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DHAGE ITERATION METHOD FOR EXISTENCE AND APPROXIMATE SOLUTIONS OF NONLINEAR QUADRATIC FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper the authors prove an algorithm for the existence as well as approximation of the positive mild solutions of the initial value problems of nonlinear quadratic fractional differential equations using the operator theoretic technique in a partially ordered metric space. The main results rely on the Dhage iteration method embodied in a recent hybrid fixed point theorem of Dhage (2014) in a partially ordered normed linear space. The approximation of the mild solutions of the considered nonlinear quadratic fractional differential equation is obtained under weaker mixed partial Lipschitz and partial compactness type conditions. Our hypotheses and the result are also illustrated by a numerical example.

1. Introduction

The Dhage iteration principle or method (in short DIP or DIM) is relatively new to the literature on nonlinear analysis, particularly in the theory of nonlinear differential and integral equations, but it has been becoming more popular among the mathematicians all over the world because of its utility of applications to nonlinear equations for different qualitative aspects of the solutions. Very recently, the above method has been applied in Dhage [3, 5, 6, 7, 8], Dhage and Dhage [12, 13] and Dhage *et.al.* [14, 15] to nonlinear ordinary differential and fractional differential equations for proving the existence and algorithms of the solutions. Similarly, DIM has also some interesting applications in the theory of nonlinear fractional equations and in the present paper we prove the existence as well as algorithms for mild solutions of the initial value problems of quadratic fractional differential equations.

Before stating the main problem of this paper, we recall the following basic definitions of fractional calculus [17, 18] which are useful in what follows.

Definition 1.1. *If $J_\infty = [t_0, \infty)$ be an interval of the real line $\mathbb{R}$ for some $t_0 \in \mathbb{R}$ with $t_0 \geq 0$, then for any $x \in C(J_\infty, \mathbb{R})$, the Riemann-Liouville fractional integral*

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of order $q > 0$ is defined as

$$I^q x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} \frac{x(s)}{(t-s)^{1-q}}\, ds, \ t \in J_\infty,$$

provided the right hand side is pointwise defined on (t_0, ∞).

Definition 1.2. *The Riemann-Liouville fractional derivative of order $q > 0$, $n-1 < q < n$, $n \in \mathbb{N}$, on the interval J_∞ of $\mathbb{R}$ is defined as*

$$D^q_{t_0+} f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_{t_0}^{t} (t-s)^{n-q-1} f(s)\, ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$ on J_∞.

Definition 1.3. *If $x \in AC^n(J_\infty, \mathbb{R})$, then the Caputo derivative ${}^cD^q x$ of x of fractional order q is defined as*

$${}^cD^q x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} x^{(n)}(s)\, ds, \ n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q, and Γ is the Euler's gamma function. Here $AC^n(J_\infty, \mathbb{R})$ denote the space of real valued functions $x(t)$ which have continuous derivatives up to order $n-1$ on J_∞ such that $x^n(t) \in AC(J_\infty, \mathbb{R})$.

Definition 1.4. *The Caputo derivative of order q for a function $f : [t_0, \infty) \to \mathbb{R}$ can be written as*

$${}^cD^q f(t) = D^q \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(t_0) \right), \quad t > 0, \quad n-1 < q < n.$$

Remark 1.5. *If $f(t) \in AC^n(J_\infty, \mathbb{R})$ then*

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} \frac{f^{(n)}(s)}{(t-s)^{q+1-n}}\, ds = I^{n-q} f^{(n)}(t), \ t > 0, \ n-1 < q < n.$$

Lemma 1.6 (Podlubny [18])**.** *For $q > 0$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$ ($n = [q]+1$).

As a consequence of Lemma 1.6, we obtain the following result concerning the Riemann-Liouville integration of Caputo fractional derivative of a function.

Lemma 1.7 (Podlubny [18])**.** *For any $q > 0$ and $x \in C^{(n)}(J_\infty, \mathbb{R})$,*

$$I^q \ {}^cD^q x(t) = x(t) + c_0 + c_1 t + \ldots + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$ ($n = [q]+1$), where $AC^n(J_\infty, \mathbb{R})$ denotes the space of n-times absolute continuously differentiable real-valued functions on $J\infty$.

Given a closed and bounded interval $J = [t_0, t_0 + a]$ of the real line $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $t_0 \geq 0$ and $a > 0$, consider the initial value problem (in short IVP) of nonlinear fractional quadratic fractional differential equation (QFDE),

$$\left.\begin{aligned} {}^cD^q \left[\frac{x(t)}{f(t, x(t))}\right] &= g(t, x(t)), \ t \in J, \\ x(t_0) &= \alpha_0 \in \mathbb{R}_+, \end{aligned}\right\} \tag{1}$$

where ${}^cD^q$ is the Caputo derivative of fractional order q, $0 < q < 1$ and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are a continuous functions.

The nonlinear QFDE (1) is well-known in the literature and has been discussed at length for the existence and other characterizations of the solutions under compactness and Lipschitz conditions which are considered to be very strong in the theory of nonlinear differential and integral equations. In the present paper we prove the existence and approximation of the mild solutions of QFDE (1) under weaker partially compactness and partially Lipschitz type conditions via Dhage iteration method and also indicate some realizations.

The rest of the paper will be organized as follows. In Section 2 we give some preliminaries and a key fixed point theorem that will be used in subsequent part of the paper. In Section 3 we discuss the main existence and approximation result for initial value problems of quadratic fractional differential equations.

2. Auxiliary Results

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$ in which the addition and the scalar multiplication by positive real numbers are preserved by $\preceq$. A few details of such partially ordered linear spaces appear in Dhage [1] and the references therein.

Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [16], Zeidler [19] and the references therein.

We need the following definitions in the sequel.

Definition 2.1. *A mapping $\mathcal{B} : E \to E$ is called* **isotone** *or* **nondecreasing** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{B}x \preceq \mathcal{B}y$ for all $x, y \in E$.*

Definition 2.2 (Dhage [1])**.** *A mapping $\mathcal{B} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{B}x - \mathcal{B}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{B}$ called a partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{B}$ is a partially continuous on E, then it is continuous on every chain C contained in E.*

Definition 2.3. *A non-empty subset S of the partially ordered Banach space E is called partially bounded if every chain C in S is bounded. A mapping $\mathcal{B} : E \to E$ is called* **partially bounded** *if $T(E)$ is partially bounded subset of E. $\mathcal{B}$ is called* **uniformly partially bounded** *if all chains C in $\mathcal{B}(E)$ are bounded by a unique constant. $\mathcal{B}$ is called* **bounded** *if $T(E)$ is a bounded subset of E.*

Definition 2.4. *A non-empty subset S of the partially ordered Banach space E is called partially compact if every chain C in S is compact. A mapping $\mathcal{B} : E \to E$ is called* **partially compact** *if $\mathcal{B}(E)$ is a partially relatively compact subset of E. $\mathcal{B}$ is called* **uniformly partially compact** *if $\mathcal{B}(E)$ is a uniformly partially bounded and partially compact subset of E. $\mathcal{B}$ is called* **partially totally bounded** *if for* *any bounded subset S of E, $\mathcal{B}(S)$ is a partially relatively compact subset of E. If $\mathcal{B}$ is partially continuous and partially totally bounded, then it is called* **partially completely continuous** *on E.*

Definition 2.5 (Dhage [1])**.** *The order relation $\preceq$ and the metric d on a non-empty set E are said to be* **compatible** *if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* implies that the original sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible. A subset S of E is called* **Janhavi** *if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are compatible in it. In particular, if $S = E$, then E is called a* **Janhavi metric** *or* **Janhavi Banach space***.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property and so is a **Janhavi Banach space**.

Definition 2.6 (Dhage [1])**.** *A upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{B} : E \to E$ is called* ***partially nonlinear $\mathcal{D}$-Lipschitz*** *if there exists a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|\mathcal{B}x - \mathcal{B}y\| \leq \psi(\|x - y\|) \tag{2}$$

for all comparable elements $x, y \in E$. If $\psi(r) = k\,r$, $k > 0$, then $\mathcal{B}$ is called a partially Lipschitz with a Lipschitz constant k. $\mathcal{T} : E \to E$ is a partially nonlinear $\mathcal{D}$-contraction if $0 < \psi(r) < r$ for $r > 0$.

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \big\{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\big\}$$

and

$$\mathcal{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } u, v \in E^+\}. \tag{3}$$

The elements of the set $\mathcal{K}$ are called the positive vectors in E. Then following lemma is immediate.

Lemma 2.7 (Dhage [1])**.** *If $u_1, u_2, v_1, v_2 \in \mathcal{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$.*

Definition 2.8. *An operator $\mathcal{B} : E \to E$ is said to be positive if the range $R(\mathcal{B})$ of $\mathcal{B}$ is such that $R(\mathcal{B}) \subseteq \mathcal{K}$.*

The common assertion developed in the hybrid fixed point theorems of Dhage [2, 3, 4, 5] is known as **Dhage iteration principle** (in short DIP) which states that "***the sequence of successive approximations of a nonlinear equation beginning with a lower or an upper solution as its first or initial approximation converges monotonically to the solution.***" This aforesaid principle forms a basic and powerful tool in the study of numerical and constructive solutions for nonlinear differential and integral equations and called the **Dhage iteration**

method for nonlinear equations. See Dhage and Dhage [11, 12] and the references therein. The following applicable hybrid fixed point theorem of Dhage [2, 10] containing the DIP is used as a key tool for the work of this paper.

Theorem 2.9 (Dhage [2]). *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear algebra such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain C of E. Let $\mathcal{A}, \mathcal{B} : E \to \mathcal{K}$ be two nondecreasing operators such that*

(a) *$\mathcal{A}$ is partially bounded and partially nonlinear $\mathcal{D}$-Lipschitz with $\mathcal{D}$-function $\psi_{\mathcal{A}}$,*
(b) *$\mathcal{B}$ is partially continuous and uniformly partially compact,*
(c) *$0 < M\psi_{\mathcal{A}}(r) < r$, $r > 0$, where $M = \sup\{\|\mathcal{B}(C)\| : C$ is a chain in $E\}$, and*
(d) *there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{A}x_0\,\mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0\,\mathcal{B}x_0$.*

Then the operator equation $\mathcal{A}x\,\mathcal{B}x = x$ has a positive solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\,\mathcal{B}x_n$, $n = 0, 1, \ldots,$ converges monotonically to x^*.

Remark 2.10. We remark that hypothesis (a) of Theorem 2.9 implies that the operator $\mathcal{A}$ is partially continuous and consequently both the operators $\mathcal{A}$ and $\mathcal{B}$ in the theorem are partially continuous on E. The regularity of E in above Theorem 2.9 may be replaced with a stronger continuity condition of the operators $\mathcal{A}$ and $\mathcal{B}$. See Dhage [1, 2] and the references therein.

Remark 2.11. The compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$. This simple fact is used to prove the desired characterization of the positive solution of the QDE (1) on J.

3. Main Existence Result

The equivalent integral form of the QFDE (1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{4}$$

and

$$x \leq y \iff x(t) \leq y(t) \tag{5}$$

for all $t \in J$. Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular and a lattice so that every pair of elements of E has a lower and an upper bound in it. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ has some nice properties w.r.t. the above order relation in it. The following lemma follows by an application of Arzellá-Ascoli theorem.

Lemma 3.1. *Let $\big(C(J, \mathbb{R}), \leq, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (4) and (5) respectively. Then every partially compact subset S of $C(J, \mathbb{R})$ $\|\cdot\|$ is Janhavi, i.e., the order relation $\leq$ and the norm $\|\cdot\|$ are compatible in S.*

Proof. The proof of the lemma is given in Dhage and Dhage [12]. Since the proof is not well-known, we give the details of it. Let S be a partially compact subset of $C(J,\mathbb{R})$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t) \leq \cdots , \tag{$\mathcal{M}$}$$

for each $t \in J$.

Suppose that a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}_{k\in\mathbb{N}}$ of the monotone real sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent. By monotone characterization, the whole sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent and converges to a point $x(t)$ in $\mathbb{R}$ for each $t \in J$. This shows that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges x point-wise in S. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence by Arzelá-Ascoli theorem. Hence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges uniformly to x. As a result, $\|\cdot\|$ and $\leq$ are compatible in S and so S is a Janhavi set in E. This completes the proof. □

We need the following definition in what follows.

Definition 3.2. *A function $u \in C^1(J,\mathbb{R})$ is said to be a lower solution of the QFDE (1) if it satisfies*

$$\left.\begin{aligned} {}^cD^q\left[\frac{u(t)}{f(t,u(t))}\right] &\leq g(t,u(t)),\ t\in J,\\ u(t_0) &\leq \alpha_0. \end{aligned}\right\} \tag{$*$}$$

Similarly, an upper solution $v \in C^1(J,\mathbb{R})$ to the QFDE (1) *is defined on J, by the above inequalities with reverse sign.*

We consider the following set of assumptions in what follows:

- (A_0) The map $x \mapsto \dfrac{x}{f(t,x)}$ is injection for each $t \in J$.
- (A_1) f defines a function $f : J \times \mathbb{R} \to \mathbb{R}_+$.
- (A_2) There exists a constant $M_f > 0$ such that $0 < f(t,x) \leq M_f$ for all $t \in J$ and $x \in \mathbb{R}$.
- (A_3) There exists a $\mathcal{D}$-function φ, such that
$$0 \leq f(t,x) - f(t,y) \leq \varphi(x-y),$$
for all $t \in J$ and $x, y \in \mathbb{R}$, $x \geq y$. Moreover,
$$\left(\left|\frac{\alpha_0}{f(t_0,\alpha_0)}\right| + \frac{a^q\, M_g}{\Gamma(q+1)}\right)\varphi(r) < r,\ r > 0.$$
- (B_1) g defines a function $g : J \times \mathbb{R} \to \mathbb{R}_+$.
- (B_2) There exists a constant $M_g > 0$ such that $0 < g(t,x) \leq M_g$ for all $t \in J$ and $x \in \mathbb{R}$.
- (B_3) $g(t,x)$ is nondecreasing in x for all $t \in J$.
- (B_4) The QDE (1) has a lower solution $u \in C(J,\mathbb{R})$.

Remark 3.3. *Notice that Hypothesis (A_0) holds in particular if the function $x \mapsto \dfrac{x}{f(t,x)}$ is increasing for each $t \in J$.*

The following lemma is useful and follows from the results given in Kilbas *et.al.* [17] and Podlubny [18].

Lemma 3.4. *Assume that the hypothesis* (A_0) *holds. If a function* $x \in C^1(J, \mathbb{R})$ *is a solution of the QFDE*

$$\left.\begin{aligned} {}^cD^q\left[\frac{x(t)}{f(t,x(t))}\right] &= g(t,x(t)),\ t \in J,\ 0<q<1, \\ x(t_0) &= \alpha_0, \end{aligned}\right\} \tag{6}$$

then it is a solution of the nonlinear quadratic fractional integral equation (QFIE),

$$x(t) = \big[f(t,x(t))\big]\left(\frac{\alpha_0}{f(t_0,\alpha_0)} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\,g(s,x(s))\,ds\right),\ t \in J. \tag{7}$$

Proof. Assume first that $x \in C^1(J, \mathbb{R})$ is a solution to the QFDE (1) defined on J. By Lemma 1.6, we have

$$\frac{x(t)}{f(t,x(t))} = \int_{t_0}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s))ds + c_0, \tag{8}$$

where $c_0 \in \mathbb{R}$. Since $x(t_0) = \alpha_0, f(t_0,\alpha_0) \neq 0$, it follows $c_0 = \dfrac{\alpha_0}{f(t_0,\alpha_0)}$. Thus (7) holds. □

Definition 3.5. *A function* $x \in C^1(J, \mathbb{R})$ *that satisfies the QFIE (8) is called a mild solution of the QFDE (1) defined on* J.

Theorem 3.6. *Assume that the hypotheses* (A_0) *through* (A_3) *and* (B_1) *through* (B_4) *hold. Then the QFDE (1) has a positive mild solution* x^* *defined on* J *and the sequence* $\{x_n\}_{n=1}^{\infty}$ *of successive approximations defined by*

$$x_{n+1}(t) = \big[f(t,x_n(t))\big]\left(\frac{\alpha_0}{f(t_0,\alpha_0)} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,x_n(s))\,ds\right), \tag{9}$$

for all $t \in \mathbb{R}$*, where* $x_1 = u$*, converges monotonically to* x^*.

Proof. By Lemma 3.4, every solution $x \in C^1(J, \mathbb{R})$ of the QFDE (1) satisfies the nonlinear quadratic fractional integral equation

$$x(t) = \big[f(t,x(t))\big]\left(\frac{\alpha_0}{f(t_0,\alpha_0)} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s))\,ds\right),\ t \in J. \tag{10}$$

Set $E = C(J, \mathbb{R})$. Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Define the operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = f(t,x(t)),\ t \in J, \tag{11}$$

and

$$\mathcal{B}x(t) = \frac{\alpha_0}{f(t_0,\alpha_0)} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s))\,ds,\ t \in J. \tag{12}$$

From the continuity of the integrals, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the maps $\mathcal{A}, \mathcal{B} : E \to E$. Then, the QFDE (1) is equivalent to the operator equation

$$\mathcal{A}x(t)\mathcal{B}x(t) = x(t),\ \ t \in J. \tag{13}$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.9. This is achieved in the series of following steps.

Step I: *$\mathcal{A}$ and $\mathcal{B}$ are nondecreasing on E.*

Let $x, y \in E$ be such that $x \geq y$. Then by hypothesis (A$_3$), we obtain

$$\mathcal{A}x(t) = f(t, x(t)) \geq f(t, y(t)) = \mathcal{A}y(t),$$

for all $t \in J$. This shows that $\mathcal{A}$ is nondecreasing operator on E into E. Similarly, we have

$$\begin{aligned} \mathcal{B}x(t) &= \frac{\alpha_0}{f(t_0,\alpha_0)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s, x(s))\, ds \\ &\geq \frac{\alpha_0}{f(t_0,\alpha_0)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s, y(s))\, ds \\ &= \mathcal{B}y(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{B}$ is nondecreasing operator on E into E.

Step II: *$\mathcal{A}$ is partially bounded and partially $\mathcal{D}$-Lipschitz on E.*

Let $x \in E$ be arbitrary. Then by (A$_2$),

$$|\mathcal{A}x(t)| \leq \big|f(t, x(t))\big| \leq M_f,$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{A}x\| \leq M_f$ and so, $\mathcal{A}$ is bounded. This further implies that $\mathcal{A}$ is partially bounded on E.

Next, let $x, y \in E$ be such that $x \geq y$. Then,

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = \big|f(t, x(t)) - f(t, y(t))\big| \leq \varphi(|x(t) - y(t)|) \leq \varphi(\|x - y\|),$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{A}x - \mathcal{A}y\| \leq \varphi(\|x - y\|)$, for all $x, y \in E$. Hence $\mathcal{A}$ is a partially $\mathcal{D}$-Lipschitz on E which further implies that $\mathcal{A}$ is a partially continuous on E.

Step III: *$\mathcal{B}$ is a partially continuous on E.*

Let $\{x_n\}$ be a sequence of points of a chain C in E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \left[\frac{\alpha_0}{f(t_0,\alpha_0)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s, x_n(s))\, ds \right] \\ &= \frac{\alpha_0}{f(t_0,\alpha_0)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \left[\lim_{n\to\infty} g(s, x_n(s))\, ds \right] ds \\ &= \frac{\alpha_0}{f(t_0,\alpha_0)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s, x(s))\, ds \\ &= \mathcal{B}x(t), \end{aligned}$$

for all $t \in J$. This shows that $\{\mathcal{B}x_n\}$ converges to $\mathcal{B}x$ pointwise on J.

Next, we will show that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned} &|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \\ &\qquad \leq \left| \frac{1}{\Gamma(q)} \int_{t_0}^{t_2} (t_2 - s)^{q-1} g(s, x_n(s))\, ds \right. \end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{\Gamma(q)} \int_{t_0}^{t_2} (t_1 - s)^{q-1} g(s, x_n(s))\, ds \Big| \\
&+ \Big| \frac{1}{\Gamma(q)} \int_{t_0}^{t_2} (t_1 - s)^{q-1} g(s, x_n(s))\, ds \\
&\quad - \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} (t_1 - s)^{q-1} g(s, x_n(s))\, ds \Big| \\
&\leq \frac{1}{\Gamma(q)} \left| \int_{t_0}^{t_2} |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}|\, |g(s, x_n(s))|\, ds \right| \\
&+ \frac{1}{\Gamma(q)} \left| \int_{t_1}^{t_2} (t_1 - s)^{q-1} |g(s, x_n(s))|\, ds \right| \\
&\leq \frac{M_g}{\Gamma(q)} \left| \int_{t_0}^{t_2} |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}|\, ds \right| \\
&+ \frac{M_g}{\Gamma(q)} \left| \int_{t_1}^{t_2} (t_1 - s)^{q-1}|\, ds \right|.
\end{aligned}
$$

Consequently,

$$
|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1
$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniformly and hence $\mathcal{B}$ is a partially continuous on E.

Step IV: *$\mathcal{B}$ is a partially compact on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $x \in C$ be arbitrary. Then,

$$
\begin{aligned}
|\mathcal{B}x(t)| &\leq \left| \frac{\alpha_0}{f(t_0, \alpha_0)} \right| + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} |g(s, x(s))|\, ds \\
&\leq \left| \frac{\alpha_0}{f(t_0, \alpha_0)} \right| + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} |g(s, x(s))|\, ds \\
&\leq \left| \frac{\alpha_0}{f(t_0, \alpha_0)} \right| + \frac{a^q\, M_g}{\Gamma(q+1)} \\
&= r,
\end{aligned}
$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{B}x\| \leq r$ for all $x \in C$. Hence $\mathcal{B}(C)$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$
\begin{aligned}
&|\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \\
&\quad \leq \left| \frac{1}{\Gamma(q)} \int_{t_0}^{t_2} (t_2 - s)^{q-1} g(s, x(s))\, ds - \frac{1}{\Gamma(q)} \int_{t_0}^{t_2} (t_1 - s)^{q-1} g(s, x(s))\, ds \right| \\
&\quad + \left| \frac{1}{\Gamma(q)} \int_{t_0}^{t_2} (t_1 - s)^{q-1} g(s, x(s))\, ds - \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} (t_1 - s)^{q-1} g(s, x(s))\, ds \right| \\
&\quad \leq \frac{1}{\Gamma(q)} \left| \int_{t_0}^{t_2} |(t_2 - s)^{q-1} - (t_1 - s)^{q-1}|\, |g(s, x(s))|\, ds \right| \\
&\quad + \frac{1}{\Gamma(q)} \left| \int_{t_1}^{t_2} (t_1 - s)^{q-1} |g(s, x(s))|\, ds \right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{M_g}{\Gamma(q)} \left| \int_{t_0}^{t_2} |(t_2-s)^{q-1} - (t_1-s)^{q-1}| \, ds \right| \\
&\quad + \frac{M_f}{\Gamma(q)} \left| \int_{t_1}^{t_2} (t_1-s)^{q-1}| \, ds \right| .
\end{aligned}
$$

Thus, we have that

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1$$

uniformly for all $x \in C$. This shows that $\mathcal{B}(C)$ is an equicontinuous set in E. Hence $\mathcal{B}(C)$ is compact subset of E and consequently $\mathcal{B}$ is a partially compact operator on E into itself.

Step V: *$\mathcal{D}$-function $\psi_{\mathcal{A}}$ satisfies the growth condition* $0 < M\psi_{\mathcal{A}}(r) < r$, $r > 0$.

From the estimate given in Step IV, it follows that

$$0 < M\psi_{\mathcal{A}}(r) \le \left(\left| \frac{\alpha_0}{f(t_0,\alpha_0)} \right| + \frac{a^q\, M_g}{\Gamma(q+1)} \right) \varphi(r) < r$$

for all $r > 0$.

Step VI: *u satisfies the operator inequality $u \le \mathcal{A}u\,\mathcal{B}u$.*

Since the hypothesis (B_4) holds, u is a lower solution of the QFDE (1) defined on J. Then,

$$ {}^cD^q \left[\frac{u(t)}{f(t,u(t))} \right] \le g(t,u(t)), \tag{14}$$

satisfying,

$$u(t_0) \le \alpha_0, \tag{15}$$

for all $t \in J$.

Operating I^q on both sides of (14), we obtain

$$u(t) \le \left[f(t,u(t)) \right] \left(\frac{\alpha_0}{f(t_0,\alpha_0)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s,u(s))\, ds \right), \tag{16}$$

for all $t \in J$. This show that u is a lower solution of the operator equation $x = \mathcal{A}x\,\mathcal{B}x$.

Thus the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.9 in view of Remark 2.11 and we apply it to conclude that the operator equation $\mathcal{A}x\,\mathcal{B}x = x$ has a positive mild solution defined on J. Consequently the integral equation and the QFDE (1) has a positive mild solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by (9) converges monotonically to x^*. This completes the proof. □

Remark 3.7. The conclusion of Theorem 3.6 also remains true if we replace the hypothesis (B_4) with the following one:

(B_4') The QFDE (1) has an upper solution $v \in C^1(J,\mathbb{R})$.

Example 3.8. *Given a closed and bounded interval $J = [0,1]$, consider the FDE,*

$$
\left.
\begin{aligned}
{}^cD^{1/2} \left[\frac{x(t)}{f(t,x(t))} \right] &= \frac{1 + \tanh x(t)}{12}, \ t \in J, \\
x(0) &= \frac{1}{3}.
\end{aligned}
\right\} \tag{17}
$$

where,

$$f(t,x)=\begin{cases} 1, & \text{if } x\le 0 \\ 1+\dfrac{x}{1+x}, & \text{if } x>0 \end{cases}$$

Clearly, the functions f is continuous and nonnegative on $J\times\mathbb{R}$ with bound $M_f=2$. It is easy to prove that the mapping

$$x\mapsto \frac{x}{f(t,x)}=\begin{cases} x, & \text{if } x\le 0, \\ \dfrac{x}{1+\frac{x}{1+x}} & \text{if } x>0, \end{cases}$$

is increasing for each $t\in J$ and so the hypothesis (A_0) is satisfied. Furthermore, the function f is $\mathcal{D}$-Lipschitz with $\mathcal{D}$-function $\psi(r)=r$, since we have

$$0\le f(t,x)-f(t,y)\le x-y$$

for $x\ge y$. Similarly, the function g is continuous and nonnegative on $J\times\mathbb{R}$ with bound $M_g=\dfrac{1}{6}$. Moreover, the function $g(t,x)=\dfrac{1+\tanh x}{12}$ is nondecreasing in x for each $t\in J$.

Finally, the QFDE (17) has a lower solution u defined by

$$u(t)=\frac{4}{15}-\frac{2}{\Gamma(1/2)}\int_0^t (t-s)^{-1/2}\,ds=\frac{4}{15}+\frac{2}{\sqrt{\pi}}t^{1/2}$$

for all $t\in J$. Further we notice that $\left|\dfrac{\alpha_0}{f(t_0,\alpha_0)}\right|+\dfrac{a^q\,M_g}{\Gamma(q+1)}=\dfrac{4}{15}+\dfrac{1}{3\sqrt{\pi}}<1$. Thus all the hypotheses of Theorem 3.6 are satisfied. Hence we conclude that the QFDE (17) has a positive mild solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by

$$x_{n+1}(t)=\big[f(t,x_n(t))\big]\left(\frac{4}{15}+\frac{1}{\sqrt{\pi}}\int_0^t (t-s)^{1/2}\left[\frac{1+\tanh x_n(s)}{12}\right]ds\right),$$

for all $t\in J$, where $\alpha_0=u$, converges monotonically to x^*.

Remark 3.9. In this paper we have proved an existence result for the mild solutions the QFDE (1) defined on J. However, other aspects of the mild solutions of the QFDE (1) such as the existence of minimal and maximal mild solutions and comparison theorems could also be proved using the same Dhage iteration method with appropriate modifications. See Dhage [7] and the references for the details. Furthermore, if the QFDE (1) has a lower solution u and an upper solution v such that $u\le v$, then the corresponding mild solutions x_* and x^* of the QFDE (1) satisfy $x_*\le x^*$ and are the minimal and maximal mild solutions in the vector segment $[u,v]$ of the Banach space $E=C(J,\mathbb{R})$, because the order relation $\le$ defined by (4) is equivalent to the order relation defined by the order cone

$$\mathcal{K}=\big\{x\in C(J,\mathbb{R})\mid x(t)\ge 0 \quad\text{for all}\quad t\in J\big\}, \tag{18}$$

which is a closed set in $C(J,\mathbb{R})$.

4. Conclusion

From the foregoing discussion it is clear that Dhage iteration method forms an interesting powerful method for discussing the existence results of certain nonlinear hybrid quadratic fractional differential equations. However, it has some limitations that unlike Picard's method, the new method does not give the rate of convergence of the sequence of successive approximations. Notwithstanding, we have been able prove the numerical mild solution of the considered nonlinear fractional differential equation. Finally, while concluding this paper we mention that the quadratic fractional differential equation considered here is of very simple nature for which we have illustrated the Dhage iteration method to obtain the algorithms for the mild solutions under weaker partially Lipschitz and compactness conditions. However, an analogous study could also be made for other complex quadratic fractional differential equations using similar method with appropriate modifications. Some of the results along this line will be reported elsewhere.

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