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ON MILD SOLUTIONS TO CAPUTO TYPE DIFFERENTIAL EQUATIONS OF ARBITRARY ORDER WITH NONLOCAL MULTI-POINT-STRIP CONDITIONS

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Abstract. We investigate the existence of mild solutions for one-dimensional nonlinear Caputo type differential equations of arbitrary order supplemented with nonlocal multipoint-strip conditions involving first-order derivative of the unknown function. The nonlocal multipoint-strip condition relates the linear combination of nonlocal values of the first-order derivative of the unknown function with its value on a strip of arbitrary size. Existence and uniqueness results for the given problem are obtained via appropriate fixed point theorems. Some examples illustrating the main results are also presented. Finally we discuss an analog Stieltjes multipoint-strip conditions case.

1. Introduction

In the recent years, the research on fractional differential equations (supplemented with a variety of initial and boundary conditions) has picked up a great momentum and the subject has been extensively developed from theoretical point of view. It has been mainly due to widespread applications of fractional calculus modelling techniques in several disciplines of applied and technical sciences. Examples include viscoelasticity, control theory, biological sciences, ecology, aerodynamics, electro-dynamics of complex medium, environmental issues, etc. For more details, we refer the reader to the works ([1]-[7]). A salient feature of fractional-order differential and integral operators is their nonlocal nature that helps to trace the past history of several materials and processes.

Fractional-order boundary value problems involving classical, nonlocal, multipoint, periodic/anti-periodic, fractional-order, and integral boundary conditions have recently been investigated by many researchers.

Nonlocal conditions, dated back to the works [8, 9, 10], are found to be more practical (than the classical initial/boundary conditions) to describe some peculiarities of physical, chemical or other processes happening inside the domain.

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Computational fluid dynamics (CFD) studies of blood flow are directly related to the boundary data and it is not always justified to assume a circular cross-section. An effective approach for handling this issue is to apply integral boundary conditions [11]. Also, integral boundary conditions are used to regularize ill-posed parabolic backward problems in time partial differential equations, see for example, mathematical models for bacterial self-regularization [12].

For some recent works on boundary value problems of fractional-order, we refer the reader to ([13]-[26]) and the references cited therein.

In this paper, we consider a new class of boundary value problems of Caputo type fractional differential equations of arbitrary order involving a nonlocal sub-strip condition given by

$$\begin{cases} {}^cD^q x(t) = f(t,x),\ n-1<q\le n,\ n\ge 2\ ,t\in[0,1], \\ x(0)=\delta x(\sigma), x'(0)=x''(0)=...=x^{(n-2)}(0)=0,\ \delta\in\mathbb{R}, \\ ax'(\zeta_1)+bx'(\zeta_2)=c\displaystyle\int_\eta^\xi x'(s)ds,\ 0<\sigma<\zeta_1<\eta<\xi<\zeta_2<1, \end{cases} \tag{1}$$

where ${}^cD^q$ denote Caputo derivative of order q and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function. The integral boundary condition in (1) implies that the linear combination of the values of the first-order derivative of the unknown function at nonlocal positions ζ_1 and ζ_2 (off the strip) is proportional to its strip contribution occupying the position (η,ξ). This work is motivated by a recent paper [17] and addresses a new situation which has practical applications in geophysics and acoustics. In the rest of the paper, by a solution of problem (1), we mean a mild solution.

The paper is organized as follows. In Section 2, we recall some definitions and establish an auxiliary lemma for the linear variant of problem (1). In Section 3, we present our main existence results. We emphasize that the tools of fixed point theory employed in this section are the standard ones; however their exposition provides a deep insight in terms of the existence criteria for solutions of the problem at hand. Section 4 is devoted to the study of Stieltjes type strip conditions.

2. Preliminaries

This section is devoted to some preliminary concepts of fractional calculus that we need in the forthcoming analysis [3, 5].

Definition 2.1. *The fractional integral of order r with the lower limit zero for a function f is defined as*

$$I^r f(t)=\frac{1}{\Gamma(r)}\int_0^t\frac{f(s)}{(t-s)^{1-r}}ds,\quad t>0,\quad r>0,$$

provided the right hand-side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(r)=\int_0^\infty t^{r-1}e^{-t}dt$.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $r>0,\ n-1<r<n,\ n\in N$, is defined as*

$$D^r_{0+}f(t)=\frac{1}{\Gamma(n-r)}\left(\frac{d}{dt}\right)^n\int_0^t(t-s)^{n-r-1}f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.3. *The Caputo derivative of order r for a function $f : [0,\infty) \to R$ can be written as*

$$^cD^r f(t) = D^r \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \quad n-1 < r < n.$$

Remark 2.4. *If $f(t) \in C^n[0,\infty)$, then*

$$^cD^r f(t) = \frac{1}{\Gamma(n-r)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{r+1-n}} ds = I^{n-r} f^{(n)}(t),\ t > 0,\ n-1 < q < n.$$

Lemma 2.5. [3] *Let $u \in C^m[0,1]$ and $v \in AC[0,1]$. Then, for $\rho \in (m-1,m), m \in \mathbb{N}$ and $t \in [0,1]$,*

- **(a):** *the general solution of the fractional differential equation $^cD^\rho u(t) = 0$ is $u(t) = b_0 + b_1 t + b_2 t^2 + ... + b_{m-1} t^{m-1}$, where $b_i \in \mathbb{R},\ i = 0,1,2,...,m-1$.*
- **(b):** $I^\rho\ {}^cD^\rho u(t) = u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u^k(0)$.
- **(c):** $^cD^\rho I^\rho v(t) = v(t)$.

Now we present an auxiliary lemma to define the solution for the problem (1).

Lemma 2.6. *Let $y \in C[0,1]$. Then the mild solution of the linear fractional differential equation*

$$^cD^q x(t) = y(t),\ \ n-1 < q \le n,\ n \ge 2\ , t \in [0,1], \tag{2}$$

supplemented with boundary conditions given in (1) is given by the integral equation

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds + \frac{\delta}{1-\delta} \int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)} y(s)ds \\ &+ \frac{1}{A}\Big(\frac{\delta\sigma^{n-1}}{1-\delta} + t^{n-1}\Big)\Big(c\int_\eta^\xi \int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} y(u)duds \\ &- a\int_0^{\zeta_1} \frac{(\zeta_1-s)^{q-2}}{\Gamma(q-1)} y(s)ds - b\int_0^{\zeta_2} \frac{(\zeta_2-s)^{q-2}}{\Gamma(q-1)} y(s)ds\Big), \end{aligned} \tag{3}$$

where

$$A = (n-1)\Big[a\zeta_1^{n-2} + b\zeta_2^{n-2} - \frac{c}{n-1}(\xi^{n-1} - \eta^{n-1})\Big] \ne 0. \tag{4}$$

Proof. By Lemma 2.5 (b), the solution of fractional differential equation (2) can be written as

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds + c_0 + c_1 t + c_2 t^2 + ... + c_{n-2}t^{n-2} + c_{n-1}t^{n-1}, \tag{5}$$

where $c_0, c_1, ..., c_{n-1} \in \mathbb{R}$ are arbitrary constants. Using the conditions: $x'(0) = x''(0) = ... = x^{(n-2)}(0) = 0$, we find that $c_1 = c_2 = ... = c_{n-2} = 0$. Thus, (5) takes the form:

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds + c_0 + c_{n-1}t^{n-1}, \tag{6}$$

which together with the conditions: $x(0) = \delta x(\sigma)$ and $ax'(\zeta_1) + bx'(\zeta_2) = c\int_\eta^\xi x'(s)ds$ yields

$$c_0 = \frac{1}{1-\delta}\Big[\delta\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)} y(s)ds + \frac{\delta\sigma^{n-1}}{A}\Big(c\int_\eta^\xi \int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} y(u)duds$$

$$- \quad a\int_0^{\zeta_1} \frac{(\zeta_1-s)^{q-2}}{\Gamma(q-1)}y(s)ds - b\int_0^{\zeta_2} \frac{(\zeta_2-s)^{q-2}}{\Gamma(q-1)}y(s)ds\Big)\Big]$$

and

$$\begin{aligned} c_{n-1} &= \frac{1}{A}\Big[c\int_\eta^\xi \int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)}y(u)duds - a\int_0^{\zeta_1} \frac{(\zeta_1-s)^{q-2}}{\Gamma(q-1)}y(s)ds \\ &- \quad b\int_0^{\zeta_2} \frac{(\zeta_2-s)^{q-2}}{\Gamma(q-1)}y(s)ds\Big], \end{aligned}$$

where A is given by (4). Substituting the values of c_0 and c_{n-1} in (6) completes the solution (3).

3. Existence results

Let $\mathcal{P} = C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from [0,1] to $\mathbb{R}$ endowed with the norm :$\|x\| = \sup\{|x(t)|, t \in [0,1]\}$.

In the rest of the paper, by a solution of problem (1), we mean a mild solution.

In Lemma 2.6, we replace $y(t)$ by $f(t,x(t))$ and define an operator $\mathcal{H}$: $\mathcal{P} \longrightarrow \mathcal{P}$ associated with problem (1) as follows:

$$\begin{aligned} (\mathcal{H}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds + \frac{\delta}{1-\delta}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds \\ &+ \quad \frac{1}{A}\Big(\frac{\delta\sigma^{n-1}}{1-\delta}+t^{n-1}\Big)\Big(c\int_\eta^\xi \int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)}f(u,x(u))duds \\ &- \quad a\int_0^{\zeta_1} \frac{(\zeta_1-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds - b\int_0^{\zeta_2} \frac{(\zeta_2-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds\Big). \end{aligned} \tag{7}$$

Observe that the problem (1) has solutions if and only if the operator $\mathcal{H}$ has fixed points.

For the sake of computational convenience, we set

$$\beta = \frac{1}{\Gamma(q+1)} + \frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)} + \Big(\frac{|\delta|\sigma^{n-1}}{|A(1-\delta)|}+\frac{1}{|A|}\Big)\Big(|c|\frac{(\xi^q-\eta^q)}{\Gamma(q+1)}+|a|\frac{\zeta_1^{q-1}}{\Gamma(q)}+|b|\frac{\zeta_2^{q-1}}{\Gamma(q)}\Big). \tag{8}$$

Now we are in a position to present the main results of our paper. The first one dealing with the existence and uniqueness of solutions for problem (1) is based on Banach's contraction mapping principle.

Theorem 3.1. *Let $f : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying the Lipschitz condition:*

(A_1)**:** $|f(t,x)-f(t,y)| \le \ell|x-y|$, $\ell > 0$, $\forall t \in [0,1], x, y \in \mathbb{R}$.

Then the problem (1) has a unique solution if $\ell\beta < 1$, where β is given by (8).

Proof. In the first step, we show that the operator $\mathcal{H}$ defined by (7) satisfies the relation: $\mathcal{H}B_r \subset B_r$, where $B_r = \{x \in \mathcal{P} : \|x\| \le r\}$, $r > \beta\alpha/(1-$

$\beta\ell)$, $\sup_{t\in[0,1]}|f(t,0)|=\alpha$. For $x\in B_r,\ t\in[0,1]$, it follows by Lipschitz condition that

$$\begin{aligned}|f(t,x(t))| &= |f(t,x(t))-f(t,0)+f(t,0)|\\ &\le |f(t,x(t))-f(t,0)|+|f(t,0)|\le \ell\|x\|+\alpha\le \ell r+\alpha.\end{aligned}$$

Then we have

$$\begin{aligned}\|(\mathcal{H}x)\| &\le \sup_{t\in[0,1]}\Big\{\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\\ &+ \frac{|\delta|}{|1-\delta|}\int_0^\sigma\frac{(\sigma-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\\ &+ \frac{1}{|A|}\Big(\frac{|\delta|\sigma^{n-1}}{|1-\delta|}+t^{n-1}\Big)\Big(|c|\int_\eta^\xi\int_0^s\frac{(s-u)^{q-2}}{\Gamma(q-1)}|f(u,x(u))|duds\\ &+ |a|\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds\\ &+ |b|\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))|ds\Big)\Big\}\\ &\le (\ell r+\alpha)\sup_{t\in[0,1]}\Big\{\frac{t^q}{\Gamma(q+1)}+\frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)}\\ &+ \Big(\frac{|\delta|\sigma^{n-1}}{|A(1-\delta)|}+\frac{t^{n-1}}{|A|}\Big)\Big(|c|\frac{(\xi^q-\eta^q)}{\Gamma(q+1)}+|a|\frac{\zeta_1^{q-1}}{\Gamma(q)}+|b|\frac{\zeta_2^{q-1}}{\Gamma(q)}\Big)\Big\}\\ &\le (\ell r+\alpha)\beta\le r,\end{aligned}$$

where we have used (8). This shows that $\mathcal{H}B_r\subset B_r$.
Now, for $x,y\in\mathbb{R}$ and for each $t\in[0,1]$, we obtain

$$\begin{aligned}&\|\mathcal{H}x-\mathcal{H}y\|\\ \le\ & \sup_{t\in[0,1]}\Big\{\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds\\ +\ & \frac{|\delta|}{|1-\delta|}\int_0^\sigma\frac{(\sigma-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds\\ +\ & \frac{1}{|A|}\Big(\frac{|\delta|\sigma^{n-1}}{|1-\delta|}+t^{n-1}\Big)\Big(|c|\int_\eta^\xi\int_0^s\frac{(s-u)^{q-2}}{\Gamma(q-1)}|f(u,x(u))-f(u,y(u))|duds\\ +\ & |a|\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))-f(s,y(s))|ds\\ +\ & |b|\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s))-f(s,y(s))|ds\Big)\Big\}\\ \le\ & \ell\|x-y\|\sup_{t\in[0,1]}\Big\{\frac{t^q}{\Gamma(q+1)}+\frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)}\\ +\ & \Big(\frac{\delta\sigma^{n-1}}{|A(1-\delta)|}+\frac{t^{n-1}}{|A|}\Big)\Big(|c|\frac{(\xi^q-\eta^q)}{\Gamma(q+1)}+|a|\frac{\zeta_1^{q-1}}{\Gamma(q)}+|b|\frac{\zeta_2^{q-1}}{\Gamma(q)}\Big)\Big\}\\ \le\ & \ell\beta\|x-y\|.\end{aligned}$$

Since $\ell\beta < 1$(given), the operator $\mathcal{H}$ is a contraction. Thus, by Banach's contraction mapping principle, there exists a unique fixed point for the operator $\mathcal{H}$ which corresponds to the unique solution for the problem (1). This completes the proof.

Our next existence result is based on Krasnoselskii's fixed point theorem [27].

Lemma 3.2. *(Krasnoselskii) Let $\mathcal{S}$ be a closed, convex, bounded and nonempty subset of a Banach space X. Let $\mathcal{G}_1, \mathcal{G}_2$ be the operators such that (i) $\mathcal{G}_1 u + \mathcal{G}_2 v \in \mathcal{S}$ whenever $u, v \in \mathcal{S}$; (ii) $\mathcal{G}_1$ is compact and continuous; and (iii) $\mathcal{G}_2$ is a contraction. Then there exists $w \in \mathcal{S}$ such that $w = \mathcal{G}_1 w + \mathcal{G}_2 w$.*

Theorem 3.3. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (A_1). In addition it is assumed that $|f(t,x)| \le \mu(t)$, $\forall (t,x) \in [0,1]\times\mathbb{R}$, and $\mu \in C([0,1], \mathbb{R}^+)$. Then the problem (1) has at least one solution on $[0,1]$ if $\ell\gamma < 1$, where*

$$\gamma = \frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)} + \Big(\frac{|\delta|\sigma^{n-1}}{|A(1-\delta)|} + \frac{1}{|A|}\Big)\Big(|c|\frac{(\xi^q - \eta^q)}{\Gamma(q+1)} + |a|\frac{\zeta_1^{q-1}}{\Gamma(q)} + |b|\frac{\zeta_2^{q-1}}{\Gamma(q)}\Big). \quad (9)$$

Proof. Let us consider a set $\mathcal{B}_\nu = \{x \in \mathcal{P} : \|x\| \le \nu\}$ with $\nu \ge \beta\|\mu\|$ ($\sup_{t\in[0,1]} |\mu(t)| = \|\mu\|$) and define the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ on $\mathcal{B}_\nu$ as

$$\begin{aligned}
(\mathcal{H}_1 x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds,\\
(\mathcal{H}_2 x)(t) &= \frac{\delta}{1-\delta}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds\\
&+ \frac{1}{A}\Big(\frac{\delta\sigma^{n-1}}{1-\delta} + t^{n-1}\Big)\Big(c\int_\eta^\xi\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} f(u,x(u))duds\\
&- a\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds - b\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds\Big).
\end{aligned}$$

For $x, y \in \mathcal{B}_\nu$, it is easy to show that $\|(\mathcal{H}_1 x) + (\mathcal{H}_2 x)\| \le \|\mu\|\beta \le \nu$ (β is given by (8)), which means that $\mathcal{H}_1 x + \mathcal{H}_2 y \in \mathcal{B}_\nu$.

Using (A_1) and (9), for $x, y \in \mathbb{R}, t \in [0,1]$, we obtain

$$\begin{aligned}
&\|(\mathcal{H}_2 x) - (\mathcal{H}_2 y)\|\\
\le\ & \sup_{t\in[0,1]}\Big\{\frac{|\delta|}{|1-\delta|}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)}|f(s,x(s)) - f(s,y(s))|ds\\
+\ & \frac{1}{|A|}\Big(\frac{|\delta|\sigma^{n-1}}{|1-\delta|} + t^{n-1}\Big)\Big(|c|\int_\eta^\xi\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)}|f(u,x(u)) - f(u,y(u))|duds\\
+\ & |a|\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s)) - f(s,y(s))|ds\\
+\ & |b|\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-2}}{\Gamma(q-1)}|f(s,x(s)) - f(s,y(s))|ds\Big)\Big\}\\
&\le \ell\|x-y\|\sup_{t\in[0,1]}\Big\{\frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)}\\
+\ & \Big(\frac{|\delta|\sigma^{n-1}}{|A(1-\delta)|} + \frac{t^{n-1}}{|A|}\Big)\Big(|c|\frac{(\xi^q-\eta^q)}{\Gamma(q+1)} + |a|\frac{\zeta_1^{q-1}}{\Gamma(q)} + |b|\frac{\zeta_2^{q-1}}{\Gamma(q)}\Big)\Big\}
\end{aligned}$$

$$\leq \ell\gamma\|x-y\|.$$

This shows that $\mathcal{H}_2$ a contraction in view of the condition $\ell\gamma < 1$.

Continuity of f implies that the operator $\mathcal{H}_1$ is continuous. Also, $\mathcal{H}_1$ is uniformly bounded on $\mathcal{B}_\nu$ as

$$\begin{aligned}\|\mathcal{H}_1 x\| &\leq \sup_{t\in[0,1]}\Big\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds\Big\} \leq \sup_{t\in[0,1]}\Big\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\mu(s)ds\Big\}\\ &\leq \|\mu\|\sup_{t\in[0,1]}\Big\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}ds\Big\} \leq \frac{\|\mu\|}{\Gamma(q+1)}.\end{aligned}$$

Moreover, with $\sup_{(t,x)\in[0,1]\times\mathcal{B}_\nu}|f(t,x)| = \overline{f} < \infty$ and $0 < t_1 < t_2 < 1$, we have

$$\begin{aligned}&\Big|(\mathcal{H}_1 x)(t_2) - (\mathcal{H}_1 x)(t_1)\Big|\\ =&\ \Big|\int_0^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds - \int_0^{t_1}\frac{(t_1-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds\Big|\\ =&\ \Big|\int_0^{t_1}\frac{[(t_2-s)^{q-1}-(t_1-s)^{q-1}]}{\Gamma(q)}f(s,x(s))ds + \int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds\Big|\\ \leq&\ \frac{\overline{f}}{\Gamma(q+1)}\Big(2|t_2-t_1|^q + |t_2^q - t_1^q|\Big),\end{aligned}$$

which tends to zero independent of x as $(t_2 - t_1) \to 0$. This implies that $\mathcal{H}_1$ is relatively compact on $\mathcal{B}_\nu$. Hence by the $Arzelá - Ascoli$ theorem, $\mathcal{H}_1$ is compact on $\mathcal{B}_\nu$. Thus the hypothesis of Krasonselskii's fixed theorem is satisfied and consequently the problem (1) has at least one solution on $[0, 1]$. This completes the proof.

Our next result relies on the following fixed point theorem [27].

Theorem 3.4. *Let X be a Banach space. Assume that $T : X \longrightarrow X$ is a completely continuous operator and the set $V = \{u \in X | u = \epsilon T u, 0 < \epsilon < 1\}$ is bounded. Then T has a fixed point in X.*

Theorem 3.5. *Assume that exists a positive constant L_1 such that $|f(t,x)| \leq L_1$ for all $t \in [0,1]$, $x \in \mathbb{R}$. Then there exists at least one solution for the problem* (1) *on* $[0,1]$.

Proof. In the first step, we show that the operator $\mathcal{H}$ is completely continuous. Clearly continuity of $\mathcal{H}$ follows from the continuity of f and it is easy to establish by the given assumption that $|(\mathcal{H}x)(t)| \leq L_1\beta = L_2$, where β is given by (8). Let $0 < t_1 < t_2 < 1$, we get

$$\begin{aligned}&|(\mathcal{H}x)(t_2) - (\mathcal{H}x)(t_1)|\\ =&\ \Big|\int_0^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds - \int_0^{t_1}\frac{(t_1-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds\\ +&\ \frac{(t_2^{n-1}-t_1^{n-1})}{A}\Big[c\int_\eta^\xi\int_0^s\frac{(s-u)^{q-2}}{\Gamma(q-1)}f(u,x(u))duds\\ -&\ a\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds - b\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds\Big]\Big|\end{aligned}$$

$$
\begin{aligned}
&\leq \quad L_1\Big[\frac{2|t_2-t_1|^q+|t_2^q-t_1^q|}{\Gamma(q+1)}+\frac{|t_2^{(n-1)}-t_1^{(n-1)}|}{|A|}\Big(|c|\frac{(\xi^q-\eta^q)}{\Gamma(q+1)}\\
&+ \quad |a|\frac{\zeta_1^{q-1}}{\Gamma(q)}+|b|\frac{\zeta_2^{q-1}}{\Gamma(q)}\Big)\Big].
\end{aligned}
$$

Clearly, the right-hand side tends to zero independently of $x \in B_\rho$ as $t_2 \longrightarrow t_1$. Thus, by the Arzelá theorem, the operator $\mathcal{H}$ is completely continuous.
Next, we consider the set $V = \{x \in \mathcal{P} : \epsilon\mathcal{H}x,\ 0 < \epsilon < 1\}$. To show that V is bounded, let $x \in [0,1]$. Then

$$
\begin{aligned}
x(t) \quad &= \quad \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds+\frac{\delta}{1-\delta}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)}f(s,x(s))ds\\
&+ \quad \frac{1}{A}\Big(\frac{\delta\sigma^{n-1}}{1-\delta}+t^{n-1}\Big)\Big(c\int_\eta^\xi\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)}f(u,x(u))duds\\
&- \quad a\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds-b\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-2}}{\Gamma(q-1)}f(s,x(s))ds\Big).
\end{aligned}
$$

As before, it can be shown that $|x(t)| = \epsilon|(\mathcal{H}x)(t)| \leq L_1\beta = L_2$. Hence, $\|x\| \leq L_2, \forall x \in V, t \in [0,1]$. So V is bounded. Thus, the conclusion of Theorem 3.4 applies and the problem (1) has at least one solution on $[0,1]$. This completes the proof.

Lemma 3.6. *(Nonlinear alternative for single valued maps [28])*
Let E be a Banach space E_1 a closed, convex subset of E, V an open subset of E_1, and $0 \in V$. Suppose that $\mathcal{U} : \overline{V} \longrightarrow E_1$ is a continuous, compact (that is, $\mathcal{U}(\overline{V})$ is a relatively compact subset ofE_1) map. Then either

- **(i):** *$\mathcal{U}$ has a fixed point in $\overline{V}$, or*
- **(ii):** *there is a $x \in \partial V$ (the boundary of V in E_1) and $\kappa \in (0,1)$ with $x = \kappa$ $\mathcal{U}(x)$.*

Theorem 3.7. *Let $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Further, it is assumed that*

- (A_2)**:** *there exist a function $p \in \mathcal{C}([0,1], \mathbb{R}^+)$and a nondecreasing function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $|f(t,x)| \leq p(t)\psi(\|x\|),\ \forall (t,x) \in [0,1] \times \mathbb{R}$;*
- (A_3)**:** *there exists a constant $M > 0$ such that*

$$
\begin{aligned}
& \quad M\Big\{\psi(M)\|p\|\Big[\frac{1}{\Gamma(q+1)}+\frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)}\\
&+ \quad \Big(\frac{|\delta|\sigma^{n-1}}{|A(1-\delta|)}+\frac{1}{|A|}\Big)\Big(|c|\frac{(\xi^q-\eta^q)}{\Gamma(q+1)}+|a|\frac{\zeta_1^{q-1}}{\Gamma(q)}+|b|\frac{\zeta_2^{q-1}}{\Gamma(q)}\Big)\Big]\Big\}^{-1} > 1.
\end{aligned}
$$

Then the problem (1) has at least one solution on $[0,1]$.

Proof. Let us consider the operator $\mathcal{H} : \mathcal{P} \longrightarrow \mathcal{P}$ defined by(7) and show that $\mathcal{H}$ maps bounded sets into bounded sets in $\mathcal{P}$. For a given positive number ρ, let $B_\rho = \{x \in \mathcal{P} : \|x\| \leq \rho\}$ be a bounded set in $\mathcal{P}$. Then, for $x \in B_\rho$ together with (A_2), we obtain

$$
|(\mathcal{H}x)(t)| \quad \leq \quad \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}p(s)\psi(\|x\|)ds+\frac{|\delta|}{|1-\delta|}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)}p(s)\psi(\|x\|)ds
$$

$$
\begin{aligned}
&+ \quad \frac{1}{|A|}\Big(\frac{|\delta|\sigma^{n-1}}{|1-\delta|}+t^{n-1}\Big)\Big(|c|\int_\eta^\xi\int_0^s\frac{(s-u)^{q-2}}{\Gamma(q-1)}p(u)\psi(\|x\|)duds \\
&+ \quad |a|\int_0^{\zeta_1}\frac{(\zeta_1-s)^{q-2}}{\Gamma(q-1)}p(s)\psi(\|x\|)ds+|b|\int_0^{\zeta_2}\frac{(\zeta_2-s)^{q-2}}{\Gamma(q-1)}p(s)\psi(\|x\|)ds\Big) \\
&\leq \quad \psi(\rho)\|p\|\Big[\frac{1}{\Gamma(q+1)}+\frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)} \\
&+ \quad \Big(\frac{|\delta|\sigma^{n-1}}{|A(1-\delta)|}+\frac{1}{|A|}\Big)\Big(|c|\frac{(\xi^q-\eta^q)}{\Gamma(q+1)}+|a|\frac{\zeta_1^{q-1}}{\Gamma(q)}+|b|\frac{\zeta_2^{q-1}}{\Gamma(q)}\Big)\Big],
\end{aligned}
$$

where A is given by (4). As in the proof of the previous result, for $0<t_1<t_2<1$ and $x\in B_\rho$. we have that the operator $\mathcal{H}$ is completely continuous. Thus, it follows that $\mathcal{H}$ maps bounded sets into equicontinuous sets of $\mathcal{P}$.
Let x be a solution for the given problem. Then, for $\lambda\in(0,1)$, as before, we obtain

$$
\begin{aligned}
|x(t)| \quad &= \quad |\lambda(\mathcal{H}x)(t)|\leq\psi(\|x\|)\|p\|\Big[\frac{1}{\Gamma(q+1)}+\frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)} \\
&+ \quad \Big(\frac{|\delta|\sigma^{n-1}}{|A(1-\delta)|}+\frac{1}{|A|}\Big)\Big(|c|\frac{(\xi^q-\eta^q)}{\Gamma(q+1)}+|a|\frac{\zeta_1^{q-1}}{\Gamma(q)}+|b|\frac{\zeta_2^{q-1}}{\Gamma(q)}\Big)\Big].
\end{aligned}
$$

which, on taking the norm for $t\in[0,1]$, yields

$$
\begin{aligned}
&\|x\|\Big\{\psi(\|x\|)\|p\|\Big[\frac{1}{\Gamma(q+1)}+\frac{|\delta|\sigma^q}{(1-\delta)\Gamma(q+1)} \\
&+ \quad \Big(\frac{1}{|A|}+\frac{|\delta|\sigma^{n-1}}{|A(1-\delta)|}\Big)\Big(|c|\frac{(\xi^q-\eta^q)}{\Gamma(q+1)}+|a|\frac{\zeta_1^{q-1}}{\Gamma(q)}+|b|\frac{\zeta_2^{q-1}}{\Gamma(q)}\Big)\Big]\Big\}^{-1}\leq 1.
\end{aligned}
$$

In view of (A_3), there exists M such that $\|x\|\neq M$. Let us choose $M_1=\{x\in\mathcal{P}:\|x\|<M+1\}$. Since the operator $\mathcal{H}:\overline{M}_1\to\mathcal{P}$ is continuous and completely continuous. From the choice of M_1, there is no $x\in\partial M_1$ such that $x=\lambda\mathcal{H}(x)$ for some $\lambda\in(0,1)$. Consequently, by Lemma 3.6, we deduce that the operator $\mathcal{H}$ has a fixed point $x\in\overline{M}_1$ which is a solution of the problem (1). This completes the proof.

Example 3.8. *Consider a fractional boundary value problem given by*

$$
\begin{cases}
{}^cD^qx(t)=f(t,x),\ 5<q\leq 6, t\in[0,1], \\
x(0)=\delta x(\sigma), x'(0)=x''(0)=x'''(0)=x^{(4)}(0)=0, \\
ax'(\zeta_1)+bx'(\zeta_2)=c\displaystyle\int_\eta^\xi x'(s)ds,\ 0<\sigma<\zeta_1<\eta<\xi<\zeta_2<1.
\end{cases}
\tag{10}
$$

Here $q=11/2,\delta=1/2,a=5,b=3,c=1,\sigma=1/6,\zeta_1=1/3,\zeta_2=3/4,\eta=1/2,\xi=2/3$, and $f(t,x)=3e^{-t}x+6t^2\sin x+\cos 3(t+1)$. With the given data, we find that $\ell=9,|A|\approx 4.954298$ and $\beta\approx 6.84021\times 10^{-3}$, where β is given by (8). Obviously all the conditions of Theorem 3.1 are satisfied with $\ell\beta<1$. Therefore, by the conclusion of Theorem 3.1, there exists a unique solution for the problem (10) on $[0,1]$.

Example 3.9. *Consider the problem* (10) *with*

$$
f(t,x)=2t^2+\frac{8}{3}t^2\sin x \tag{11}
$$

Clearly $|f(t,x)| \leq p(t)\psi(|x|)$ with $p(t) = 2t^2$, $\psi(|x|) = 1+\frac{4}{3}x$. By the assumption (A_3) of Theorem 3.7, we find that $M > 0.0139346$. Thus, by Theorem 3.7, there exists at least one solution for the problem (10) with $f(t,x)$ given by (11).

4. Stieltjes type multi-point-strip conditions

In this section, we consider a Caputo type fractional differential equation

$$ {}^cD^q x(t) = f(t,x), \quad n-1 < q \leq n,\ n \geq 2\ , t \in [0,1], \tag{12}$$

supplemented with Stieltjes type multi-point-strip conditions

$$\begin{cases} x(0) = \delta x(\sigma), x'(0) = x''(0) = ... = x^{(n-2)}(0) = 0, \\ ax'(\zeta_1) + bx'(\zeta_2) = c\displaystyle\int_\eta^\xi x'(s)d\varphi(s), 0 < \sigma < \zeta_1 < \eta < \xi < \zeta_2 < 1, \end{cases} \tag{13}$$

where $\varphi(s)$ is a function of bounded variation.
Relative to the problem (12)-(13), we have an operator $\mathcal{H}_s$: $\mathcal{P} \longrightarrow \mathcal{P}$ defined by

$$\begin{aligned} (\mathcal{H}_s x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \frac{\delta}{1-\delta}\int_0^\sigma \frac{(\sigma-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ &+ \frac{1}{A_s}\Big(\frac{\delta\sigma^{n-1}}{1-\delta} + t^{n-1}\Big)\Big(c\int_\eta^\xi\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} f(u,x(u))dud\varphi(s) \\ &- a\int_0^{\zeta_1} \frac{(\zeta_1-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds - b\int_0^{\zeta_2} \frac{(\zeta_2-s)^{q-2}}{\Gamma(q-1)} f(s,x(s))ds\Big), \end{aligned} \tag{14}$$

where

$$A_s = (n-1)\Big[a\zeta_1^{n-2} + b\zeta_2^{n-2} - c\int_\eta^\xi s^{n-2}d\varphi(s)\Big] \neq 0 \tag{15}$$

and

$$\begin{aligned} \beta_s &= \frac{1}{\Gamma(q+1)} + \frac{|\delta|\sigma^q}{|1-\delta|\Gamma(q+1)} \\ &+ \frac{1}{|A_s|}\Big(\frac{|\delta|\sigma^{n-1}}{|1-\delta|} + 1\Big)\Big(|c|\int_\eta^\xi\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} dud\varphi(s) + |a|\frac{\zeta_1^{q-1}}{\Gamma(q)} + |b|\frac{\zeta_2^{q-1}}{\Gamma(q)}\Big). \end{aligned} \tag{16}$$

With the help of the operator $\mathcal{H}_s$ and the constant β_s, we can establish the existence results for the problem (12)-(13) similar to the ones obtained in the previous section.

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