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APPROXIMATE SOLUTION OF THE FRACTIONAL SUSCEPTIBLE-INFECTED-RECOVERED MODEL BY MODIFIED VARIATIONAL ITERATION METHOD

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Abstract. The article presents the solution of the fractional Susceptible-Infected-Recovered model by modified variational iteration method. By using initial values, the explicit solutions of fractional SIR model for different values of fractional derivatives have been deduced. Modified variational iteration method performs extremely well in terms of efficiency that accelerates rapid convergence of series solution and also overcome the problem of crucial way of determining the Lagrange multipliers that arises in classical VIM method. Main feature of the present paper is to demonstrate the efficiency of modified variation method and exhibit system behavior for different fractional orders that are depicted graphically.

1. Introduction

Since fractional order differential equations are related to systems with memory that occurs in most biological systems and also related to fractals that are abundant in these systems. Fractional differential equations enable us to describes all the previous history of $f(t)$ while calculating time fractional derivatives $f(t)$ at time $t = t_1$ and also reduces the errors arising from the neglected parameters in real life modeling. The membranes of cells of biological organism must have fractional order conductance and then classified in non- integer model further fractional order derivatives enables to describe hereditary properties of process involved and embody essential feature of cell rheological behavior. Hence fractional models are more approximate than their integer order form. We investigate an approximate solution of fractional Susceptible-Infected-Recovered model via modified variational iteration method. In this paper we will adopt fractional derivatives in Caputo's sense due to its advantage over Riemann-Liouville definition to dealing initial value problems.

A novel modification of the variational iteration method was proposed by Guo-Cheng Wu and Dumitru Baleanu [1] through Laplace transform by using Lagrange multipliers. The Lagrange multiplier methods have been widely used to solve a

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number of nonlinear fractional differential equations which arises in mathematical physics and other related areas. The variational iteration method was developed by He [2, 3] to solve nonlinear problems. The method has been applied to initial boundary value problems [4]-[9], fractal initial problems [10, 11]. Further effect of fractional order on fractional order chaotic and hyper-chaotic systems discussed by Hegazi, E Ahmed, AE Matouk [12], and analytical and numerical solutions of multi-term nonlinear fractional orders differential equations obtained by EI-Sayed, IL El-Kalla, EAA Ziada [13], Wu, GC studied q-difference equations [14]. Due to flexibility, reliability and efficiency of the VIM method it has become popular amongst researches. This algorithm makes use of Lagrange multiplier and generally following three steps; establishing of correlation functional; identify the Lagrange multipliers; determining initial iteration. The application of the method use Lagrange multiplier in ordinary differential equation which yields poor convergence. Modified VIM method has overcome this problem and defined Lagrange multiplier from Laplace transform and can easily be apply to fractional differential equations with initial value problems.

In the present paper this modified VIM method is applied to solve fractional Susceptible-Infected-Recovered model. The SIR model was introduced by W.O. Kermack [15] which has played significant role in mathematical epidemiology to compute the amount of susceptible, infected, recovered people in population.

2. Fractional SIR Model

Definition 2.1. The Riemann-Liouville fractional integral operator [1] of order $\alpha > 0$ of function $f : R^+ \to R$ is defined as

$$I^\alpha f(\mathrm{x}) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt$$

Definition 2.2. The Caputo fractional derivative [1] of order$\alpha > 0$,$n-1 < \alpha \leq n, n \in \mathbb{N}$ is defined as

$$D^\alpha f(x) = I^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt$$

where $f(t)$ has absolute continuous derivatives up to order $(n-1)$.

The basic SIR model works under following assumption [16].

(a)The population is fixed.

(b)The only way a person can leave the susceptible case is to become infected and the only way to leave infected group is to recover from the disease.

(c)Age, sex, social status and race do not affect the probability of being infected.

(d)There is no inherited immunity.

Fractional model start with some basic notations:

$S(t)$ denote the number of susceptible people at time t,

$I(t)$ denote the number of infected people at time t,

$R(t)$ denote the number of recovered people at time t,

$$S(t) + I(t) + R(t) = N$$

where N is the total population size .

The assumptions lead us to introduce fractional order SIR model as follows

$$D_t^{\alpha_1} x(t) = -\beta \ x(t) \ \ y(t) \tag{1}$$

$$D_t^{\alpha_2} y(t) = \beta \ x(t) \ \ y(t) - k \ y(t) \tag{2}$$

$$D_t^{\alpha_3} z(t) = k \ y(t) \tag{3}$$

where $x(t), y(t)$ and $z(t)$ represents $S(t), I(t)$ and $R(t)$ respectively with $0 \leq \alpha_i \leq 1$ for $i = 1, 2, 3$;
k is the recovery rate (with greater or equal to zero), β is the average number of transmission from an infected person in a time t (with greater or equal to zero). Fractional differential equations have gained much attention due to provide exact solutions of various nonlinear phenomena.

3. Modified Variational Iterational Method

Let's consider the following nonlinear differential equation [1], in order to better understanding of MVIM method.

$$\frac{d^m u}{dt^m} + R_1(u) + N_1(u) = f(t) \tag{4}$$

with the initial conditions

$$u^{(k)}(0) = u_0^k \tag{5}$$

for $k = 0, 1, \ldots.., m-1$ where $u = u(t)$, R_1 is a linear bounded operator, N_1 is a nonlinear bounded operator, $f(t)$ is known continuous function.

Now take Laplace transform of equation (4), to construct correction functional is

$$U_{n+1}(s)$$

$$= U_n(s) + \ \lambda(s)(s^m U_n(s) - s^{m-1} u(0) - \cdots - u^{m-1}(0) + L[R_1(u_n) + N_1(u_n) - f(t) \tag{6}$$

Regarding the terms $L[R_1(u_n) + N_1(u_n) - f(t)]$ as restricted variations to make (6) stationary with respect to U_n.

$$\delta U_{n+1}(s) = \delta U_n(s) + \ \lambda(s)(s^m \delta U_n(s)) \tag{7}$$

From (7) we calculate Lagrange multiplier as

$$\lambda(s) = -\frac{1}{s^m}$$

The successive approximations can be obtained by taking inverse Laplace transform of equation (6)

$$u_{n+1}(t) = u_n(t)$$

$$-L^{-1}\{\frac{1}{s^m}(s^m U_n(s) - s^{m-1} u(0) - \cdots - u^{m-1}(0) + L[R_1(u_n) + N_1(u_n) - f(t)])\}$$

$$= L^{-1}(\frac{u(0)}{s} + \ldots \cdots + \frac{u^{(m-1)}(0)}{s^m}) - L^{-1}[\frac{1}{s^m}(L[R_1(u_n) + N_1(u_n) - f(t)])] \tag{8}$$

with initial conditions

$$\mathrm{u}_0\,(t) = L^{-1}(\tfrac{u(0)}{s} + \ldots\ldots + \tfrac{u^{(m-1)}(0)}{s^m}) = \mathrm{u}\,(0) + u^{'}(0)\,\mathrm{t} + \ldots + u^{(m-1)}\,(0)\,\tfrac{t^{m-1}}{(m-1)!} \quad (9)$$

4. Solution of Fractional SIR Model by MVIM Method

In this section, the approximate analytical solution of Fractional SIR system (equations (1) to (3)) is obtained through powerful mathematical tool modified VIM method with initial conditions $x(0) = \delta$, $y(0) = \gamma$, $z(0) = \mu$. The numerical calculations of FSIR for different fractional order time derivatives are carried out, which are depicted graphically.

According to MVIM method the correction function for equation (1) has constructed as follows

$$X_{n+1}(s) = X_n\,(s) + \lambda(s)(s^{\alpha_1} X_n\,(s) - s^{\alpha_1 - 1} x\,(0) + L[\beta\ x_n\,(t)\,y_n\,(t)]) \quad (10)$$

for $\lambda\,(s) = -\frac{1}{s^{\alpha_1}}$ and taking Laplace inverse

$$x_{n+1}(t) = L^{-1}(s^{-1} x\,(0) - \frac{1}{s^{\alpha_1}} L[\beta\ x_n\,(t)\,y_n\,(t)]) \quad (11)$$

Case I: when $n = 0$, equation (11) gives first iteration

$$x_1(t) = L^{-1}(s^{-1}\delta - \frac{1}{s^{\alpha_1}} L[\beta\delta\gamma])$$

$$x_1(t) = \delta - (\beta\delta\gamma)\frac{t^{\alpha_1}}{\Gamma\,(\alpha_1 + 1)} \quad (12)$$

Similarly the correction function for equation (2) can be given as

$$Y_{n+1}(s) = Y_n\,(s) + \lambda(s)(s^{\alpha_2} Y_n\,(s) - s^{\alpha_2 - 1} y\,(0) - L[\beta\ x_n\,(t)\,y_n\,(t) - k y_n\,(t)]). \quad (13)$$

for $\lambda\,(s) = -\frac{1}{s^{\alpha_2}}$, n = 0 and taking inverse Laplace to equation (13) we get

$$y_1(t) = L^{-1}(s^{-1}\gamma + \frac{1}{s^{\alpha_2}} L[\beta\delta\gamma - k\gamma])$$

$$y_1(t) = \gamma + L^{-1}(\frac{1}{s^{\alpha_2}} L[\beta\delta\gamma - k\gamma])$$

$$y_1(t) = \gamma + [\beta\delta\gamma - k\gamma]\ \frac{t^{\alpha_2}}{\Gamma\,(\alpha_2 + 1)} \quad (14)$$

Further

$$Z_{n+1}(s) = Z_n\,(s) + \lambda(s)(s^{\alpha_3} Z_n\,(s) - s^{\alpha_3 - 1} z\,(0) - L[k y_n\,(t)]) \quad (15)$$

For $\lambda\,(s) = -\frac{1}{s^{\alpha_3}}$, we get first iteration for $z(t)$ on similar lines as follows

$$z_1(t) = \mu + (k\gamma\ )\frac{t^{\alpha_3}}{\Gamma\,(\alpha_3 + 1)} \quad (16)$$

Case II: when $n = 1$, we can obtain next components for $x(t), y(t), z(t)$ as

$$x_2(t) = L^{-1}(s^{-1}\delta - \frac{1}{s^{\alpha_1}} L[\beta x_1(t)\, y_1(t)])$$

$$x_2(t) = L^{-1}(s^{-1}\delta - \frac{\beta}{s^{\alpha_1}} L[(\delta - (\beta\delta\gamma)\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)})\ (\gamma + [\beta\delta\gamma - k\gamma]\ \frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)})]\ )$$

$$x_2(t) = \delta - \beta\delta\gamma\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)} - \beta\delta\left[\beta\delta\gamma - k\gamma\right]\frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}$$
$$+\beta^2\delta\gamma^2\frac{t^{2\alpha_1}}{\Gamma(2\alpha_1+1)} + \beta^2\delta\gamma\left[\beta\delta\gamma - k\gamma\right]\frac{\Gamma(\alpha_1+\alpha_2+1)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(2\alpha_1+\alpha_2+1)}t^{2\alpha_1+\alpha_2} \tag{17}$$

Similarly

$$y_2(t) = \gamma + \beta\delta\gamma\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} + \beta\delta\left[\beta\delta\gamma - k\gamma\right]\frac{t^{2\alpha_2}}{\Gamma(2\alpha_2+1)} - \beta^2\gamma^2\delta\frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}$$
$$-\beta^2\delta\gamma\left[\beta\delta\gamma - k\gamma\right]\frac{\Gamma(\alpha_1+\alpha_2+1)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(2\alpha_2+\alpha_1+1)}t^{2\alpha_2+\alpha_1}$$
$$-\ k\ \gamma\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} - k\left[\beta\delta\gamma - k\gamma\right]\frac{t^{2\alpha_2}}{\Gamma(2\alpha_2+1)} \tag{18}$$

and

$$z_2(t) = \mu + k\gamma\frac{t^{\alpha_3}}{\Gamma(\alpha_3+1)} + k\left[\beta\delta\gamma - k\gamma\right]\frac{t^{\alpha_3+\alpha_2}}{\Gamma(\alpha_3+\alpha_2+1)} \tag{19}$$

In similar manner, rest of components can be obtained and $x_n(t)$, $y_n(t), z_n(t)$ rapidly converges to exact solution as $n \to \infty$, rapidly converges means that only few terms are required to get approximate solutions.

5. Numerical Results and Discussion

In this section numerical results of fractional SIR model for different values of $\alpha_1, \alpha_2, \alpha_3$ and for the standard case of $\alpha_1 = \alpha_2 = \alpha_3 = 1$ calculated for various values of time t.

Case I: with initial condition

$x(0) = \delta = 990, y(0) = \gamma = 10, z(0) = \mu = 0, \alpha_1 = \alpha_2 = \alpha_3 = 1$ and for parameter $k = 1, \beta = 0.003$, the equation (17) to (19) gives

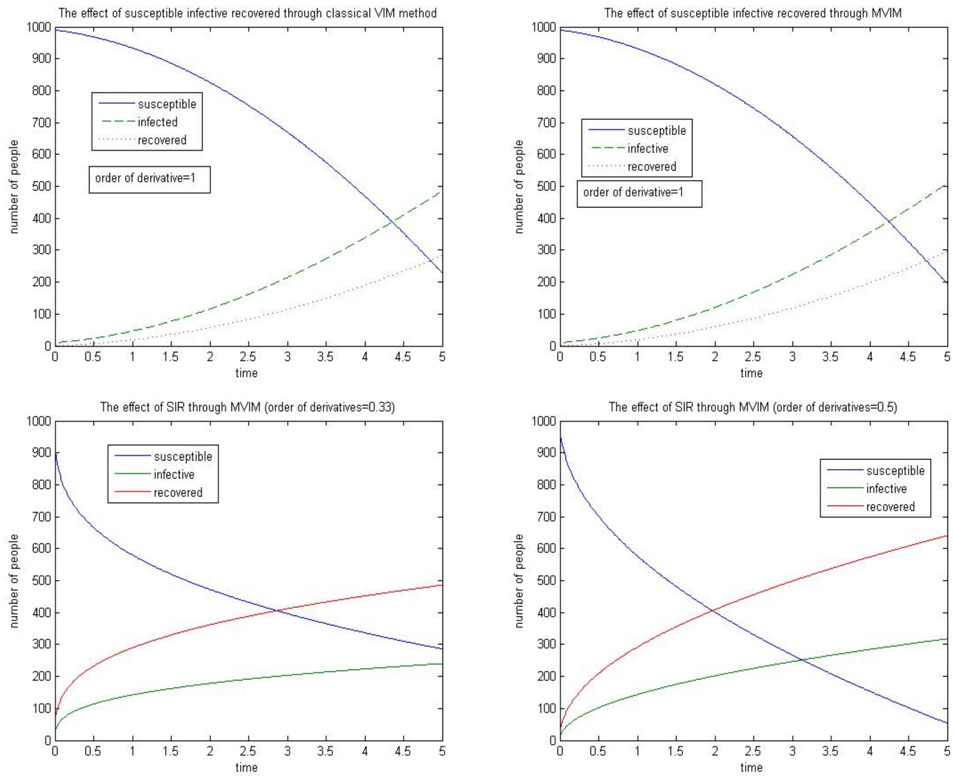
$$x_2(t) = 990 - 29.7t - 28.809\ t^2 + 0.58509t^3 \tag{20}$$

$$y_2(t) = 10 + 19.70\ t + 18.959\ t^2 - 0.58509\ t^3 \tag{21}$$

$$z_2(t) = 10\ t + 9.85\ t^2 \tag{22}$$

It can be observed that above results of the fractional SIR model for the classical integer order one are in complete agreement with the results obtained by F.S. Akinboro et al. [17] through basic VIM and differential transform method as shown graphically by Fig.1 and Fig.2 below. Better results obtained merely in second iteration clearly indicate that the present method is more powerful tool to solve nonlinear fractional differential equations.

Further the results for $\alpha_1 = \alpha_2 = \alpha_3 = 0.33$; $\alpha_1 = \alpha_2 = \alpha_3 = 0.5$with the parameters $x(0) = \delta = 1000, y(0) = \gamma = 2.5, z(0) = \mu = 0, \mathrm{k} = 1, \beta = 0.15$, are shown in Fig.3 and Fig.4 respectively.



(Figure 1, 2 above: comparison of SIR between classical VIM and MVIM and Figure 3, 4 below: effect of SIR for different fractional order derivatives)

6. Conclusion

A literature survey on SIR type model reveals that a considerable amount of work has been done. However the concept of solving Fractional SIR model through modified VIM method is original one. The proposed method gives the analytical approximate solution of Fractional SIR model in which the Lagrange's multiplier can be identified easily and new iteration can be deduced without using linearization, discretization or critical assumptions and requires less computational work compared to existing analytical methods. This method gives better realistic series solutions which converge rapidly and results obtained just at second iteration are in excellent agreement with results given by different authors. The results reveal that solution continuously depends on time fractional derivatives and valid for long time in integer case. The population of the susceptible people decreases with time while the recovered group population increases due to inclusion of vaccinated susceptible group. The above graph shows that recovery rate is very fast for fractional order in comparison of integer order and population will generally remain disease free with all the time and endemic equilibrium remain stable. There is vast scope of further study of this kind of problems with higher number of population that will certainly motivate researchers who are working in field of fractional models.

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