$$\mathcal{T}(n,p,\lambda,\alpha) \equiv \mathcal{T}(n,p,\lambda,\alpha,0),$$

was studied earlier by Altintas et al. [1]. The special classes

$$\mathcal{T}_\alpha(n,p) \equiv \mathcal{T}(n,p,0,\alpha,0),$$

and

$$\mathcal{C}_\alpha(n,p) \equiv \mathcal{T}(n,p,1,\alpha,0),$$

are the classes of p–valently starlike functions of order α and p–valently convex functions of order α in $\mathcal{U}$ $(0 \le \alpha < p)$; respectively, that is,

$$\mathcal{T}_\alpha(n,p) = \left\{ f\ :\ f \in \mathcal{T}(n,p)\ \ and\ \ \Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha\ \ (0 \le \alpha < p;\ z \in \mathcal{U}) \right\},$$

and

$$\mathcal{C}_\alpha(n,p) = \left\{ f\ :\ f \in \mathcal{T}(n,p)\ \ and\ \ \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha\ \ (0 \le \alpha < p;\ z \in \mathcal{U}) \right\}.$$

Note that he classes

$$\mathcal{T}(n,1,0,\alpha,0) = \mathcal{T}_\alpha(n) \quad \text{and} \quad \mathcal{T}(n,1,1,\alpha,0) = \mathcal{C}_\alpha(n),$$

were studied earlier by Srivastava et al. [6]. In fact, Silverman [5] is the first researcher who studied the classes

$$\mathcal{T}(1,1,0,\alpha,0) = \mathcal{T}^*(\alpha) = T_\alpha(1) \quad \text{and} \quad \mathcal{T}(1,1,1,\alpha,0) = \mathcal{C}(\alpha),$$

Finally, a function $f \in \mathcal{T}(n,p)$ is said to be p–valently close–to–convex of order α if it satisfies the condition

$$\Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \quad (0 \leq \alpha < p;\ z \in \mathcal{U};\ p \in \mathbb{N}).$$

The object of the present paper is to investigate various interesting properties of functions belonging to the class $\mathcal{T}(n,p,\lambda,\alpha,\xi)$.

2. Coefficient bounds

In the following theorem, we obtain a necessary and sufficient condition for a function f to belong to the class $\mathcal{T}(n,p,\lambda,\alpha,\xi)$.

Theorem 1. A function $f \in \mathcal{T}(n,p)$ is in $\mathcal{T}(n,p,\lambda,\alpha,\xi)$ if and only if

$$\sum_{k=n}^{\infty} U(k)\frac{p+k-\xi-\alpha}{p-\xi-\alpha}a_{k+p} \leq 1 \tag{2.1}$$

$$(0 \leq \alpha < p;\ 0 \leq \lambda \leq 1;\ n \in \mathbb{N};\ p \in \mathbb{N};\ T \geq p-\xi),$$

where

$$U(k) = \frac{\Gamma(p+k+1)\Gamma(p-\xi+1)\left[1+\lambda(p+k-\xi-1)\right]}{\Gamma(p+k-\xi+1)\Gamma(p+1)\left[1+\lambda\left(p-\xi-1\right)\right]}, \tag{2.2}$$

and

$$T = \frac{\Gamma(p+1)}{\Gamma(p-\xi+1)}\left[1+\lambda(p-\xi-1)\right](p-\xi-\alpha). \tag{2.3}$$

Proof: Let

$$f(z) = z^p - \sum_{k=n}^{\infty} a_{k+p}z^{k+p} \in \mathcal{T}(n,p,\lambda,\alpha,\xi).$$

Then

$$D_z^{\xi}f(z) = \frac{\Gamma(p+1)}{\Gamma(p-\xi+1)}z^{p-\xi} - \sum_{k=n}^{\infty}\frac{\Gamma(p+k+1)}{\Gamma(p+k-\xi+1)}a_{k+p}z^{k+p-\xi}, \tag{2.4}$$

$$D_z^{\xi+1}f(z) = \frac{\Gamma(p+1)}{\Gamma(p-\xi)}z^{p-\xi-1} - \sum_{k=n}^{\infty}\frac{\Gamma(p+k+1)}{\Gamma(p+k-\xi)}a_{k+p}z^{k+p-\xi-1}, \tag{2.5}$$

and

$$D_z^{\xi+2}f(z) = \frac{\Gamma(p+1)}{\Gamma(p-\xi-1)}z^{p-\xi} - \sum_{k=n}^{\infty}\frac{\Gamma(p+k+1)}{\Gamma(p+k-\xi-1)}a_{k+p}z^{k+p-\xi-2}. \tag{2.6}$$

From (1.4), (2.4), (2.5) and (2.6), we get

$$\Re\left\{\frac{\frac{\Gamma(p+1)}{\Gamma(p-\xi)}\left[1+\lambda\left(p-\xi-1\right)\right]z^{p-\xi} - \sum\limits_{k=n}^{\infty}\frac{\Gamma(p+k+1)}{\Gamma(p+k-\xi)}\left[1+\lambda\left(p+k-\xi-1\right)\right]a_{k+p}z^{p+k-\xi}}{\frac{\Gamma(p+1)}{\Gamma(p-\xi+1)}\left[1+\lambda\left(p-\xi-1\right)\right]z^{p-\xi} - \sum\limits_{k=n}^{\infty}\frac{\Gamma(p+k+1)}{\Gamma(p+k-\xi+1)}\left[1+\lambda\left(p+k-\xi-1\right)\right]a_{k+p}z^{p+k-\xi}}\right\} > \alpha.$$

If we choose z to be real and let $z \to 1-$, we get

$$\frac{\frac{T(p-\xi)}{p-\xi-\alpha} - \sum\limits_{k=n}^{\infty}\frac{\Gamma(p+k+1)}{\Gamma(p+k-\xi)}\left[1+\lambda\left(p+k-\xi-1\right)\right]a_{k+p}}{\frac{T}{p-\xi-\alpha} - \sum\limits_{k=n}^{\infty}\frac{\Gamma(p+k+1)}{\Gamma(p+k-\xi+1)}\left[1+\lambda\left(p+k-\xi-1\right)\right]a_{k+p}} \geq \alpha,$$

where T is defined by (2.3), or, equivalently,

$$\sum_{k=n}^{\infty} \frac{\Gamma(p+k+1)}{\Gamma(p+k-\xi+1)} \left[1+\lambda(p+k-\xi-1)\right](p+k-\xi-\alpha)\, a_{k+p} \leq T,$$

which leads us to the assertion (2.1).
Conversely, let the inequality (2.1) holds true and

$$z \in \partial\mathcal{U} = \{z : \ z \in \mathbb{C} \ and \ |z| = 1\}.$$

It is sufficient to prove that

$$\left| \frac{zD_z^{\xi+1}f(z) + \lambda z^2 D_z^{\xi+2} f(z)}{(1-\lambda)D_z^{\xi}f(z) + \lambda z D_z^{\xi+1} f(z)} - T \right| \leq T - \alpha. \tag{2.7}$$

Suppose that

$$\left| \frac{zD_z^{\xi+1}f(z) + \lambda z^2 D_z^{\xi+2} f(z)}{(1-\lambda)D_z^{\xi}f(z) + \lambda z D_z^{\xi+1} f(z)} - T \right| = \left| \frac{M(z)}{N(z)} \right|, \tag{2.8}$$

where

$$M(z) = (1-\lambda T)\, zD_z^{\xi+1}f(z) - (1-\lambda)\, T D_z^{\xi} f(z) + \lambda z^2 D_z^{\xi+2} f(z),$$

and

$$N(z) = (1-\lambda)D_z^{\xi}f(z) + \lambda z D_z^{\xi+1} f(z).$$

Using (2.4), (2.5) and (2.6), we get

$$\begin{aligned} M(z) &= \frac{\Gamma(p+1)}{\Gamma(p-\xi+1)} (p-\xi-T)\left(1+\lambda\left(p-\xi-1\right)\right) z^{p-\xi} \\ &- \sum_{k=n}^{\infty} \frac{\Gamma(p+k+1)}{\Gamma(p+k-\xi+1)} (p+k-\xi-T)\left(1+\lambda\left(p+k-\xi-1\right)\right) a_{k+p} z^{k+p-\xi}, \end{aligned}$$

and

$$\begin{aligned} N(z) &= \frac{\Gamma(p+1)}{\Gamma(p-\xi+1)} \left(1+\lambda\left(p-\xi-1\right)\right) z^{p-\xi} \\ &- \sum_{k=n}^{\infty} \frac{\Gamma(p+k+1)}{\Gamma(p+k-\xi+1)} \left(1+\lambda\left(p+k-\xi-1\right)\right) a_{k+p} z^{k+p-\xi}. \end{aligned}$$

Since $T \geq p - \xi$, we get

$$\left| \frac{M(z)}{N(z)} \right| \leq \frac{\frac{T(p-\xi-T)}{p-\xi-\alpha} + \sum\limits_{k=n}^{\infty} \frac{\Gamma(p+k+1)}{\Gamma(p+k-\xi+1)} (p+k-\xi-T)\left(1+\lambda\left(p+k-\xi-1\right)\right) a_{k+p}}{\frac{T}{p-\xi-\alpha} - \sum\limits_{k=n}^{\infty} \frac{\Gamma(p+k+1)}{\Gamma(p+k-\xi+1)} \left(1+\lambda\left(p+k-\xi-1\right)\right) a_{k+p}}.$$

Thus, from (2.8), we have the desired inequality (2.7) if

$$\sum_{k=n}^{\infty} \frac{\Gamma(p+k+1)\left[1+\lambda(p+k-\xi-1)\right](p+k-\xi-\alpha)}{\Gamma(p+k-\xi+1)} a_{k+p} \leq \frac{\Gamma(p+1)\left[1+\lambda(p-\xi-1)\right](p-\xi-\alpha)}{\Gamma(p-\xi+1)},$$

which leads to the inequality (2.1). In view of the maximum modulus theorem, we find that $f(z) \in \mathcal{T}(n, p, \lambda, \alpha, \xi)$.
The condition (2.1) is sharp for the function f given by

$$f(z) = z^p - \frac{p-\xi-\alpha}{U(k)(p+k-\xi-\alpha)} z^{k+p},$$

where $U(k)$ is defined by (2.2).
As an immediate consequence of Theorem 1, we have the following result.
Corollary 1([1, p. 10, Theorem 1]). A function $f \in \mathcal{T}(n,p)$ is in the class $\mathcal{T}(n,p,\lambda,\alpha)$ if and only if

$$\sum_{k=n}^{\infty}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)a_{k+p} \leq (p-\alpha)(1+\lambda p-\lambda)$$

$$(0 \leq \alpha < 1;\ 0 \leq \lambda \leq 1;\ p \in \mathbb{N}\ (p \neq 1);\ n \in \mathbb{N};\ \lambda(p-1)(p-\alpha) \geq \alpha\ ).$$

In its special case, when $\xi = 1$, Theorem 1 yields the following result.
Corollary 2. Let the function f be in the class $\mathcal{T}(n,p)$. Then

$$\Re\left\{\frac{zf''(z)+\lambda z^2 f'''(z)}{(1-\lambda)f'(z)+\lambda z f''(z)}\right\} > \alpha,$$

if and only if

$$\sum_{k=n}^{\infty}\frac{(p+k)\left[1+\lambda(p+k-2)\right](p+k-1-\alpha)}{p\left[1+\lambda(p-2)\right](p-\alpha-1)} \leq 1$$

$$(0 \leq \alpha < 1;\ 0 \leq \lambda \leq 1;\ p \in \mathbb{N}\ (p \neq 1);\ n \in \mathbb{N};\ p\left[1+\lambda(p-2)\right](p-\alpha-1) \geq p-1).$$

Theorem 2. Let the function $f_j\ \ (1 \leq j \leq m;\ j \in \mathbb{N})$, defined by

$$f_j(z) = z^p - \sum_{k=n}^{\infty} a_{k+p,j}z^{k+p} \quad (a_{k+p,j} \geq 0;\ p \in \mathbb{N};\ n \in \mathbb{N}),$$

be in the class $\mathcal{T}(n,p,\lambda,\alpha,\xi)$. Then the function

$$h(z) = \sum_{j=1}^{\infty}\gamma_j f_j(z) \quad \left(\gamma_j \geq 0,\ \sum_{j=1}^{m}\gamma_j = 1\right),$$

is also in the class $\mathcal{T}(n,p,\lambda,\alpha,\xi)$.
Proof: Let

$$h(z) = z^p - \sum_{k=n}^{\infty} b_{k+p}z^{k+p},$$

where

$$b_{k+p} = \sum_{j=1}^{m}\gamma_j a_{k+p,j} \quad (k \in \mathbb{N};\ p \in \mathbb{N};\ m \in \mathbb{N}).$$

Making use of Theorem 1, it is sufficient to prove that

$$\sum_{k=n}^{\infty}U(k)\frac{p+k-\xi-\alpha}{p-\xi-\alpha}b_{k+p} \leq 1$$

$$(0 \leq \alpha < 1;\ 0 \leq \lambda \leq 1;\ p \in \mathbb{N}\ (p \neq 1);\ n \in \mathbb{N};\ T \geq p-\xi),$$

where $U(k)$ and T are defined by (2.2) and (2.3), respectively. Under the hypotheses, $f_j \in \mathcal{T}(n,p,\lambda,\alpha,\xi)$ $(1 \leq j \leq m)$, from Theorem 1, we get

$$\sum_{k=n}^{\infty}U(k)\frac{p+k-\xi-\alpha}{p-\xi-\alpha}a_{k+p,j} \leq 1 \tag{2.9}$$

$$(0 \leq \alpha < 1;\ 0 \leq \lambda \leq 1;\ p \in \mathbb{N}\ (p \neq 1);\ n \in \mathbb{N};\ T \geq p-\xi;\ 1 \leq j \leq m).$$

Since $\sum_{j=1}^{m} \gamma_j = 1$, from (2.9), we obtain

$$\sum_{k=n}^{\infty} U(k)\frac{p+k-\xi-\alpha}{p-\xi-\alpha}b_{k+p} = \sum_{j=1}^{m}\gamma_j\sum_{k=n}^{\infty} U(k)\frac{p+k-\xi-\alpha}{p-\xi-\alpha} \leq \sum_{j=1}^{m}\gamma_j = 1,$$

which completes the proof of Theorem 2.
Setting

$$\gamma_1 = 1-\beta, \quad \gamma_2 = \beta, \quad f_1(z) = f(z), \quad and \quad f_2(z) = g(z),$$

in Theorem 2, we obtain the following.
Corollary 3 ([1, p. 11, Theorem 2]). Let the function f be defined by (1.3), and suppose the function g given by

$$g(z) = z^p - \sum_{k=n}^{\infty} b_{k+p}z^{k+p} \quad (b_{k+p} \geq 0;\ p \in \mathbb{N}, n \in \mathbb{N}),$$

is in the class $T(n,p,\lambda,\alpha)$. Then the function h defined by

$$h(z) = (1-\beta)f(z) + \beta g(z),$$

is also in the class $T(n,p,\lambda,\alpha)$.
Theorem 3. Let the function f defined by (1) be in the class $T(n,p,\lambda,\alpha,\xi)$. Then

$$\frac{f(tz)}{t^p} \in T(n,p,\lambda,\alpha,\xi),$$

where $0 \leq t \leq 1$.
Proof: Let

$$g(z) = \frac{f(tz)}{t^p} = z^p - \sum_{k=n}^{\infty} b_{k+p}z^{k+p},$$

$$\left(b_{k+p} = a_{k+p}t^k;\ n \in \mathbb{N};\ p \in \mathbb{N}\right).$$

Making use of Theorem 1, we get

$$\sum_{k=n}^{\infty} U(k)\frac{p+k-\xi-\alpha}{p-\xi-\alpha}b_{k+p} \leq \sum_{k=n}^{\infty} U(k)\frac{p+k-\xi-\alpha}{p-\xi-\alpha}a_{k+p} \leq 1.$$

This completes the proof.

3. Convolution theorem

Considering the two functions f and g of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

let $f*g$ denote the *convolution* (or *Hadamard product*) of f and g defined by

$$(f*g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n \quad (z \in\ C).$$

Theorem 4. If $f, g \in \mathcal{T}(n,p,\lambda,\alpha,\xi)$, then $f*g \in \mathcal{T}(n,p,\lambda,\delta,\xi)$, where

$$\delta \leq p-\xi-\frac{(p-\xi-\alpha)^2}{U(n)(p+n-\xi-\alpha)} \quad (n \in \mathbb{N};\ p \in \mathbb{N}) \qquad (3.1))$$

$U(n)$ is defined by (2.2).
Proof: In view of Theorem 1, we have

$$\sum_{k=n}^{\infty} U(k)\frac{p+k-\xi-\alpha}{p-\xi-\alpha}a_{k+p} \leq 1, \tag{3.2}$$

and

$$\sum_{k=n}^{\infty} U(k)\frac{p+k-\xi-\alpha}{p-\xi-\alpha}b_{k+p} \leq 1, \tag{3.3}$$

where $U(k)$ is defined by (2.2). We find the largest δ such that

$$\sum_{k=n}^{\infty} U(k)\frac{p+k-\xi-\delta}{p-\xi-\delta}a_{k+p}b_{k+p} \leq 1. \tag{3.4}$$

Using Cauchy–Schwarz inequality, (3.2), (3.3), and (3.4) yields

$$\sum_{k=n}^{\infty} U(k)\frac{p+k-\xi-\alpha}{p-\xi-\alpha}\sqrt{a_{k+p}b_{k+p}} \leq 1.$$

It therefore follows that (3.4) is true if

$$\sqrt{a_{k+p}b_{k+p}} \leq \frac{p-\xi-\delta}{p-\xi-\alpha} \quad \left(k \geq n;\ n \in \mathbb{N};\ p \in \mathbb{N}\right). \tag{3.5}$$

But (3.5) is satisfied if

$$\frac{p-\xi-\alpha}{U(k)(p+k-\xi-\alpha)} \leq \frac{p-\xi-\delta}{p-\xi-\alpha}.$$

Note that

$$\phi(k) = p-\xi-\frac{(p-\xi-\alpha)^2}{U(k)(p+k-\xi-\alpha)} \quad (k \geq n)$$

is an increasing function of k. This proves (3.1).
The result is sharp for the functions f and g given by

$$f(z) = g(z) = z^p - \frac{p-\xi-\alpha}{(p+n-\xi-\alpha)U(n)}z^{p+n} \quad \left(n \in \mathbb{N};\ p \in \mathbb{N}\right).$$

For $\xi = 0$, Theorem 4 yields the following.
Corollary 4 ([1, p. 12, Theorem 3]). If each of functions f and g is in the class $\mathcal{T}(n,p,\lambda,\alpha)$, then

$$f * g \in \mathcal{T}(n,p,\lambda,\delta),$$

where

$$\delta \leq p - \frac{(p-\alpha)^2(1+\lambda p-\lambda)}{(p+n-\alpha)(\lambda p+\lambda n-\lambda+1)} \quad \left(n \in \mathbb{N};\ p \in \mathbb{N}\right).$$

Special cases of the above results can be found in several research articles (e.g., [5], [7]).

4. Radii of starlikeness and convexity

In this section, we find the radii of p–valently starlikeness and convexity for the class $\mathcal{T}(n,p,\lambda,\alpha,\xi)$.

Theorem 5. Let the function f defined by (1.3) be in the class $\mathcal{T}(n,p,\lambda,\alpha,\xi)$. Then f is p–valently starlike of order α $(0 \le \alpha < p)$ in the disk $|z| < r_1$, where

$$r_1 = r_1(p,\lambda,\alpha,\xi) = inf_k \left\{ U(k) \frac{(p+k-\xi-\alpha)(p-\alpha)}{(p-\xi-\alpha)(k+p-\alpha)} \right\}^{\frac{1}{k}} \tag{4.1}$$

$$(k \ge n;\ n \in \mathbb{N};\ p \in \mathbb{N};\ 0 \le \alpha < p;\ 0 \le \lambda \le 1;\ T \ge p-\xi),$$

where $U(k)$ and T are defined by (2.2) and (2.3), respectively.

Proof: Suffices to prove that

$$\left| \frac{zf'(z)}{f(z)} - p \right| < p - \alpha \quad (0 \le \alpha < p). \tag{4.2}$$

For the left–hand side of (4.2), we obtain

$$\left| \frac{zf'(z)}{f(z)} - p \right| \le \frac{\sum\limits_{k=n}^{\infty} k a_{k+p} |z|^k}{1 - \sum\limits_{k=n}^{\infty} a_{k+p} |z|^k}.$$

This last expression is less than $p - \alpha$ if

$$\sum_{k=n}^{\infty} \frac{(k+p-\alpha)}{(p-\alpha)} a_{k+p} |z|^k \le 1. \tag{4.3}$$

In view of Theorem 1, (4.3) holds if

$$\frac{(k+p-\alpha)}{(p-\alpha)} |z|^k \le U(k) \frac{k+p-\xi-\alpha}{p-\xi-\alpha}$$

$$(k \ge n;\ 0 \le \alpha < 1;\ 0 \le \lambda \le 1;\ n \in \mathbb{N};\ p \in \mathbb{N};\ T \ge p-\xi),$$

where $U(k)$ and T are defined by (2.2) and (2.3), respectively. The last inequality implies (4.1). This complete the proof.

By putting $\xi = 0$ in Theorem 5, we deduce the following result.

Corollary 5. Let the function f defined by (1.3) be in the class $\mathcal{T}(n,p,\lambda,\alpha)$. Then f is p–valently starlike of order α $(0 \le \alpha < p)$ in the disk

$$|z| < \left(\frac{1+\lambda(p+n-1)}{1+\lambda(p-1)} \right)^{\frac{1}{n}}$$

$$(k \ge n;\ n \in \mathbb{N};\ p \in \mathbb{N};\ 0 \le \lambda \le 1;\ \lambda(p-1)(p-\alpha) \ge \alpha).$$

Theorem 6. Let the function f defined by (1.3) be in the class $\mathcal{T}(n,p,\lambda,\alpha,\xi)$. Then f is p–valently convex of order α $(0 \le \alpha < p)$ in the disk $|z| < r_2$, where

$$r_2 = r_2(p,\lambda,\alpha,\xi) = inf_k \left\{ U(K) \frac{p(p-\alpha)(p+k-\xi-\alpha)}{(k+p)(k+p-\alpha)(p-\xi-\alpha)} \right\}^{\frac{1}{k}} \tag{4.4}$$

$$(k \ge n;\ n \in \mathbb{N};\ p \in \mathbb{N};\ 0 \le \alpha < p;\ 0 \le \lambda \le 1;\ T \ge p-\xi),$$

where $U(k)$ and T are defined by (2.2) and (2.3), respectively.

Proof: It is sufficient to prove that

$$\left| 1 + \frac{zf'(z)}{f(z)} - p \right| < p - \alpha \quad (0 \le \alpha < p). \tag{4.5}$$

For the left–hand side of (4.5), we obtain

$$\begin{aligned}\left|1+\frac{zf''(z)}{f'(z)}-p\right| &= \left|\frac{p(p-1)-\sum\limits_{k=n}^{\infty}(k+p)(k+p-1)a_{k+p}z^k}{p-\sum\limits_{k=n}^{\infty}(k+p)a_{k+p}z^k}+1-p\right| \\ &\leq \frac{\sum\limits_{k=n}^{\infty}k(k+p)a_{k+p}|z|^k}{p-\sum\limits_{k=n}^{\infty}(k+p)a_{k+p}|z|^k}.\end{aligned}$$

This last expression is less than $p-\alpha$ if

$$\sum_{k=n}^{\infty}\frac{(k+p)(k+p-\alpha)}{p(p-\alpha)}a_{k+p}|z|^k \leq 1. \tag{4.6}$$

In view of Theorem 1, (4.6) hold if

$$\frac{(k+p)(k+p-\alpha)}{p(p-\alpha)}|z|^k \leq U(k)\frac{k+p-\xi-\alpha}{p-\xi-\alpha}$$

$$(k \geq n;\ n\in\mathbb{N};\ p\in\mathbb{N};\ 0\leq\alpha<p;\ 0\leq\lambda\leq1;\ T\geq p-\xi),$$

where $U(k)$ and T are defined by (2.2) and (2.3), respectively. The last inequality implies (4.4). This complete the proof.

Theorem 7. Let the function f defined by (1.3) be in the class $\mathcal{T}(n,p,\lambda,\alpha,\xi)$. Then f is p–valently close–to–convex of order α $(0\leq\alpha<p)$ in the disk $|z|<r_3$, where

$$r_3 = r_3(p,\lambda,\alpha,\xi) = inf_k\left\{U(K)\frac{(p+k-\xi-\alpha)(p-\alpha)}{(k+p)(p-\xi-\alpha)}\right\}^{\frac{1}{k}} \tag{4.7}$$

$$(k \geq n;\ n\in\mathbb{N};\ p\in\mathbb{N};\ 0\leq\alpha<p;\ 0\leq\lambda\leq1;\ T\geq p-\xi),$$

where $U(k)$ and T are defined by (2.2) and (2.3), respectively.

Proof: It is sufficient to prove that

$$\left|\frac{zf'(z)}{z^{p-1}}-p\right| \leq p-\alpha \quad (0\leq\alpha<p) \tag{4.8}$$

For the left–hand side of (4.8), we obtain

$$\left|\frac{zf'(z)}{z^{p-1}}-p\right| \leq \sum_{k=n}^{\infty}(k+p)a_{k+p}|z|^k.$$

Since $f\in\mathcal{T}(n,p,\lambda,\alpha,\xi)$, it follows by Theorem 1 that (4.8) holds if

$$\frac{k+p}{p-\alpha}|z|^k \leq U(k)\frac{k+p-\xi-\alpha}{p-\xi-\alpha}$$

$$(k \geq n;\ n\in\mathbb{N};\ p\in\mathbb{N};\ 0\leq\alpha<p;\ 0\leq\lambda\leq1;\ T\geq p-\xi),$$

where $U(k)$ and T are defined by (2.2) and (2.3), respectively. The last inequality implies (4.7). This completes the proof.

Upon setting $\xi=0$ in Theorem 6, we arrive at the following result.

Corollary 6. Let the function f defined by (1.3) be in the class $\mathcal{T}(n,p,\lambda,\alpha,\xi)$. Then f is p–valently close–to–convex of order α $(0 \le \alpha < p)$ in the disk

$$|z| < \left[\frac{(1+\lambda(p+n-1))\,(p-\alpha)}{(1+\lambda(p-1))(p+n)}\right]^{\frac{1}{n}}$$

$$(k \ge n;\; n \in \mathbb{N};\; p \in \mathbb{N};\; 0 \le \alpha < p;\; 0 \le \lambda \le 1;\; \lambda(p-1)(p-\alpha) \ge \alpha)\,.$$

5. Distortion theorems involving operators of fractional calculus

In this section, we obtain various distortion inequalities for fractional calculus of functions in the class $\mathcal{T}(n,p,\lambda,\alpha,\xi)$ are given.

Theorem 8. If $f \in \mathcal{T}(n,p,\lambda,\alpha,\xi)$, then

$$\left|D_z^{-\mu} f(z)\right| \le |z|^{p+\mu}\left[\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} + \frac{\Gamma(p+n+1)(p-\xi-\alpha)}{U(n)(p+n-\xi-\alpha)\Gamma(p+n+\mu+1)}|z|\right], \tag{5.1}$$

and

$$\left|D_z^{-\mu} f(z)\right| \ge |z|^{p+\mu}\left[\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} - \frac{\Gamma(p+n+1)(p-\xi-\alpha)}{U(n)(p+n-\xi-\alpha)\Gamma(p+n+\mu+1)}|z|\right], \tag{5.2}$$

for $\mu > 0$, $n \in \mathbb{N}$, $p \in \mathbb{N}$ and for all $z \in \mathcal{U}$. The function $U(n)$ is defined by (2.2). The result is sharp for the function f given by

$$f(z) = z^p - \frac{p-\xi-\alpha}{U(n)(p+n-\xi-\alpha)} z^{n+p} \qquad (n \in \mathbb{N};\; p \in \mathbb{N})\,. \tag{5.3}$$

Proof: Making use of Theorem 1, we find that

$$\sum_{k=n}^{\infty} a_{k+p} \le \frac{p-\xi-\alpha}{U(n)(p+n-\xi-\alpha)}, \tag{5.4}$$

where $U(n)$ is defined by (2.2). From (1.1), we have

$$D_z^{-\mu} f(z) = z^{p+\mu}\left[\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} - \sum_{k=n}^{\infty}\omega(k)a_{k+p}z^k\right] \qquad (\mu > 0;\; n \in \mathbb{N},\; p \in \mathbb{N})\,,$$

where

$$\omega(k) = \frac{\Gamma(p+k+1)}{\Gamma(p+k+\mu+1)}.$$

Since $\omega(k)$ is a decreasing function of k, we find that

$$\left|D_z^{-\mu} f(z)\right| \le |z|^{p+\mu}\left[\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} + |z|\omega(n)\sum_{k=n}^{\infty} a_{k+p}\right], \tag{5.5}$$

and, that

$$\left|D_z^{-\mu} f(z)\right| \ge |z|^{p+\mu}\left[\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} - |z|\omega(n)\sum_{k=n}^{\infty} a_{k+p}\right]. \tag{5.6}$$

From (5.4), the inequalities (5.5) and (5.6) yield the assertions (5.1) and (5.2), respectively.

Setting $\xi = 0$ in Theorem 8, we get the following.
Corollary 7. ([1, p.13, Theorem 4]). If $f \in \mathcal{T}(n,p,\lambda,\alpha,\xi)$, then

$$\left|D_z^{-\mu} f(z)\right| \leq |z|^{p+\mu}\left[\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} + \frac{\Gamma(p+n+1)(p-\alpha)(1+\lambda p-\lambda)}{(p+n-\alpha)\Gamma(p+n+\mu+1)(1+\lambda p+\lambda n-\lambda)}|z|\right],$$

and

$$\left|D_z^{-\mu} f(z)\right| \geq |z|^{p+\mu}\left[\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} - \frac{\Gamma(p+n+1)(p-\alpha)(1+\lambda p-\lambda)}{(p+n-\alpha)\Gamma(p+n+\mu+1)(1+\lambda p+\lambda n-\lambda)}|z|\right],$$

for $\mu > 0$, $n \in \mathbb{N}$, $p \in \mathbb{N}$ and for all $z \in \mathcal{U}$. The result is sharp for the function f given by

$$f(z) = z^p - \frac{(p-\alpha)(1+\lambda p-\lambda)}{(p+n-\alpha)(1+\lambda p+\lambda n-\lambda)} z^{n+p} \qquad (n \in \mathbb{N};\ p \in \mathbb{N}). \tag{5.7}$$

Theorem 9. If $f \in \mathcal{T}(n,p,\lambda,\alpha,\xi)$, then

$$|D_z^{\mu} f(z)| \leq |z|^{p-\mu}\left[\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} + \frac{\Gamma(p+n+1)(p-\xi-\alpha)}{U(n)(p+n-\xi-\alpha)\Gamma(p+n-\mu+1)}|z|\right], \tag{5.8}$$

and

$$|D_z^{\mu} f(z)| \geq |z|^{p-\mu}\left[\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} - \frac{\Gamma(p+n+1)(p-\xi-\alpha)}{U(n)(p+n-\xi-\alpha)\Gamma(p+n-\mu+1)}|z|\right], \tag{5.9}$$

for $0 \leq \mu < 1$, $n \in \mathbb{N}$, $p \in \mathbb{N}$ and for all $z \in \mathcal{U}$. The function $U(n)$ is defined by (2.2). The result is sharp for the function f given by (5.3).
Proof: Making use of Theorem 1, we find that

$$\sum_{k=n}^{\infty}(p+k)a_{k+p} \leq \frac{(p-\xi-\alpha)(p+n)}{U(n)(p+n-\xi-\alpha)}, \tag{5.10}$$

where $U(n)$ is defined by (2.2). From (1.2), we have

$$D_z^{\mu} f(z) = z^{p-\mu}\left[\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} - \sum_{k=n}^{\infty}(k+p)\eta(k)a_{k+p}z^k\right] \quad (0 \leq \mu < 1;\ n \in \mathbb{N},\ p \in \mathbb{N};\ z \in \mathcal{U}),$$

where

$$\eta(k) = \frac{\Gamma(p+k)}{\Gamma(p+k-\mu+1)}.$$

Since $\eta(k)$ is a decreasing function of k, we find that

$$|D_z^{\mu} f(z)| \leq |z|^{p-\mu}\left[\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} + |z|\eta(n)\sum_{k=n}^{\infty}(k+p)a_{k+p}\right], \tag{5.11}$$

and, that

$$|D_z^{\mu} f(z)| \geq |z|^{p-\mu}\left[\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} - |z|\eta(n)\sum_{k=n}^{\infty}(k+p)a_{k+p}\right]. \tag{5.12}$$

From (5.10), the inequalities (5.11) and (5.12) yield the assertions (5.8) and (5.9), respectively.

The following result is an obvious variant of Theorem 9 as following.
Corollary 8 ([1, p. 13, Theorem 5]). If $f \in \mathcal{T}(n,p,\lambda,\alpha)$, then

$$|D_z^{\mu} f(z)| \le |z|^{p-\mu}\left[\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} + \frac{\Gamma(p+n+1)(p-\alpha)(1+\lambda p-\lambda)}{(p+n-\alpha)\Gamma(p+n-\mu+1)(1+\lambda p+\lambda n-\lambda)}|z|\right],$$

and

$$|D_z^{\mu} f(z)| \ge |z|^{p-\mu}\left[\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} - \frac{\Gamma(p+n+1)(p-\alpha)(1+\lambda p-\lambda)}{(p+n-\alpha)\Gamma(p+n-\mu+1)(1+\lambda p+\lambda n-\lambda)}|z|\right],$$

for $0 \le \mu < 1$, $n \in \mathbb{N}$, $p \in \mathbb{N}$ and for all $z \in \mathcal{U}$. The result is sharp for the function f given by (5.7).
Letting $\mu = 0$ in Theorem 9, we obtain the following result.
Corollary 9. If $f \in \mathcal{T}(n,p,\lambda,\alpha,\xi)$, then

$$|f(z)| \le |z|^p\left[1 + \frac{p-\xi-\alpha}{U(n)(p+n-\xi-\alpha)}|z|\right],$$

and

$$|f(z)| \ge |z|^p\left[1 - \frac{p-\xi-\alpha}{U(n)(p+n-\xi-\alpha)}|z|\right],$$

for $n \in \mathbb{N}$, $p \in \mathbb{N}$ and for all $z \in \mathcal{U}$. The result is sharp for the function f given by (5.3).
On the other hand, by setting $\mu = 1$ in Theorem 9, we get the following corollary.
Corollary 10. If $f \in \mathcal{T}(n,p,\lambda,\alpha,\xi)$, then

$$|f'(z)| \le |z|^{p-1}\left[p + \frac{(p+n)(p-\xi-\alpha)}{U(n)(p+n-\xi-\alpha)}|z|\right],$$

and

$$|f'(z)| \ge |z|^{p-1}\left[p - \frac{(p+n)(p-\xi-\alpha)}{U(n)(p+n-\xi-\alpha)}|z|\right],$$

for $n \in \mathbb{N}$, $p \in \mathbb{N}$ and for all $z \in \mathcal{U}$. The result is sharp for the function f given by (5.3).
By setting $\xi = 0$ in Corollary 9 and Corollary 10, we are thus led to Corollary 11 and Corollary 12 below.
Corollary 11 ([1, p. 15, Corollary 2]). If $f \in \mathcal{T}(n,p,\lambda,\alpha)$, then

$$|f(z)| \le |z|^p\left[1 + \frac{(p-\alpha)(1+\lambda p-\lambda)}{(p+n-\alpha)(1+\lambda n+\lambda p-\lambda)}|z|\right],$$

and

$$|f(z)| \ge |z|^p\left[1 - \frac{(p-\alpha)(1+\lambda p-\lambda)}{(p+n-\alpha)(1+\lambda n+\lambda p-\lambda)}|z|\right],$$

for $n \in \mathbb{N}$, $p \in \mathbb{N}$ and for all $z \in \mathcal{U}$. The result is sharp for the function f given by (5.7).
Corollary 12 ([1, p. 15, Corollary 3]). If $f \in \mathcal{T}(n,p,\lambda,\alpha)$, then

$$|f'(z)| \le |z|^{p-1}\left[p + \frac{(p+n)(p-\alpha)(1+\lambda p-\lambda)}{(p+n-\alpha)(1+\lambda p+\lambda n-\lambda)}|z|\right],$$

and

$$|f'(z)| \ge |z|^{p-1}\left[p - \frac{(p+n)(p-\alpha)(1+\lambda p-\lambda)}{(p+n-\alpha)(1+\lambda p+\lambda n-\lambda)}|z|\right],$$

for $n \in \mathbb{N}$, $p \in \mathbb{N}$ and for all $z \in \mathcal{U}$. The result is sharp for the function f given by (5.7).

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Sh. Khosravianarab
Ruhr University, Bochum, Germany
E-mail address: `shahnaz.khosravianarab@rub.de`

S.R. Kulkarni
Department of Mathematics, Fergusson College, Pune, India
E-mail address: `kulkarni_ferg@yahoo.com`

O.P. Ahuja
Department of Mathematical Sciences, Kent State University, 14111 Claridon Troy Road, Burton, Ohio 44021-9500, USA
E-mail address: `oahuja@kent.edu`