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GENERALIZED CONVOLUTION PROPERTIES FOR SUBCLASSES OF HARMONIC UNIVALENT FUNCTIONS

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Abstract. Studies of convolution play an important role in Geometric Function Theory. In this paper, We studied the generalized convolution of the subclasses $V_H(\beta)$, $U_H(\beta)$ and $R_H(\beta)$ harmonic univalent functions analogous to analytic univalent functions.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain $\mathbb{D}$ is said to be harmonic in $\mathbb{D}$ if both u and v are real harmonic in $\mathbb{D}$. In any simply connected domain $\mathbb{D}$, we can write $f = h+\overline{g}$, where h and g are analytic in $\mathbb{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$, $z \in \mathbb{D}$ (see[2]).

Denote by S_H the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f'_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k, \qquad |b_1| < 1. \tag{1}$$

Note that S_H reduces to the class of normalized analytic functions if the co-analytic part of its member is zero.

For $1 < \beta \le 4/3$ and $z \in U$, suppose that $M_H(\beta)$ denote the family of harmonic functions $f = h + \overline{g}$ of the form (1) satisfying the condition

$$\frac{\partial}{\partial\theta}\left(arg f(z)\right) = Re\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} \le \beta, \qquad z \in U,$$

and $L_H(\beta)$ denote the family of harmonic functions of the form (1) satisfying the condition

$$\frac{\partial}{\partial\theta}\left\{arg\left(\frac{\partial}{\partial\theta} f(z)\right)\right\} = Re\left\{1 + \frac{z^2h''(z) + \overline{2zg'(z) + z^2g''(z)}}{zh'(z) - \overline{zg'(z)}}\right\} \le \beta, \qquad z \in U.$$

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A necessary condition for a function $f(z) = h + \overline{g}$, where $h(z)$ and $g(z)$ are of the form (1) belong to the classes $M_H(\beta)$ and $L_H(\beta)$ (see [7])

$$\sum_{k=2}^{\infty}(k-\beta)|a_k| + \sum_{k=1}^{\infty}(k+\beta)|b_k| \leq \beta - 1, \tag{2}$$

where $1 < \beta \leq \frac{4}{3}$ and

$$\sum_{k=2}^{\infty}k(k-\beta)|a_k| + \sum_{k=1}^{\infty}k(k+\beta)|b_k| \leq \beta - 1, \tag{3}$$

where $1 < \beta \leq \frac{4}{3}$.

Further, we let V_H and U_H be the subclasses of S_H consisting of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty}|a_k|z^k - \sum_{k=1}^{\infty}|b_k|\overline{z}^k, \tag{4}$$

and

$$f(z) = z + \sum_{k=2}^{\infty}|a_k|z^k + \sum_{k=1}^{\infty}|b_k|\overline{z}^k, \tag{5}$$

respectively.

Let $V_H(\beta) \equiv M_H(\beta) \cap V_H$, and $U_H(\beta) \equiv L_H(\beta) \cap U_H$. A necessary and sufficient condition for a function $f(z)$ is of the form (4) belong to the classes $V_H(\beta)$ (see [7])

$$\sum_{k=2}^{\infty}(k-\beta)|a_k| + \sum_{k=1}^{\infty}(k+\beta)|b_k| \leq \beta - 1, \tag{6}$$

where $1 < \beta \leq \frac{4}{3}$.

Also, a necessary and sufficient condition for a function $f(z)$ is of the form (5) belong to the classes $U_H(\beta)$ (see [7])

$$\sum_{k=2}^{\infty}k(k-\beta)|a_k| + \sum_{k=1}^{\infty}k(k+\beta)|b_k| \leq \beta - 1, \tag{7}$$

where $1 < \beta \leq \frac{4}{3}$.

We note that for $g = 0$ the classes $M_H(\beta) \equiv M(\beta)$, $L_H(\beta) \equiv L(\beta)$, $V_H(\beta) \equiv V(\beta)$ and $U_H(\beta) \equiv U(\beta)$ [8]. A function $f(z)$ of the form (5) in S_H is said to be in the class $R_H(\beta)$, $(1 < \beta \leq 2)$, if and only if

$$Re\{h'(z) + g'(z)\} < \beta, \qquad z \in U. \tag{8}$$

A necessary and sufficient condition for a function $f(z)$ is of the form (5) belong to the classes $R_H(\beta)$ [3]

$$\sum_{k=2}^{\infty}k|a_k| + \sum_{k=1}^{\infty}k|b_k| \leq \beta - 1, \qquad (1 < \beta \leq 2). \tag{9}$$

We note that the class $R_H(\beta)$ reduces to the class $R(\beta)$ if co-analytic part of f is zero i.e. $g \equiv 0$ (see [9]).

Let $f_j(z)\,(j = 1, 2)$ in S_H be given by

$$f_j(z) = z + \sum_{k=2}^{\infty}|a_{k,j}|z^k + \sum_{k=1}^{\infty}|b_{k,j}|\overline{z}^k. \tag{10}$$

Then the convolution $f_1 * f_2$ is defined by

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} |a_{k,1}a_{k,2}|z^k + \sum_{k=1}^{\infty} |b_{k,1}b_{k,2}|\overline{z}^k. \tag{11}$$

Furthermore, for any $p > 0$ and $q > 0$, we define the generalized convolution $(f_1 \Delta f_2)(p, q; z)$ by

$$(f_1 \Delta f_2)(p, q; z) = z + \sum_{k=2}^{\infty} |a_{k,1}|^p |a_{k,2}|^q z^k + \sum_{k=1}^{\infty} |b_{k,1}|^p |b_{k,2}|^q \overline{z}^k. \tag{12}$$

In the special case, if we take $p = q = 1$, then we have

$$(f_1 \Delta f_2)(1, 1; z) = (f_1 * f_2)(z), \qquad (z \in U).$$

Studies of convolution play an important role in Geometric function theory. It has attracted large number of researchers. By making use of convolution, several new and interesting subclasses have been defined and studies in the direction of subordination, partial sums, neighborhood, argument problems, integral mean inequalities and some other related interesting properties. For detailed study in ([1], [4], [5], [6]) and others.

In the present paper, we studied the generalized convolution of harmonic univalent functions analogous to analytic univalent functions.

2. Main results

Theorem 1. *If the functions $f_j(z)$ $(j = 1, 2)$ defined by (4) with $b_{1,j} = 0$ $(j = 1, 2)$ are in the classes $V_H(\beta_j)$ for each j and the condition*

$$|a_{k,1}|^{\frac{1}{p}} |a_{k,2}|^{\frac{1}{q}} + |b_{k,1}|^{\frac{1}{p}} |b_{k,2}|^{\frac{1}{q}} \le (|a_{k,1}| + |b_{k,1}|)^{\frac{1}{p}} (|a_{k,2}| + |b_{k,2}|)^{\frac{1}{q}}, \quad (k = 2, 3, ...), \tag{13}$$

is satisfied then

$$(f_1 \Delta f_2)\left(\frac{1}{p}, \frac{1}{q}; z\right) \in V_H(\beta),$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta = 1 + \frac{1}{1+\left(\frac{2-\beta_1}{\beta_1-1}\right)^{\frac{1}{p}} \left(\frac{2-\beta_2}{\beta_2-1}\right)^{\frac{1}{q}}}$.

Proof. Since $f_j(z) \in V_H(\beta_j)$, by using (6), we have

$$\sum_{k=2}^{\infty} \left(\frac{k - \beta_j}{\beta_j - 1}\right) (|a_{k,j}| + |b_{k,j}|) \le 1, \qquad (j = 1, 2). \tag{14}$$

then from eq.(14) we obtain

$$\left\{\sum_{k=2}^{\infty} \left(\frac{k - \beta_1}{\beta_1 - 1}\right) (|a_{k,1}| + |b_{k,1}|)\right\}^{\frac{1}{p}} \le 1, \tag{15}$$

and

$$\left\{\sum_{k=2}^{\infty} \left(\frac{k - \beta_2}{\beta_2 - 1}\right) (|a_{k,2}| + |b_{k,2}|)\right\}^{\frac{1}{q}} \le 1. \tag{16}$$

Now

$$\sum_{k=2}^{\infty}\left(\frac{k-\beta_1}{\beta_1-1}\right)^{\frac{1}{p}}\left(\frac{k-\beta_2}{\beta_2-1}\right)^{\frac{1}{q}}(|a_{k,1}|^{\frac{1}{p}}|a_{k,2}|^{\frac{1}{q}}+|b_{k,1}|^{\frac{1}{p}}|b_{k,2}|^{\frac{1}{q}})$$
$$\leq \sum_{k=2}^{\infty}\left(\frac{k-\beta_1}{\beta_1-1}\right)^{\frac{1}{p}}\left(\frac{k-\beta_2}{\beta_2-1}\right)^{\frac{1}{q}}(|a_{k,1}|+|b_{k,1}|)^{\frac{1}{p}}(|a_{k,2}|+|b_{k,2}|)^{\frac{1}{q}}$$
$$\leq \left\{\sum_{k=2}^{\infty}\left(\frac{k-\beta_1}{\beta_1-1}\right)(|a_{k,1}|+|b_{k,1}|)\right\}^{\frac{1}{p}}\left\{\sum_{k=2}^{\infty}\left(\frac{k-\beta_2}{\beta_2-1}\right)(|a_{k,2}|+|b_{k,2}|)\right\}^{\frac{1}{q}}\leq 1.$$

Since

$$(f_1\Delta f_2)(\frac{1}{p},\frac{1}{q};z)=z+\sum_{k=2}^{\infty}|a_{k,1}|^{\frac{1}{p}}|a_{k,2}|^{\frac{1}{q}}z^k-\sum_{k=1}^{\infty}|b_{k,1}|^{\frac{1}{p}}|b_{k,2}|^{\frac{1}{q}}\overline{z}^k,$$

it suffices to show that $(f_1\Delta f_2)(\frac{1}{p},\frac{1}{q};z)\in V_H(\beta)$ if

$$\sum_{k=2}^{\infty}\left(\frac{k-\beta}{\beta-1}\right)\left(|a_{k,1}|^{\frac{1}{p}}|a_{k,2}|^{\frac{1}{q}}+|b_{k,1}|^{\frac{1}{p}}|b_{k,2}|^{\frac{1}{q}}\right)\leq 1. \tag{17}$$

For this we have to show that L.H.S. of (17) is bounded above by

$$\sum_{k=2}^{\infty}\left(\frac{k-\beta_1}{\beta_1-1}\right)^{\frac{1}{p}}\left(\frac{k-\beta_2}{\beta_2-1}\right)^{\frac{1}{q}}(|a_{k,1}|^{\frac{1}{p}}|a_{k,2}|^{\frac{1}{q}}+|b_{k,1}|^{\frac{1}{p}}|b_{k,2}|^{\frac{1}{q}})\leq 1,$$

which is equivalent to $\beta\geq 1+\frac{1}{1+\left(\frac{2-\beta_1}{\beta_1-1}\right)^{\frac{1}{p}}\left(\frac{2-\beta_2}{\beta_2-1}\right)^{\frac{1}{q}}}$. ∎

Corollary 1. *If the functions $f_j(z)$ $(j=1,2)$ defined by (4) with $b_{1,j}=0$ $(j=1,2)$ are in the classes $V_H(\beta)$ for each j and the condition (13) is satisfied then*

$$(f_1\Delta f_2)\left(\frac{1}{p},\frac{1}{q};z\right)\in V_H(\beta),$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $f_j(z)\in V_H(\beta)$, by using (6), Corollary 1 follows readily from Theorem 1 in special case when $\beta_j=\beta$. ∎

Theorem 2. *If the functions $f_j(z)$ $(j=1,2)$ defined by (10) with $b_{1,j}=0$ $(j=1,2)$ are in the classes $U_H(\beta_j)$ for each j and the condition (13) is satisfied then*

$$(f_1\Delta f_2)\left(\frac{1}{p},\frac{1}{q};z\right)\in U_H(\beta),$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1$ and $\beta=1+\frac{1}{1+\left(\frac{2-\beta_1}{\beta_1-1}\right)^{\frac{1}{p}}\left(\frac{2-\beta_2}{\beta_2-1}\right)^{\frac{1}{q}}}$.

Proof. Since $f_j(z)\in U_H(\beta_j)$, by using (7), we have

$$\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_j)}{\beta_j-1}\right)(|a_{k,j}|+|b_{k,j}|)\leq 1, \qquad (j=1,2). \tag{18}$$

then from eq.(18) we obtain

$$\left\{\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_1)}{\beta_1-1}\right)(|a_{k,1}|+|b_{k,1}|)\right\}^{\frac{1}{p}}\leq 1, \tag{19}$$

and

$$\left\{\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_2)}{\beta_2-1}\right)(|a_{k,2}|+|b_{k,2}|)\right\}^{\frac{1}{q}}\leq 1. \tag{20}$$

Now

$$\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_1)}{\beta_1-1}\right)^{\frac{1}{p}}\left(\frac{k(k-\beta_2)}{\beta_2-1}\right)^{\frac{1}{q}}(|a_{k,1}|^{\frac{1}{p}}|a_{k,2}|^{\frac{1}{q}}+|b_{k,1}|^{\frac{1}{p}}|b_{k,2}|^{\frac{1}{q}})$$

$$\leq\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_1)}{\beta_1-1}\right)^{\frac{1}{p}}\left(\frac{k(k-\beta_2)}{\beta_2-1}\right)^{\frac{1}{q}}(|a_{k,1}|+|b_{k,1}|)^{\frac{1}{p}}(|a_{k,2}|+|b_{k,2}|)^{\frac{1}{q}}$$

$$\leq\left\{\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_1)}{\beta_1-1}\right)(|a_{k,1}|+|b_{k,1}|)\right\}^{\frac{1}{p}}\left\{\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_2)}{\beta_2-1}\right)(|a_{k,2}|+|b_{k,2}|)\right\}^{\frac{1}{q}}\leq 1.$$

Since

$$(f_1\Delta f_2)(\frac{1}{p},\frac{1}{q};z)=z+\sum_{k=2}^{\infty}|a_{k,1}|^{\frac{1}{p}}|a_{k,2}|^{\frac{1}{q}}z^k+\sum_{k=1}^{\infty}|b_{k,1}|^{\frac{1}{p}}|b_{k,2}|^{\frac{1}{q}}\overline{z}^k,$$

it suffices to show that $(f_1\Delta f_2)(\frac{1}{p},\frac{1}{q};z)\in U_H(\beta)$ if

$$\sum_{k=2}^{\infty}\left(\frac{k(k-\beta)}{\beta-1}\right)\left(|a_{k,1}|^{\frac{1}{p}}|a_{k,2}|^{\frac{1}{q}}+|b_{k,1}|^{\frac{1}{p}}|b_{k,2}|^{\frac{1}{q}}\right)\leq 1. \tag{21}$$

For this we have to show that L.H.S. of (21) is bounded above by

$$\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_1)}{\beta_1-1}\right)^{\frac{1}{p}}\left(\frac{k(k-\beta_2)}{\beta_2-1}\right)^{\frac{1}{q}}(|a_{k,1}|^{\frac{1}{p}}|a_{k,2}|^{\frac{1}{q}}+|b_{k,1}|^{\frac{1}{p}}|b_{k,2}|^{\frac{1}{q}})\leq 1,$$

which is equivalent to $\beta\geq 1+\frac{1}{1+\left(\frac{2-\beta_1}{\beta_1-1}\right)^{\frac{1}{p}}\left(\frac{2-\beta_2}{\beta_2-1}\right)^{\frac{1}{q}}}$. ■

Corollary 2. *If the functions $f_j(z)$ $(j=1,2)$ defined by (10) with $b_{1,j}=0$ $(j=1,2)$ are in the classes $U_H(\beta)$ for each j and the condition (13) is satisfied then*

$$(f_1\Delta f_2)\left(\frac{1}{p},\frac{1}{q};z\right)\in U_H(\beta),$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $f_j(z)\in U_H(\beta)$, by using (7), Corollary 2 follows readily from Theorem 2 in special case when $\beta_j=\beta$. ■

Theorem 3. *If the functions $f_j(z)$ $(j=1,2)$ defined by (10) with $b_{1,j}=0$ $(j=1,2)$ are in the classes $R_H(\beta_j)$ for each j and the condition (13) is satisfied then*

$$(f_1\Delta f_2)\left(\frac{1}{p},\frac{1}{q};z\right)\in R_H(\beta),$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\beta=1+(\beta_1-1)^{\frac{1}{p}}(\beta_2-1)^{\frac{1}{q}}$.

Proof. Since $f_j(z) \in R_H(\beta_j)$, by using (9), we have

$$\sum_{k=2}^{\infty}\left(\frac{k}{\beta_j-1}\right)(|a_{k,j}|+|b_{k,j}|) \le 1, \qquad (j=1,2). \tag{22}$$

then from eq.(22) we obtain

$$\left\{\sum_{k=2}^{\infty}\left(\frac{k}{\beta_1-1}\right)(|a_{k,1}|+|b_{k,1}|)\right\}^{\frac{1}{p}} \le 1, \tag{23}$$

and

$$\left\{\sum_{k=2}^{\infty}\left(\frac{k}{\beta_2-1}\right)(|a_{k,2}|+|b_{k,2}|)\right\}^{\frac{1}{q}} \le 1. \tag{24}$$

Now

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left(\frac{k}{\beta_1-1}\right)^{\frac{1}{p}}\left(\frac{k}{\beta_2-1}\right)^{\frac{1}{q}}(|a_{k,1}|^{\frac{1}{p}}|a_{k,2}|^{\frac{1}{q}}+|b_{k,1}|^{\frac{1}{p}}|b_{k,2}|^{\frac{1}{q}})\\
&\le \sum_{k=2}^{\infty}\left(\frac{k}{\beta_1-1}\right)^{\frac{1}{p}}\left(\frac{k}{\beta_2-1}\right)^{\frac{1}{q}}(|a_{k,1}|+|b_{k,1}|)^{\frac{1}{p}}(|a_{k,2}|+|b_{k,2}|)^{\frac{1}{q}}\\
&\le \left\{\sum_{k=2}^{\infty}\left(\frac{k}{\beta_1-1}\right)(|a_{k,1}|+|b_{k,1}|)\right\}^{\frac{1}{p}}\left\{\sum_{k=2}^{\infty}\left(\frac{k}{\beta_2-1}\right)(|a_{k,2}|+|b_{k,2}|)\right\}^{\frac{1}{q}} \le 1.
\end{aligned}$$

Since

$$(f_1\Delta f_2)(\frac{1}{p},\frac{1}{q};z) = z + \sum_{k=2}^{\infty}|a_{k,1}|^{\frac{1}{p}}|a_{k,2}|^{\frac{1}{q}}z^k + \sum_{k=1}^{\infty}|b_{k,1}|^{\frac{1}{p}}|b_{k,2}|^{\frac{1}{q}}\overline{z}^k,$$

it suffices to show that $(f_1\Delta f_2)(\frac{1}{p},\frac{1}{q};z) \in R_H(\beta)$ if

$$\sum_{k=2}^{\infty}\left(\frac{k}{\beta-1}\right)\left(|a_{k,1}|^{\frac{1}{p}}|a_{k,2}|^{\frac{1}{q}}+|b_{k,1}|^{\frac{1}{p}}|b_{k,2}|^{\frac{1}{q}}\right) \le 1. \tag{25}$$

For this we have to show that L.H.S. of (25) is bounded above by

$$\sum_{k=2}^{\infty}\left(\frac{k}{\beta_1-1}\right)^{\frac{1}{p}}\left(\frac{k}{\beta_2-1}\right)^{\frac{1}{q}}\left(|a_{k,1}|^{\frac{1}{p}}|a_{k,2}|^{\frac{1}{q}}+|b_{k,1}|^{\frac{1}{p}}|b_{k,2}|^{\frac{1}{q}}\right) \le 1,$$

which is equivalent to $\beta \ge 1+(\beta_1-1)^{\frac{1}{p}}(\beta_2-1)^{\frac{1}{q}}$. ∎

Corollary 3. *If the functions $f_j(z)$ $(j=1,2)$ defined by (10) with $b_{1,j}=0$ $(j=1,2)$ are in the classes $R_H(\beta)$ for each j and the condition (13) is satisfied then*

$$(f_1\Delta f_2)\left(\frac{1}{p},\frac{1}{q};z\right) \in R_H(\beta),$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. In view of eq. (9), Corollary 3 follows readily from Theorem 3 in special case when $\beta_j=\beta$. ∎

Theorem 4. *If the functions $f_j(z)$ $(j = 1, 2, ..., m)$ defined by (4) with $b_{1,j} = 0$ $(j = 1, 2, ..., m)$ are in the classes $V_H(\beta_j)$ for each j and $F_m(z)$ defined in*

$$F_m(z) = z + \sum_{k=2}^{\infty}\left(\sum_{j=1}^{m}|a_{k,j}|^p\right)z^k - \sum_{k=2}^{\infty}\left(\sum_{j=1}^{m}|b_{k,j}|^p\right)\overline{z}^k, \quad (p \geq 1), \tag{26}$$

then $F_m(z) \in V_H(\beta_m)$ where $\beta_m \geq 1 + \frac{1}{1+\frac{1}{m}(\frac{2-\beta}{\beta-1})^p}$, $\beta = \max_{1\leq j\leq m}\beta_j$ and $m(\beta - 1)^p \leq 2(2-\beta)^p$.

Proof. Since $f_j(z) \in V_H(\beta_j)$ by using (6), we have

$$\sum_{k=2}^{\infty}\left(\frac{k-\beta_j}{\beta_j - 1}\right)(|a_{k,j}| + |b_{k,j}|) \leq 1, \qquad (j = 1, 2, ..., m) \tag{27}$$

Now

$$\begin{aligned}\sum_{k=2}^{\infty}\left(\frac{k-\beta_j}{\beta_j-1}\right)^p(|a_{k,j}|^p + |b_{k,j}|^p) &\leq \sum_{k=2}^{\infty}\left(\frac{k-\beta_j}{\beta_j-1}\right)^p(|a_{k,j}| + |b_{k,j}|)^p \\ \leq \left\{\sum_{k=2}^{\infty}\left(\frac{k-\beta_j}{\beta_j-1}\right)(|a_{k,j}| + |b_{k,j}|)\right\}^p &\leq 1.\end{aligned} \tag{28}$$

It follows from (28)

$$\sum_{k=2}^{\infty}\left(\frac{1}{m}\sum_{j=1}^{m}\left(\frac{k-\beta_j}{\beta_j-1}\right)^p(|a_{k,j}|^p + |b_{k,j}|^p)\right) \leq 1.$$

Putting $\beta = \max_{1\leq j\leq m}\beta_j$ we find that

$$\sum_{k=2}^{\infty}\left(\frac{k-\beta_m}{\beta_m-1}\right)\left(\sum_{j=1}^{m}|a_{k,j}|^p + |b_{k,j}|^p\right) \leq \sum_{k=2}^{\infty}\frac{1}{m}\left(\frac{k-\beta}{\beta-1}\right)^p\left(\sum_{j=1}^{m}(|a_{k,j}|^p + |b_{k,j}|^p)\right) \leq 1. \tag{29}$$

which is equivalent to $\beta_m \geq 1 + \frac{1}{1+\frac{1}{m}(\frac{2-\beta}{\beta-1})^p}$.

Now let $g(k) = 1 + \frac{k-1}{1+\frac{1}{m}(\frac{k-\beta}{\beta-1})^p}$, it is easy to verify that $g(k)$ is decreasing function of k for $p \geq 1$. Therefore

$$\beta_m = \max_{k\geq 2} g(k) = g(2) = 1 + \frac{1}{1+\frac{1}{m}(\frac{2-\beta}{\beta-1})^p}.$$

This completes the proof. ∎

Theorem 5. *If the functions $f_j(z)$ $(j = 1, 2, ..., m)$ defined by (10) with $b_{1,j} = 0$ $(j = 1, 2, ..., m)$ are in the classes $U_H(\beta_j)$ for each j and*

$$F_m(z) = z + \sum_{k=2}^{\infty}\left(\sum_{j=1}^{m}|a_{k,j}|^p\right)z^k + \sum_{k=2}^{\infty}\left(\sum_{j=1}^{m}|b_{k,j}|^p\right)\overline{z}^k, \quad (p \geq 1), \tag{30}$$

then $F_m(z) \in U_H(\beta_m)$ where $\beta_m \geq 1 + \frac{1}{1+\frac{2^{p-1}}{m}(\frac{2-\beta}{\beta-1})^p}$, $\beta = \max_{1\leq j\leq m}\beta_j$ and $m(\beta - 1)^p \leq 2^p(2-\beta)^p$.

Proof. Since $f_j(z) \in U_H(\beta_j)$ by using (7), we have

$$\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_j)}{\beta_j-1}\right)(|a_{k,j}|+|b_{k,j}|)\leq 1, \qquad (j=1,2,...,m) \tag{31}$$

Now

$$\begin{aligned}\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_j)}{\beta_j-1}\right)^p(|a_{k,j}|^p+|b_{k,j}|^p)&\leq\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_j)}{\beta_j-1}\right)^p(|a_{k,j}|+|b_{k,j}|)^p\\ &\leq\left\{\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_j)}{\beta_j-1}\right)(|a_{k,j}|+|b_{k,j}|)\right\}^p\leq 1.\end{aligned} \tag{32}$$

It follows from (32)

$$\sum_{k=2}^{\infty}\left(\frac{1}{m}\sum_{j=1}^{m}\left(\frac{k(k-\beta_j)}{\beta_j-1}\right)^p(|a_{k,j}|^p+|b_{k,j}|^p)\right)\leq 1.$$

Putting $\beta=\max_{1\leq j\leq m}\beta_j$ we find that

$$\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_m)}{\beta_m-1}\right)\left(\sum_{j=1}^{m}|a_{k,j}|^p+|b_{k,j}|^p\right)\leq\sum_{k=2}^{\infty}\frac{1}{m}\left(\frac{k(k-\beta)}{\beta-1}\right)^p\left(\sum_{j=1}^{m}|a_{k,j}|^p+|b_{k,j}|^p\right)\leq 1. \tag{33}$$

which is equivalent to $\beta_m\geq 1+\frac{1}{1+\frac{2^{p-1}}{m}(\frac{2-\beta}{\beta-1})^p}$.

Now let $g(k)=1+\frac{k-1}{1+\frac{k^{p-1}}{m}(\frac{k-\beta}{\beta-1})^p}$, it is easy to verify that $g(k)$ is decreasing function of k for $p\geq 1$. Therefore

$$\beta_m=\max_{k\geq 2}g(k)=g(2)=1+\frac{1}{1+\frac{2^{p-1}}{m}(\frac{2-\beta}{\beta-1})^p}.$$

This completes the proof. ■

Theorem 6. *If the functions $f_j(z)$ $(j=1,2,...,m)$ defined by (10) with $b_{1,j}=0$ $(j=1,2,...,m)$ are in the classes $R_H(\beta_j)$ for each j and $F_m(z)$ defined in (30), Then $F_m(z)\in R_H(\beta_m)$ where $\beta_m=1+\frac{m(\beta-1)^p}{2^{p-1}}$, $\beta=\max_{1\leq j\leq m}\beta_j$ and $m(\beta-1)^p\leq 2^{p-1}$.*

Proof. Since $f_j(z)\in R_H(\beta_j)$ by using (9), we have

$$\sum_{k=2}^{\infty}\left(\frac{k}{\beta_j-1}\right)(|a_{k,j}|+|b_{k,j}|)\leq 1, \qquad (j=1,2,...,m) \tag{34}$$

Now

$$\begin{aligned}\sum_{k=2}^{\infty}\left(\frac{k}{\beta_j-1}\right)^p(|a_{k,j}|^p+|b_{k,j}|^p)&\leq\sum_{k=2}^{\infty}\left(\frac{k}{\beta_j-1}\right)^p(|a_{k,j}|+|b_{k,j}|)^p\\ &\leq\left\{\sum_{k=2}^{\infty}\left(\frac{k}{\beta_j-1}\right)(|a_{k,j}|+|b_{k,j}|)\right\}^p\leq 1.\end{aligned} \tag{35}$$

It follows from (35)

$$\sum_{k=2}^{\infty}\left(\frac{1}{m}\sum_{j=1}^{m}\left(\frac{k}{\beta_j-1}\right)^p(|a_{k,j}|^p+|b_{k,j}|^p)\right)\leq 1.$$

Putting $\beta=\max_{1\leq j\leq m}\beta_j$ we find that

$$\sum_{k=2}^{\infty}\left(\frac{k}{\beta_m-1}\right)\left(\sum_{j=1}^{m}|a_{k,j}|^p+|b_{k,j}|^p\right)\leq\sum_{k=2}^{\infty}\frac{1}{m}\left(\frac{k}{\beta-1}\right)^p\left(\sum_{j=1}^{m}(|a_{k,j}|^p+|b_{k,j}|^p)\right)\leq 1. \tag{36}$$

which is equivalent to $\beta_m\geq 1+\frac{m(\beta-1)^p}{k^{p-1}}$.

Now let $g(k)=1+\frac{m(\beta-1)^p}{k^{p-1}}$, it is easy to verify that $g(k)$ is decreasing function of k for $p\geq 1$. Therefore

$$\beta_m=\max_{k\geq 2}g(k)=g(2)=1+\frac{m(\beta-1)^p}{2^{p-1}}.$$

This completes the proof. ■

3. Application on convolution

Theorem 7. *If the functions $f_j(z)$ $(j=1,2)$ defined by (10) with $b_{1,j}=0$ $(j=1,2)$ are in the classes $U_H(\beta_j)$ for each j and the condition (13) is satisfied then*

$$(f_1\Delta f_2)\left(\frac{1}{p},\frac{1}{q};z\right)\in R_H(\beta),$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\beta=1+\frac{1}{\left(\frac{2-\beta_1}{\beta_1-1}\right)^{\frac{1}{p}}\left(\frac{2-\beta_2}{\beta_2-1}\right)^{\frac{1}{q}}}$.

Proof. Since $f_j(z)\in U_H(\beta_j)$, by using (7), we have

$$\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_j)}{\beta_j-1}\right)(|a_{k,j}|+|b_{k,j}|)\leq 1,\qquad (j=1,2). \tag{37}$$

then from eq.(37) we obtain

$$\left\{\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_1)}{\beta_1-1}\right)(|a_{k,1}|+|b_{k,1}|)\right\}^{\frac{1}{p}}\leq 1, \tag{38}$$

and

$$\left\{\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_2)}{\beta_2-1}\right)(|a_{k,2}|+|b_{k,2}|)\right\}^{\frac{1}{q}}\leq 1. \tag{39}$$

Now

$$\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_1)}{\beta_1-1}\right)^{\frac{1}{p}}\left(\frac{k(k-\beta_2)}{\beta_2-1}\right)^{\frac{1}{q}}(|a_{k,1}|^{\frac{1}{p}}|a_{k,2}|^{\frac{1}{q}}+|b_{k,1}|^{\frac{1}{p}}|b_{k,2}|^{\frac{1}{q}})$$

$$\leq\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_1)}{\beta_1-1}\right)^{\frac{1}{p}}\left(\frac{k(k-\beta_2)}{\beta_2-1}\right)^{\frac{1}{q}}(|a_{k,1}|+|b_{k,1}|)^{\frac{1}{p}}(|a_{k,2}|+|b_{k,2}|)^{\frac{1}{q}}$$

$$\leq\left\{\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_1)}{\beta_1-1}\right)(|a_{k,1}|+|b_{k,1}|)\right\}^{\frac{1}{p}}\left\{\sum_{k=2}^{\infty}\left(\frac{k(k-\beta_2)}{\beta_2-1}\right)(|a_{k,2}|+|b_{k,2}|)\right\}^{\frac{1}{q}}\leq 1.$$

Since

$$(f_1 \Delta f_2)(\frac{1}{p}, \frac{1}{q}; z) = z + \sum_{k=2}^{\infty} |a_{k,1}|^{\frac{1}{p}} |a_{k,2}|^{\frac{1}{q}} z^k + \sum_{k=1}^{\infty} |b_{k,1}|^{\frac{1}{p}} |b_{k,2}|^{\frac{1}{q}} \overline{z}^k,$$

it suffices to show that $(f_1 \Delta f_2)(\frac{1}{p}, \frac{1}{q}; z) \in R_H(\beta)$ if

$$\sum_{k=2}^{\infty} \left(\frac{k}{\beta - 1}\right) \left(|a_{k,1}|^{\frac{1}{p}} |a_{k,2}|^{\frac{1}{q}} + |b_{k,1}|^{\frac{1}{p}} |b_{k,2}|^{\frac{1}{q}}\right) \leq 1. \tag{40}$$

For this we have to show that L.H.S. of (40) is bounded above by

$$\sum_{k=2}^{\infty} \left(\frac{k(k-\beta_1)}{\beta_1 - 1}\right)^{\frac{1}{p}} \left(\frac{k(k-\beta_2)}{\beta_2 - 1}\right)^{\frac{1}{q}} (|a_{k,1}|^{\frac{1}{p}} |a_{k,2}|^{\frac{1}{q}} + |b_{k,1}|^{\frac{1}{p}} |b_{k,2}|^{\frac{1}{q}}) \leq 1,$$

which is equivalent to $\beta \geq 1 + \frac{1}{\left(\frac{2-\beta_1}{\beta_1 - 1}\right)^{\frac{1}{p}} \left(\frac{2-\beta_2}{\beta_2 - 1}\right)^{\frac{1}{q}}}$. ■

Theorem 8. *If the functions $f_j(z)$ $(j = 1, 2, ..., m)$ defined by (10) with $b_{1,j} = 0$ $(j = 1, 2, ..., m)$ are in the classes $U_H(\beta_j)$ for each j and $F_m(z)$ defined in (30), then $F_m(z) \in R_H(\beta_m)$ where $\beta_m \geq 1 + \frac{m(\beta-1)^p}{2^{p-1}(2-\beta)^p}$, $\beta = \max_{1 \leq j \leq m} \beta_j$ and $m(\beta - 1)^p \leq 2^{p-1}(2 - \beta)^p$.*

Proof. Since $f_j(z) \in U_H(\beta_j)$ by using (7), we have

$$\sum_{k=2}^{\infty} \left(\frac{k(k-\beta_j)}{\beta_j - 1}\right) (|a_{k,j}| + |b_{k,j}|) \leq 1, \qquad (j = 1, 2, ..., m) \tag{41}$$

Now

$$\begin{aligned} &\sum_{k=2}^{\infty} \left(\frac{k(k-\beta_j)}{\beta_j - 1}\right)^p (|a_{k,j}|^p + |b_{k,j}|^p) \leq \sum_{k=2}^{\infty} \left(\frac{k(k-\beta_j)}{\beta_j - 1}\right)^p (|a_{k,j}| + |b_{k,j}|)^p \\ &\leq \left\{\sum_{k=2}^{\infty} \left(\frac{k(k-\beta_j)}{\beta_j - 1}\right) (|a_{k,j}| + |b_{k,j}|)\right\}^p \leq 1. \end{aligned} \tag{42}$$

It follows from (42)

$$\sum_{k=2}^{\infty} \left(\frac{1}{m} \sum_{j=1}^{m} \left(\frac{k(k-\beta_j)}{\beta_j - 1}\right)^p (|a_{k,j}|^p + |b_{k,j}|^p)\right) \leq 1.$$

Putting $\beta = \max_{1 \leq j \leq m} \beta_j$ we find that

$$\sum_{k=2}^{\infty} \left(\frac{k}{\beta_m - 1}\right) \left(\sum_{j=1}^{m} |a_{k,j}|^p + |b_{k,j}|^p\right) \leq \sum_{k=2}^{\infty} \frac{1}{m} \left(\frac{k(k-\beta)}{\beta - 1}\right)^p \left(\sum_{j=1}^{m} |a_{k,j}|^p + |b_{k,j}|^p\right) \leq 1. \tag{43}$$

which is equivalent to $\beta_m \geq 1 + \frac{m(\beta-1)^p}{2^{p-1}(2-\beta)^p}$. This completes the proof. ■

4. Conclusion

In this work, we obtain the conditions on the parameter β such that the generalized convolution of two functions belongs to the subclasses $V_H(\beta)$, $U_H(\beta)$ and $R_H(\beta)$ of harmonic univalent functions. Also, we gave some application for convolution of two functions belongs to the subclasses $U_H(\beta)$ to be in $R_H(\beta)$.

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