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# FEKETE- SZEGÖ INEQUALITY FOR CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract. In the present investigation, we obtain Fekete-Szego inequality for certain normalized analytic function $(z)$ defined in the open unit disc for which

$$\frac{(1-\lambda)z\left(D^{n}\left(z\right)\right)'+\lambda z\left(D^{n+m}\left(z\right)\right)'}{(1-\lambda)D^{n}\left(z\right)+\lambda D^{n+m}\left(z\right)}(\lambda\geq 0)$$

lines in a region starlike with respect to 1 and is symmetric with respect to the real axis. Also certain applications of the main result for a class of functions defined by convolution are given. As a special case of this result, Fekete-Szego inequality for a class of functions defined through fractional derivatives is obtained. The motivation of this paper is to give a generalization of the Fekete-Szego inequalities obtained by Salagean differential operator.

## 1. Introduction

Let $A$ be class of functions $(z)$ of the form:

$$(z)=z+\sum_{k=2}^{\infty}a_{k}z^{k} \tag{1}$$

which are analytic in the open disc $U=\{z:z\in\mathbb{C}and\,|z\;|<1\}$. Further, let $S$ denote the class of functions which are univalent in $U$. For a function $(z)$ in $A$, we define

$$D^{0}\left(z\right)=\left(z\right),D^{1}\left(z\right)=z'\left(z\right),$$

$$D^{n}\left(z\right)=D\left(D^{n-1}\left(z\right)\right)\qquad\left(n\in\mathbb{N}=\{1,2,3,...\}\right).$$

Note that

$$D^{n}\left(z\right)=z+\sum_{k=2}^{\infty}k^{n}a_{k}z^{k},\left(n\in\mathbb{N}_{0}=\mathbb{N}\cup\{0\}\right). \tag{2}$$

The differential operator $D^n$ was introduced by Sălăgean [5].

Let $\phi(z)$ be an analytic function with positive real part on $U$ with $\phi(0)=1$, $\phi'(0)>0$ which maps the unit disk $U$ onto a region starlike with respect to 1 which

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is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $(z) \in S$ for which

$$\frac{z^{'}(z)}{(z)} \prec \phi(z), \qquad (z \in U),$$

and $C(\phi)$ be the class of function in $(z) \in S$ for which

$$1 + \frac{z^{''}(z)}{{}'(z)} \prec \phi(z), \qquad (z \in U),$$

where $\prec$ denotes the subordination between analytic functions. These classes were investigated and studied by Ma and Minda [3]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $\in C(\phi)$ if and only if $z^{'}(z) \in S^*(\phi)$, we get the Fekete-Szegö inequality for functions in the class $S^*(\phi)$. For a brief history of the Fekete-Szegö problem for class of starlike, convex, and close-to convex functions, see the recent paper by Srivastava et al. [8].

In the present paper, we obtain the Fekete-Szegö inequality for functions in a more general class $G_{\lambda,n,m}(\phi)$ of functions which we define below. Also we give applications of our results to certain functions defined through convolution (or the Hadamard product) and in particular we consider a class $G^{\gamma}_{\lambda,n,m}(\phi)$ of functions deifned by fractional derivatives. The motivation of this paper is to give a generalization of the Fekete-Szegö inequalities of Srivastava and Mishra [7].

Let $\phi(z)$ be a univalent starlike function with respect to 1which maps the unit disc $U$ onto a region in the right half plane which is symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. A function $\in A$ is in the class $G_{\lambda,n,m}(\phi)$ if and only if

$$\left\{ \frac{(1-\lambda)z\left(D^n(z)\right)' + \lambda z\left(D^{n+m}(z)\right)'}{(1-\lambda)D^n(z) + \lambda D^{n+m}(z)} \right\} \prec \phi(z) \qquad (\lambda \geq 0),$$

where $D^{n+m}(z)$ was studied by Sekine [6] and $D^n(z)$ denote Salagean operator of $(z)$ [5]. For fixed $g \in A$, we define the class $G^{g}_{\lambda,n,m}(\phi)$ to be class of function $\in A$ for which $(*g) \in G_{\lambda,n,m}(\phi)$. In order to derive our main results, we have to recall here the following Lemma [3].

If $p_1 = 1 + c_1 z + c_2 z^2 + ...$,is an analytic function with positive real part in $U$, then

$$\left|c_2 - vc_1^2\right| \leq \begin{cases} -4v+2 & \text{if} \quad v \leq 0; \\ 2 & \text{if} \quad 0 \leq v \leq 1; \\ 4v-2 & \text{if} \quad v \geq 1 \end{cases}$$

when $v < 0$ or $v > 1$, the equality holds if and only if $p_1(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < v < 1$, then the quality holds if and only if $p_1(z)$ is $\left(1+z^2\right)/(1-z^2)$ or one of its rotations. If $v = 0$, the quality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\gamma\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\gamma\right)\frac{1-z}{1+z} \qquad (0 \leq \gamma \leq 1)$$

or one of its rotations. If $v = 1$, the quality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$. Also the above upper bound is sharp, and it can be improved as follows when $0 < v < 1$.

$$\left|c_2 - vc_1^2\right| + v\left|c_1\right|^2 \leq 2 \qquad \left(0 < v \leq \frac{1}{2}\right)$$

and

$$\left|c_2 - vc_1^2\right| + (1-v)\left|c_1\right|^2 \leq 2 \qquad \left(\frac{1}{2} < v \leq 1\right).$$

## 2. Fekete- Szegö Problem

Our main result is the following:
Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ .If

$$(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

belongs to $G_{\lambda,n,m}(\phi)$ , then

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{B_2}{3^n[2+2\lambda(3^m-1)]} - \frac{\mu}{2^{2n}[1+\lambda(2^m-1)]^2}B_1^2 + \frac{1}{3^n[2+2\lambda(3^m-1)]}B_1^2 & if \quad \mu \geq \sigma_1; \\ \frac{B_1}{3^n[2+2\lambda(3^m-1)]} & if \quad \sigma_1 \leq \ \mu \leq \sigma_2; \\ -\frac{B_2}{3^n[2+2\lambda(3^m-1)]} + \frac{\mu}{2^{2n}[1+\lambda(2^m-1)]^2}B_1^2 - \frac{1}{3^n[2+2\lambda(3^m-1)]}B_1^2 & if \quad \mu \geq \sigma_2, \end{cases} \tag{3}$$

where

$$\sigma_1 = \frac{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2\left\{\left(B_2 - B_1\right) + B_1^2\right\}}{3^n\left[2+2\lambda\left(3^m-1\right)\right]B_1^2}$$

$$\sigma_2 = \frac{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2\left\{\left(B_2 + B_1\right) + B_1^2\right\}}{3^n\left[2+2\lambda\left(3^m-1\right)\right]B_1^2}.$$

The result is sharp.

*Proof.* For $(z) \in G_{\lambda,n,m}(\phi)$ , let

$$p(z) = \frac{(1-\lambda)z\left(D^n(z)\right)' + \lambda z\left(D^{n+m}(z)\right)'}{(1-\lambda)D^n(z) + \lambda D^{n+m}(z)} = 1 + b_1 z + b_2 z^2 + \ldots \tag{4}$$

From $(4)$ , we obtain

$$2^n\left[1+\lambda\left(2^m-1\right)\right]a_2 = b_1 \text{ and } 3^n\left[2+2\lambda\left(3^m-1\right)\right]a_3 = b_2 + 2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2 a_2^2. \tag{5}$$

Sine $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1+\phi^{-1}(p(z))}{1-\phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \ldots$$

is analytic and has a positive real part in $U$. Also we have

$$p(z) = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) \tag{6}$$

and from this equation (4),

$$\begin{aligned}1+b_1z+b_2z^2+... &= \phi\left(\frac{c_1z+c_2z^2+...}{2+c_1z+c_2z^2+...}\right)= \\ &= \phi\left[\frac{1}{2}c_1z+\frac{1}{2}\left(c_2-\frac{1}{2}c_1^2\right)z^2+...\right] \\ &= 1+B_1\frac{1}{2}c_1z+B_1\frac{1}{2}\left(c_2-\frac{1}{2}c_1^2\right)z^2+...+B_2\frac{1}{4}c_1^2z^2+...\end{aligned}$$

we obtian

$$b_1=\frac{1}{2}B_1c_1 \text{ and } b_2=\frac{1}{2}B_1\left(c_2-\frac{1}{2}c_1^2\right)+\frac{1}{4}B_2c_1^2.$$

Therefore we have

$$a_3-\mu a_2^2=\frac{B_1}{2.3^n\left[2+2\lambda\left(3^m-1\right)\right]}$$

$$\times\left\{c_2-c_1^2\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{3^n\left[2+2\lambda\left(3^m-1\right)\right]\mu-2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}B_1\right)\right\}$$

$$a_3-\mu a_2^2=\frac{B_1}{2.3^n\left[2+2\lambda\left(3^m-1\right)\right]}\left\{c_2-vc_1^2\right\} \tag{7}$$

where

$$v:=\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{3^n\left[2+2\lambda\left(3^m-1\right)\right]\mu-2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}B_1\right).$$

If $\mu\leq\sigma_1$, then by applying Lemma 1, we get

$$\left|a_3-\mu a_2^2\right|=\frac{B_1}{2.3^n\left[2+2\lambda\left(3^m-1\right)\right]}$$

$$\times\left|c_2-c_1^2\left\{\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{3^n\left[2+2\lambda\left(3^m-1\right)\right]\mu-2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}B_1\right)\right\}\right|$$

$$\left|a_3-\mu a_2^2\right|\leq\frac{B_2}{3^n\left[2+2\lambda\left(3^m-1\right)\right]}-\frac{\mu}{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}B_1^2+\frac{1}{3^n\left[2+2\lambda\left(3^m-1\right)\right]}B_1^2,$$

which is the first part of assertion (3).

Naxt, if $\mu\geq\sigma_2$, by applying Lemma 1, we get

$$\left|a_3-\mu a_2^2\right|=\frac{B_1}{2.3^n\left[2+2\lambda\left(3^m-1\right)\right]}$$

$$\times\left|c_2-c_1^2\left\{\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{3^n\left[2+2\lambda\left(3^m-1\right)\right]\mu-2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}B_1\right)\right\}\right|$$

$$\left|a_3-\mu a_2^2\right|\leq-\frac{B_2}{3^n\left[2+2\lambda\left(3^m-1\right)\right]}+\frac{\mu}{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}B_1^2-\frac{1}{3^n\left[2+2\lambda\left(3^m-1\right)\right]}B_1^2.$$

If $\mu=\sigma_1$, then equality holds if and only if

$$p_1(z)=\left(\frac{1+\gamma}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-\gamma}{2}\right)\frac{1-z}{1+z} \qquad (0\leq\gamma\leq1; z\in U)$$

or one of its rotations.

If $\mu=\sigma_2$, then

$$\frac{1}{2}\left[1-\frac{B_2}{B_1}+\frac{3^n\left[2+2\lambda\left(3^m-1\right)\right]\mu-2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}B_1\right]=0.$$

Therefore,

$$\frac{1}{p_1(z)}=\left(\frac{1+\gamma}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-\gamma}{2}\right)\frac{1-z}{1+z}\qquad(0\leq\gamma\leq1;z\in U).$$

Finaly, we see that

$$\left|a_3-\mu a_2^2\right|=\frac{B_1}{2.3^n\left[2+2\lambda\left(3^m-1\right)\right]}.$$

$$\left|c_2-c_1^2\left\{\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{3^n\left[2+2\lambda\left(3^m-1\right)\right]\mu-2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}B_1\right)\right\}\right|$$

and

$$\max\left|\frac{1}{2}\left[1-\frac{B_2}{B_1}+\frac{3^n\left[2+2\lambda\left(3^m-1\right)\right]\mu-2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}B_1\right]\right|\leq1,$$

$$(\sigma_1\leq\mu\leq\sigma_2).$$

Therefore using Lemma 1. , we get

$$\left|a_3-\mu a_2^2\right|=\frac{B_1}{2.3^n\left[2+2\lambda\left(3^m-1\right)\right]}\leq\frac{B_1}{3^n\left[2+2\lambda\left(3^m-1\right)\right]},\quad(\sigma_1\leq\mu\leq\sigma_2).$$

If $\sigma_1<\mu<\sigma_2$, then we have

$$p_1(z)=\frac{1+\gamma z^2}{1-\gamma z^2},\qquad(0\leq\gamma\leq1).$$

Our result now follows by an application of Lemma 1. To show that the bounds are sharp, we define the functions $K_\alpha^{\phi,\delta}$ $(\delta=2,3,..)$ by

$$\frac{(1-\lambda)z\left[D^nK_\alpha^{\phi,\delta}\right]'(z)+\lambda z\left[D^{n+m}K_\alpha^{\phi,\delta}\right]'}{(1-\lambda)\left[D^nK_\alpha^{\phi,\delta}\right](z)+\lambda\left[D^{n+m}K_\alpha^{\phi,\delta}\right](z)}=\phi\left(z^{\delta-1}\right),$$

$$K_\alpha^{\phi,\delta}(0)=0=\left[K_\alpha^{\phi,\delta}\right]'(0)-1$$

and function $F_\alpha^\gamma$ and $W_\alpha^\gamma$ ( $0\leq\gamma\leq1$)by

$$\frac{(1-\lambda)z\left[D^nF_\alpha^\gamma\right]'(z)+\lambda z\left[D^{n+m}F_\alpha^\gamma\right]'(z)}{(1-\lambda)\left[D^nW_\alpha^\gamma\right](z)+\lambda\left[D^{n+m}W_\alpha^\gamma\right](z)}=\phi\left[\frac{z(z+\gamma)}{1+\gamma z}\right]$$

$$F^\gamma(0)=0=\left(F^\gamma\right)'(0)-1)$$

and

$$\frac{(1-\lambda)z\left[D^nW_\alpha^\gamma\right]'(z)+\lambda z\left[D^{n+m}W_\alpha^\gamma\right]'(z)}{(1-\lambda)\left[D^nW_\alpha^\gamma\right](z)+\lambda\left[D^{n+m}W_\alpha^\gamma\right](z)}=\phi\left[-\frac{z(z+\gamma)}{1+\gamma z}\right]$$

$$W^\beta(0)=0=\left(W^\beta\right)'(0)-1).$$

Clearly the functions $K_\alpha^{\phi,\delta}$ , $F_\alpha^\gamma$ ,$W_\alpha^\gamma\in G_{\lambda,n,m}(\phi)$. Also we write $K_\alpha^\phi=K_\alpha^{\phi2}$. If $\mu<\sigma_1$ or $\mu>\sigma_2$, then the equality holds if and only if is $K_\alpha^\phi$ or one of its rotations. When $\mu<\sigma_1<\sigma_2$, the equality holds if and only if is $K_\alpha^{\phi3}$or one of its rotations. If $\mu=\sigma_1$ then the equality holds if and only if is $F_\alpha^\gamma$ or one of its rotations. If $\mu=\sigma_2$ then the equality holds if and only if is $W_\alpha^\gamma$ or one of its rotations. □

If $\sigma_1 \le \mu \le \sigma_2$ then, in view of Lemma 1 ,2 can be improved. Let $\sigma_3$ be given by

$$\sigma_3 = \frac{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2\left\{B_1^2+B_2\right\}}{3^n\left[2+2\lambda\left(3^m-1\right)\right]B_1^2}.$$

If $\sigma_1 \le \mu \le \sigma_3$, then

$$\left|a_3-\mu a_2^2\right| + \frac{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}{3^n\left[2+2\lambda\left(3^m-1\right)\right]B_1^2}$$
$$\left[B_1-B_2+\frac{3^n\mu\left[2+2\lambda\left(3^m-1\right)\right]-2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}B_1^2\right]\left|a_2\right|^2$$
$$\le \frac{B_1}{3^n\left[2+2\lambda\left(3^m-1\right)\right]}.$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$\left|a_3-\mu a_2^2\right| + \frac{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}{3^n\left[2+2\lambda\left(3^m-1\right)\right]B_1^2}$$
$$\left[B_1+B_2+\frac{3^n\mu\left[2+2\lambda\left(3^m-1\right)\right]-2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}B_1^2\right]\left|a_2\right|^2$$
$$\le \frac{B_1}{3^n\left[2+2\lambda\left(3^m-1\right)\right]}.$$

*Proof.* For the values of $\sigma_1 \le \mu \le \sigma_3$, we have

$$\left|a_3-\mu a_2^2\right| + (\mu-\sigma_1)\left|a_2\right|^2 =$$
$$\frac{B_1}{3^n 2\left[2+2\lambda\left(3^m-1\right)\right]}\left|c_2-vc_1^2\right| + (\mu-\sigma_1)\frac{B_1^2}{4.2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}\left|c_1\right|^2 =$$
$$\frac{B_1}{3^n 2\left[2+2\lambda\left(3^m-1\right)\right]}\left|c_2-vc_1^2\right| +$$
$$\left(\mu-\frac{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2\left\{(B_2-B_1)+B_1^2\right\}}{3^n\left[2+2\lambda\left(3^m-1\right)\right]B_1^2}\right)\frac{B_1^2}{4.2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}\left|c_1\right|^2 =$$
$$\frac{B_1}{3^n\left[2+2\lambda\left(3^m-1\right)\right]}\left\{\frac{1}{2}\left[\left|c_2-vc_1^2\right|+v\left|c_1\right|^2\right]\right\} \le \frac{B_1}{3^n\left[2+2\lambda\left(3^m-1\right)\right]}.$$

Similary, for the values of $\sigma_3 \le \mu \le \sigma_2$, we write

$$\left|a_3-\mu a_2^2\right| + (\sigma_2-\mu)\left|a_2\right|^2 =$$
$$= \frac{B_1}{3^n 2\left[2+2\lambda\left(3^m-1\right)\right]}\left|c_2-vc_1^2\right| + (\sigma_2-\mu)\frac{B_1^2}{4.2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}\left|c_1\right|^2$$
$$= \frac{B_1}{3^n 2\left[2+2\lambda\left(3^m-1\right)\right]}\left|c_2-vc_1^2\right|$$
$$+\left(\frac{2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2\left\{(B_2+B_1)+B_1^2\right\}}{3^n\left[2+2\lambda\left(3^m-1\right)\right]B_1^2}-\mu\right)\frac{B_1^2}{2.2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}\left|c_1\right|^2$$
$$= \frac{B_1}{3^n\left[2+2\lambda\left(3^m-1\right)\right]}\left\{\frac{1}{2}\left[\left|c_2-vc_1^2\right|+(1-v)\left|c_1\right|^2\right]\right\} \le \frac{B_1}{3^n\left[2+2\lambda\left(3^m-1\right)\right]}.$$

Thus, the proof of Remark 2 is evidently completed. □

## 3. Applications to Functions Defined by Fractional Derivatives

In order to introduce the class $G^{\gamma}_{\lambda,n,m}(\phi)$, we need the following:

Let $(z)$ be analytic in a simply connected region of the $z-$plane containing the origin. The fractional derivative of order $\gamma$ is defined by

$$D^{\gamma}_{z}(z) := \frac{1}{\Gamma(1-\gamma)}\frac{d}{dz}\int_0^z \frac{(\zeta)}{(z-\zeta)^{\gamma}}d\zeta \qquad (0 \le \gamma < 1),$$

where the multiplicity of $(z-\zeta)$ is removed by requiring that $\log(z-\zeta)^{\gamma}$ is real for $z-\zeta > 0$. Using the above Definition 3. and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [1] introduced the operator $\Psi^{\gamma} : A \to A$ defined by

$$(\Psi^{\gamma})(z) = \Gamma(2-\gamma)z^{\gamma}D^{\gamma}_{z}(z) \qquad (\gamma \ne 2.3.4,...).$$

The class $G^{\gamma}_{\lambda,n,m}(\phi)$ consists of functions $\in A$ for which $\Psi^{\gamma} \in G_{\lambda,n,m}(\phi)$. Note that $G^{*}_{0,0,m}(\phi) = S^{*}(\phi)$ and $G^{\gamma}_{\lambda,n,m}(\phi)$ is the special case of the class $G^{g}_{\lambda,n,m}(\phi)$ when

$$g(z) = z + \sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)}z^k. \tag{8}$$

Let

$$g(z) = z + \sum_{k=2}^{\infty} g_k z^k \qquad (g_k > 0).$$

Since

$$D^n(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \in G^{g}_{\lambda,n,m}(\phi)$$

If and only if

$$D^n(*g)(z) = z + \sum_{k=2}^{\infty} k^n g_k z^k \in G_{\lambda,n,m}(\phi),$$

we obtain the coefficient estimate for functions in the class $G^{g}_{\lambda,n,m}(\phi)$, from the corresponding estimate for functions in the class $G_{\lambda,n,m}(\phi)$. Applying Theorem 2 for the function $(f*g)(z) = z + 2^n g_2 a_2 z^2 + 3^n g_3 a_3 z^3 + ...$, we get the following Theorem 3 after an obvious change of the parameter $\mu$:

Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + ....$and let $\in G^{g}_{\lambda,n,m}(\phi)$, then $\mu \in \mathbb{C}$

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{1}{g_3}\left[\frac{B_2}{3^n[2+2\lambda(3^m-1)]} - \frac{\mu g_3}{2^{2n}[1+\lambda(2^m-1)]^2 g_2^2}B_1^2 + \frac{1}{3^n[2+2\lambda(3^m-1)]}B_1^2\right] & \text{if } \mu \le \sigma_1; \\ \frac{1}{g_3}\left[\frac{B_1}{3^n[2+2\lambda(3^m-1)]}\right] & \text{if } \sigma_1 \le \mu \le \sigma_2; \\ \frac{1}{g_3}\left[-\frac{B_2}{3^n[2+2\lambda(3^m-1)]} + \frac{\mu g_3}{2^{2n}[1+\lambda(2^m-1)]^2 g_2^2}B_1^2 - \frac{1}{3^n[2+2\lambda(3^m-1)]}B_1^2\right] & \text{if } \mu \ge \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{g_2^2 2^{2n}\left[1+\lambda(2^m-1)\right]^2}{g_3}\left[\frac{(B_2 - B_1) + B_1^2}{3^n\left[2+2\lambda(3^m-1)\right]B_1^2}\right],$$

$$\sigma_2 = \frac{g_2^2 2^{2n}\left[1+\lambda(2^m-1)\right]^2}{g_3}\left[\frac{(B_2 + B_1) + B_1^2}{3^n\left[2+2\lambda(3^m-1)\right]B_1^2}\right].$$

The result is sharp.

Since

$$g\left(z\right)=z+\sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)}k^{n+m}z^{k}$$

we have

$$g_2:=\frac{\Gamma(3)\Gamma(2-\gamma)}{\Gamma(3-\gamma)}=\frac{2}{2-\gamma} \tag{9}$$

and

$$g_3:=\frac{\Gamma(4)\Gamma(2-\gamma)}{\Gamma(4-\gamma)}=\frac{6}{\left(2-\gamma\right)\left(3-\gamma\right)}. \tag{10}$$

For $g_2$ and $g_3$ given by (9) and (10), Theorem 3 reduces to the following:

Let the function $\phi(z)$ be given by $\phi(z)=1+B_1z+B_2z^2+....$and let $\in G^{g}_{\lambda,n,m}\left(\phi\right)$, then $\mu\in\mathbb{C}$

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{(2-\gamma)(3-\gamma)}{6}\left[\frac{B_2}{3^n[2+2\lambda(3^m-1)]}-\frac{\mu}{2^{2n}[1+\lambda(2^m-1)]^2}B_1^2+\frac{1}{3^n[2+2\lambda(3^m-1)]}B_1^2\right] & if\ \mu\geq\sigma_1;\\ \frac{(2-\gamma)(3-\gamma)}{6}\left[\frac{B_1}{3^n[2+2\lambda(3^m-1)]}\right] & if\ \ \sigma_1\leq\ \mu\leq\sigma_2;\\ \frac{(2-\gamma)(3-\gamma)}{6}\left[-\frac{B_2}{3^n[2+2\lambda(3^m-1)]}+\frac{\mu}{2^{2n}[1+\lambda(2^m-1)]^2}B_1^2-\frac{1}{3^n[2+2\lambda(3^m-1)]}B_1^2\right] & if\ \mu\geq\sigma_2,\end{cases}$$

where

$$\sigma_1=\frac{2(3-\gamma)2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}{3(2-\gamma)}\left[\frac{(B_2-B_1)+B_1^2}{3^n\left[2+2\lambda\left(3^m-1\right)\right]B_1^2}\right],$$

$$\sigma_2=\frac{2(3-\gamma)2^{2n}\left[1+\lambda\left(2^m-1\right)\right]^2}{3(2-\gamma)}\left[\frac{(B_2+B_1)+B_1^2}{3^n\left[2+2\lambda\left(3^m-1\right)\right]B_1^2}\right].$$

The result is sharp.

When $\lambda=0,n,m=0,B_1=8/\pi^2$ and $B_2=16/3\pi$ the above Theorem 3 reduces to a recent result of Srivastava and Mishra [1, Theorem 8, p. 64] for a class of functions for which $\left(\Psi^{\gamma}\right)(z)$is a parabolic starlike functions [2, 4].

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