Journal of Fractional Calculus and Applications
Vol. 8(1) Jan. 2017, pp. 1-5.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

ZALCMAN CONJECTURE FOR SOME SUBCLASS OF ANALYTIC FUNCTIONS

DEEPAK BANSAL AND JANUSZ SOKÓL

Abstract. In the present investigation sharp upper bound of Zalcman functional $|a_n^2 - a_{2n-1}|$ for functions belonging to classe $\mathcal{M}$ and $\mathcal{N}$ for $n = 3$ is investigated.

1. Introduction

Let $\mathcal{H}(\mathbb{U})$ denote the class of functions which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ and $\mathcal{A}$ be the class of functions $f \in \mathcal{H}(\mathbb{U})$, normalized by $f(0) = 0;\ f'(0) = 1$ and having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{U}. \tag{1.1}$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$. A function $f \in \mathcal{S}$ is called starlike (with respect to origin 0), denoted by $f \in \mathcal{S}^*$ if $tw \in f(\mathbb{U})$ whenever $w \in f(\mathbb{U})$ and $t \in [0, 1]$. A function $f \in \mathcal{S}$ maps the unit disk $\mathbb{U}$ onto a convex domain is called convex function. A function $f \in \mathcal{S}$ is called starlike function of order λ $(0 \le \lambda < 1)$, denoted by $\mathcal{S}^*(\lambda)$, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \lambda, \qquad z \in \mathbb{U}. \tag{1.2}$$

A function $f \in \mathcal{S}$ is called convex function of order λ $(0 \le \lambda < 1)$, denoted by $\mathcal{K}(\lambda)$, if and only if $zf'(z) \in \mathcal{S}^*(\lambda)$. Nishiwaki and Owa [16] studied a class of function $f \in \mathcal{A}$ satisfying (1.2) with opposite inequality, *i. e.* denoted by $\mathcal{M}(\lambda)$, $\lambda > 1$ is the class of function $f \in \mathcal{A}$ satisfying the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < \lambda, \quad z \in \mathbb{U} \tag{1.3}$$

and let $\mathcal{N}(\lambda)$, $\lambda > 1$ is the class of function $f \in \mathcal{A}$ satisfying the inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \lambda \quad z \in \mathbb{U}. \tag{1.4}$$

2010 *Mathematics Subject Classification.* 30C45, 30C50, 15B05.

Key words and phrases. Analytic, Starlike and Convex functions; Fekete-Szegö functional; Hankel determinants.
Submitted .

For convenience, we put $\mathcal{M}(3/2) = \mathcal{M}$ and $\mathcal{N}(3/2) = \mathcal{N}$. For $1 < \lambda \leq 4/3$, the classes $\mathcal{M}(\lambda)$ and $\mathcal{N}(\lambda)$ were investigated by Uralegaddi *et al.* [24]. In an earlier study, Ozaki [**?**] proved that functions in $\mathcal{N}$ are *univalent* in $\mathbb{U}$. Singh and Singh [23, Theorem 6] proved that function in $\mathcal{N}$ are *starlike* in $\mathbb{U}$. Saitoh *et al.* [22] and Nunokawa [17] have improved the result of Singh and Singh [23, Theorem 6].

For $f \in \mathcal{A}$ of the form (1.1), the classical *Fekete-Szegö functional* $\Phi_\lambda(f) := a_3 - \lambda a_2^2$ plays an important role in the function theory. A classical problem settled by Fekete and Szegö [4] is to find for each $\lambda \in [0,1]$ the maximum value of the $|\Phi_\lambda(f)|$ over the function $f \in \mathcal{S}$. By applying the *Löewner* method they proved that

$$\max_{f\in\mathcal{S}} |\Phi_\lambda(f)| = \begin{cases} 1 + 2\exp\{-2\lambda/(1-\lambda)\}, & \lambda \in [0,1], \\ 1, & \lambda = 1. \end{cases}$$

The problem of calculating $\max_{f\in\mathcal{F}} |\Phi_\lambda(f)|$ for various compact subfamilies $\mathcal{F}$ of $\mathcal{A}$, as well as λ being an arbitrary real or complex number, was considered by many authors (see e.g. [5, 6, 9]).

In 1960, Lawrence Zalcman posed a conjecture that the coefficients of $\mathcal{S}$ satisfy the inequality

$$|a_n^2 - a_{2n-1}| \leq (n-1)^2, \tag{1.5}$$

with equality only for Koebe function $k(z) = z/(1-z)^2$ and its rotations. We call $J_f(n) = a_n^2 - a_{2n-1}$ the Zalcman functional for $f \in \mathcal{S}$. This remarkable conjecture was investigated by many mathematicians, and remain open for all $n > 6$. The case $n = 2$ is the elementary well-known Fekete-Szegö inequality. The Zalcman coefficient inequality (1.5) for $n = 3$ was established in [10] and also for the special cases $n = 4, 5, 6$ in [11]. This conjecture was proved for certain special subclasses of $\mathcal{S}$ in [2, 14], (see also [12] and [15]), and an observation demonstrates that the Zalcman coefficient conjecture is asymptotically true.
In the present paper, we investigate the validity of Zalcman conjecture for $n = 3$ for the functions belonging to the classes $\mathcal{M}$ and $\mathcal{N}$ defined above. In our study we shall need the *Carathéodory functions* $\mathcal{P}$ (see, Duren [3]), which is the class of functions $p \in \mathcal{H}(\mathbb{U})$ with $\Re(p(z)) > 0,\ z \in \mathbb{U}$ and having the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \qquad z \in \mathbb{U}. \tag{1.6}$$

Lemma 1.1. ([3]) *If $p \in \mathcal{P}$ is of the form (1.6). Then for all $n \geq 1$ and $s \geq 1$, we have*

$$|p_n| \leq 2 \quad \text{and} \quad |p_n - p_s p_{n-s}| \leq 2. \tag{1.7}$$

These inequalities are sharp for all n and for all s, equality being attained for each n and for each s by the function $p(z) = (1+z)/(1-z)$.

The second inequality in Lemma 1.1 is due to Livingston [13].

Lemma 1.2. ([18]) *If $f(z) \in \mathcal{M}$ be given by (1.1), then $|a_n| \leq \frac{1}{n-1}$, $n \geq 2$. The result is sharp for the function $g_n(z) = z(1 - z^{n-1})^{1/(n-1)}$, $n \geq 2$.*

Lemma 1.3. ([21, Theorem 2.1]) *Let the function $f \in \mathcal{M}$ be given by (1.1), then*

$$|a_3 - a_2^2| \leq 1. \tag{1.8}$$

The result (2.1) is sharp and equality in (2.1) is attended for the function $e_1(z) = z - z^2$.

As it is known that, if $f(z) \in \mathcal{N}$ then $zf'(z) \in \mathcal{M}$, therefore from Lemma 1.2, we conclude that

Lemma 1.4. ([18, Theorem 1]) *If $f \in \mathcal{N}$ be given by (1.1), then*

$$|a_n| \leq \frac{1}{n(n-1)}, \quad n \geq 2.$$

The result is sharp for the function f_n such that $f_n'(z) = (1-z^{n-1})^{1/(n-1)}$, $n \geq 2$.

Lemma 1.5. ([18, Corollary 2]) *If $f \in \mathcal{N}$ be given by (1.1), then $|a_3 - a_2^2| \leq 1/4$. Equality is attended for the function f such that $f'(z) = (1-z^2e^{i\theta})^{1/2}$, $\theta \in [0, 2\pi]$.*

2. Main Results

Our first main result is contained in the following theorem:

Theorem 2.1. *Let the function $f \in \mathcal{M}$ be given by (1.1), then*

$$|a_3^2 - a_5| \leq \frac{3}{8}. \tag{2.1}$$

The result is sharp.

Proof. Let $f \in \mathcal{M}$ be given by (1.1), then there exists a function $p \in \mathcal{P}$ of the form (1.6), such that

$$\frac{zf'(z)}{f(z)} = \frac{1}{2}(3 - p(z)),$$

which in terms of power series is equivalent to

$$2\sum_{n=1}^{\infty} na_n z^n = \left(\sum_{n=1}^{\infty} a_n z^n\right)\left(2 - \sum_{n=1}^{\infty} p_n z^n\right).$$

Comparing coefficient of z^n

$$a_n = \frac{-1}{2(n-1)}\left[p_{n-1} + a_2 p_{n-2} + \ldots + a_{n-1}p_1\right] \ (n = 2, 3, \ldots). \tag{2.2}$$

A simple calculation gives

$$a_2 = -\frac{1}{2}p_1, \quad a_3 = \frac{1}{8}(p_1^2 - 2p_2), \quad a_4 = \frac{1}{48}(6p_1p_2 - 8p_3 - p_1^3) \tag{2.3}$$

and

$$a_5 = \frac{1}{384}\left({p_1}^4 + 12p_2^2 + 32p_1p_3 - 48p_4 - 12p_1^2p_2\right). \tag{2.4}$$

By using (2.3), (2.4) and Lemma 1.1, one can easily see that

$$\begin{aligned}
\left|a_3^2 - a_5\right| &= \frac{1}{384}\left|5{p_1}^4 - 12p_1^2p_2 + 12p_2^2 - 32p_1p_3 + 48p_4\right| \\
&= \frac{1}{384}\left|5(p_2 - p_1^2)^2 + 7p_2^2 - 2p_1^2p_2 + 32(p_4 - p_1p_3) + 16p_4\right| \\
&\leq \frac{1}{384}\left(5|p_2 - p_1^2|^2 + 2|p_2||p_2 - p_1^2| + 5|p_2|^2 + 32|p_4 - p_1p_3| + 16|p_4|\right) \\
&= \frac{1}{384}(5 \times 4 + 2 \times 2 \times 2 + 5 \times 4 + 32 \times 2 + 16 \times 2) = \frac{3}{8}.
\end{aligned}$$

To show that (2.1) is sharp consider $f \in \mathcal{M}$ such that

$$\frac{zf'(z)}{f(z)} = \frac{1}{2}(3 - q(z)) = \frac{1}{2}\left(3 - \frac{1+z^2}{1-z^2}\right),$$

then $q \in \mathcal{P}$ and

$$q(z) = 1 + 2z^2 + 2z^4 + 2z^6 + \ldots = 1 + 2\sum_{n=1}^{\infty} q_n z^n \quad z \in \mathbb{U}.$$

Then, we have $q_1 = q_3 = 0$ and $q_2 = q_4 = 2$, hence

$$\begin{aligned} \left|a_3^2 - a_5\right| &= \frac{1}{384}\left|5{q_1}^4 - 12q_1^2 q_2 + 12q_2^2 - 32q_1 p_3 + 48q_4\right| \\ &= \frac{1}{384}\left|12q_2^2 + 48q_4\right| = \frac{3}{8}. \end{aligned}$$

□

Theorem 2.2. *Let the function $f \in \mathcal{N}$ be given by* (1.1), *then*

$$|a_3^2 - a_5| \le \frac{1}{15}. \tag{2.5}$$

Proof. Let the function $f \in \mathcal{N}$ be given by (1.1), then by definitions it is clear that $f(z) \in \mathcal{N}$ if and only if $zf'(z) \in \mathcal{M}$, thus replacing a_n by na_n in (2.2), we get

$$a_2 = -\frac{1}{4}p_1, \quad a_3 = \frac{1}{24}(p_1^2 - 2p_2), \quad a_4 = \frac{1}{192}(6p_1p_2 - 8p_3 - p_1^3) \tag{2.6}$$

and

$$a_5 = \frac{1}{1920}\left({p_1}^4 + 12p_2^2 + 32p_1p_3 - 48p_4 - 12p_1^2 p_2\right). \tag{2.7}$$

By using (2.6), (2.7) and Lemma 1.3, one can easily see that

$$\begin{aligned} \left|a_3^2 - a_5\right| &= \frac{1}{5760}\left|7{p_1}^4 - 4p_1^2 p_2 + 4p_2^2 - 96p_1p_3 + 144p_4\right| \qquad (2.8) \\ &= \frac{1}{5760}|4(p_1^2 - p_2)^2 + 3p_1^4 + 4p_1^2 p_2 + 96(p_4 - p_1p_3) + 48p_4| \\ &\le \frac{1}{5760}(4|p_1^2 - p_2|^2 + 3|p_1|^4 + 4|p_1^2 p_2| + 96|p_4 - p_1p_3| + 48|p_4|) \\ &= \frac{1}{5760}(4 \times 4 + 3 \times 16 + 4 \times 8 + 96 \times 2 + 48 \times 2) = \frac{12}{180}. \end{aligned}$$

□

We have $\frac{12}{180} = \frac{384}{5760}$. The function $g(z)$ such that

$$\frac{z(zg'(z))'(z)}{zg'(z)} = \frac{1}{2}(3 - q(z)) = \frac{1}{2}\left(3 - \frac{1+z^2}{1-z^2}\right),$$

is in the class $\mathcal{N}$ and when $g(z) = z + a_2 z^2 + \cdots$, we have

$$\begin{aligned} \left|a_3^2 - a_5\right| &= \frac{1}{5760}\left|7{p_1}^4 - 4p_1^2 p_2 + 4p_2^2 - 96p_1p_3 + 144p_4\right| \\ &= \frac{304}{5760}. \end{aligned}$$

This suggest the following conjecture.

Conjecture If $f \in \mathcal{N}$, then

$$|a_3^2 - a_5| \le \frac{304}{5760} = \frac{19}{360}.$$

References

[1] H. R. Abdet Gawad, D. K. Thomas, The Fekete-Szegö problem for strongly close-to-convex functions, Proc. Amer. Math. Soc. 114(1992) 345–349.
[2] J. E. Brown and A. Tsao , On the Zalcman conjecture for starlikeness and typically real functions, Math. Z. 191 (1986), 467–474.
[3] P. L. Duren, Univalent Functions, Springer Verlag, New Yark Inc. 1983.
[4] M. Fekete, G. Szegö, Eine Benberkung uber ungerada Schlichte funktionen, J. Lond. Math. Soc. 8(1933) 85–89.
[5] F. R. Keogh, E. P. Merkes, A Coefficient Inequality for Certain Classes of Analytic Functions, Proc. Amer. Math. Soc. 20(1969) 8–12.
[6] W. Koepf, On the Fekete-Szegö problem for close-to-convex functions, Proc. Amer. Math. Soc. 101(1987) 85–95.
[7] C. R. Leverenz, Hermitian forms in function theory, Trans. Amer. Math. Soc. 286(2)(1984) 675–688.
[8] R. J. Libera, E. J. Złotkiewicz, Early coefficients of the inverse of a regular convex function, Proc. Amer. Math. Soc. 85(1982) 225–230.
[9] R. R. London, Fekete-Szegö inequalities for close-to-convex functions, Proc. Amer. Math. Soc. 117(1993) 947–950.
[10] S. L. Krushkal, Univalent functions and holomorphic motions, J. Analyse Math. 66 (1995), 253–275.
[11] S. L. Krushkal, Proof of the Zalcman conjecture for initial coefficients, Georgian Math. J. 17 (2010), 663–681.
[12] S. L. Krushkal, Hyperbolic metrics, homogeneous holomorphic functionals and Zalcmans conjecture, http://arxiv.org/pdf/1109.4646v2
[13] A. E. Livingston, The coefficients of multivalent close-to-convex functions, Proc. Amer. Math. Soc. 21 (1969), 545–552.
[14] W. Ma , The Zalcman conjecture for close-to-convex functions, Proc. Amer. Math. Soc. 104 (1988), 741–744.
[15] W. Ma , Generalized Zalcman conjecture for starlike and typically real functions, J. Math. Ana. Appl. 234 (1999), 328–339.
[16] J. Nishiwaki, S. Owa, Coefficient inequalities for certain analytic functions, Int. J. Math. Math. Sci. 29(2002) 285–290.
[17] M. Nunokawa, A sufficient condition for univalence and starlikeness, Proc. Japan Acad. Ser. A. 65(1989) 163–164.
[18] M. Obradoviȉ, S. Ponnusamy, K. J. Wirths, Coefficient characterizations and sections for some univalent functions, Sib. Math. J. 54(2013) 679–696.
[19] Ch. Pommerenke , Univalent functions, Vandenhoeck and Ruprecht, Göttingen, 1975.
[20] Y. A. Muhanna, L. Li.and S. Ponnusamy, Extremal problem on the class of convex functions of order $-1/2$, Archiv der Mathematik. 103(201???),461–471.
[21] J. K. Prajapat, Deepak Bansal and Sudhananda Maharana, Bounds for the third order Hankel determinant for certain classes of analytic functions (Communicated).
[22] H. Saitoh, M. Nunokawa, S. Fukui, S. Owa, A remark on close-to-convex and starlike functions, Bull. Soc. Roy. Sci. Liege 57(1988) 137–141.
[23] R. Singh, S. Singh, Some sufficient conditions for univalence and starlikeness, Collect. Math. 47(1982) 309–314.
[24] B. A. Uralegaddi, M. D. Ganigi, S. M. Sarangi, Univalent functions with positive coefficients, Tamkang J. Math. 25(1994) 225–230.

Deepak Bansal
Department of Mathematics, Govt. College of Enggineering and Technology, Bikaner-334004, Rajasthan, India
E-mail address: deepakbansal_79@yahoo.com

Janusz Sokół
Department of Mathematics, Rzeszów University of Technology, al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland
E-mail address: jsokol@prz.edu.pl