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SOLUTION OF THE FRACTIONAL EPIDEMIC MODEL BY L-ADM

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Abstract. In this paper the fractional order epidemic model of a non-fatal disease in a population which is assumed to have a constant size over the period of the epidemic is considered. Laplace-adomian decomposition method (for short L-ADM) is used to compute an analytical solution of the system of nonlinear fractional differential equations governing the problem. The results are compared with the results obtained by the classical Runge-Kutta method in the case of integer-order derivatives.

1. Introduction

A simple deterministic model predicting the behavior of epidemic outbreaks was formulated by A. G. McKendrick and W. O. Kermack in 1927 (see [1]). In their mathematical epidemic model, called the Susceptible-Infected-Recovered (which this is called an SIR model, or the xyz model), to describe the spread of diseases, McKendrick and Kermack proposed the following nonlinear system of ordinary differential equations [1]

$$\begin{cases} \frac{dx(t)}{dt} = -\beta x(t)y(t), \\ \frac{dy(t)}{dt} = \beta x(t)y(t) - \gamma y(t), \\ \frac{dz(t)}{dt} = \gamma y(t). \end{cases} \tag{1.1}$$

In this model the fixed population consists of three types where, at time t, $x(t)$ is the number of susceptible individuals, $y(t)$ is the number of infected individuals, able to spread the disease by contact with susceptible, $z(t)$ is the number of isolated individuals, who cannot get or transmit the disease for various reasons. Moreover, the constant β and γ give the transition rates between compartments. The transition rate between x (Susceptible) and y (infected) is βy, where β is the contact rate, which takes into account the probability of getting the disease in a contact between a susceptible and an infectious subject [2]. The transition rate

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between y (Infected) and z (recovered), is γ, which has the meaning of the rate of recovery or death. Since β and γ are interpreted as transition rates (probabilities), their range is $0 \leq \beta \leq 1$ and $0 \leq \gamma \leq 1$. The problem (1.1) was solved by Biazar [3] using adomian decomposition method (ADM), Rafei et al. [4] using homotopy perturbation method (HPM), Rafei et al. [5] by variation iteration method (VIM), Fadi et al. [6] using homotopy analysis method (HAM) and Abdul-Monim et al. [7] by the differential transformation method (DTM).

The fractional order extension of this model have been first studied in [9, 11]. The reason of using fractional differential quations (FDEs) is that FDEs are naturally related to systems with memory which exists in most biological system. Also they show the realistic biphasic decline behavior of infection of diseases but at a slower rate. The new system is described by the following system of fractional differential quations (FDEs).

$$\begin{cases} D^{\alpha_1}x(t) = -\beta x(t)y(t), \\ D^{\alpha_2}y(t) = \beta x(t)y(t) - \gamma y(t), \text{ where } \alpha_1, \alpha_2, \alpha_3 > 0 \\ D^{\alpha_3}z(t) = \gamma y(t), \end{cases} \tag{1.2}$$

subject to the initial conditions

$$x(0) = N_1, \quad y(0) = N_2, \quad z(0) = N_3. \tag{1.3}$$

For this model the initial conditions are not independent, since they must satisfy the condition $N_1 + N_2 + N_3 = N$, where N is the total fixed number of the individuals in the given population.

The motivation of this paper is to find analytical solution for general class of fraction order model of non-fatal epidemic by using the L-ADM .

2. Preliminary

Here, we present some necessary definitions and notations related to fractional calculus (see e.g. [10]). The most commonly used definitions are Riemann-Liouville and Caputo.

Definition 2.1. The Riemann-Liouville fractional integeration of order α is defined as:

$$\begin{aligned} (J^{\alpha}_{t_0}f)(t) &= \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s)ds, \ \alpha > 0, \ t > t_0, \\ (J^{0}_{t_0}f)(t) &= f(t). \end{aligned}$$

The Riemann-Liouville derivative has certain disadvantages such that the fractional derivative of a constant is not zero. Therefore, we will make use of Caputo's definition owing to its convenience for initial conditions of the fractional differential equations.

Definition 2.2. Riemann-Liouville and Caputo fractional derivatives of order α can be defined respectively as:

$$\begin{aligned} D^{\alpha}f(t) &= D^{n}(J^{n-\alpha}f(t)), \\ D^{\alpha}_{*}f(t) &= J^{n-\alpha}(D^{n}f(t)), \end{aligned}$$

where $n-1<\alpha\leq n,\ n\in\mathbb{N}$, f is a given function, and $\Gamma(\cdot)$ denotes the gamma function. It is known that $(D^{\alpha}_{t_0}f)(t)\to f'(t)$ as $\alpha\to 1$.

Now, we recall the definitions of Laplace transform of Caputo's derivative and Mittag-Leffler function in two arguments.

Definition 2.3.

$$\mathcal{L}\{D^{\alpha}f(t),s\} \quad = \quad s^{\alpha}F(s)-\sum_{i=0}^{n-1}s^{\alpha-i-1}f^{(i)}(0),\ (n-1<\alpha\leq n);\ n\in\mathbb{N}.$$

3. The Laplace-Adomian Decomposition Method (L-ADM)

Consider the fractional-order epidemic model (1.2) subject to the initial condition (1.3). The nonlinear term in this model Eqs. (1.2) is xy and β,γ are known costants. For $\alpha_1=\alpha_2=\alpha_3=1$ the fractional order model converts to the classical epidemic model (see e.g. [8]). Applying the Laplace transform on both sides of Eqs. (1.2)

$$\begin{cases}\mathcal{L}\{D^{\alpha_1}(x)\}=-\beta\mathcal{L}\{x(t)y(t)\},\\ \mathcal{L}\{D^{\alpha_2}(y)\}=\beta\mathcal{L}\{x(t)y(t)\}-\gamma\mathcal{L}\{y(t)\},\\ \mathcal{L}\{D^{\alpha_3}(z)\}=\gamma\mathcal{L}\{y(t)\},\end{cases}\tag{3.1}$$

using property of the Laplace transform, we get

$$\begin{cases}S^{\alpha_1}\mathcal{L}\{x\}-S^{\alpha_1-1}x(0)=-\beta\mathcal{L}\{x(t)y(t)\},\\ S^{\alpha_2}\mathcal{L}\{y\}-S^{\alpha_2-1}y(0)=\beta\mathcal{L}\{x(t)y(t)\}-\gamma\mathcal{L}\{y(t)\},\\ S^{\alpha_3}\mathcal{L}\{z\}-S^{\alpha_3-1}z(0)=\gamma\mathcal{L}\{y(t)\},\end{cases}\tag{3.2}$$

using initial condition from (1.3)

$$\begin{cases}\mathcal{L}\{x\}=\frac{N_1}{S}-\frac{\beta}{S^{\alpha_1}}\mathcal{L}\{x(t)y(t)\},\\ \mathcal{L}\{y\}=\frac{N_2}{S}+\frac{\beta}{S^{\alpha_2}}\mathcal{L}\{x(t)y(t)\}-\frac{\gamma}{S^{\alpha_2}}\mathcal{L}\{y(t)\},\\ \mathcal{L}\{z\}=\frac{N_3}{S}+\frac{\gamma}{S^{\alpha_3}}\mathcal{L}\{y(t)\}.\end{cases}\tag{3.3}$$

The method assumes the solution as an infinite series:

$$x=\sum_{k=0}^{\infty}x_k,\quad y=\sum_{k=0}^{\infty}y_k,\quad z=\sum_{k=0}^{\infty}z_k.\tag{3.4}$$

The nonlinearity xy is decomposed as

$$xy=\sum_{k=0}^{\infty}A_k,$$

where A_k so-called Adomian Polynomials given as

$$A_k=\frac{1}{k!}\frac{d^k}{d\lambda^k}\left[\sum_{j=0}^{k}\lambda^j x_j\sum_{j=0}^{k}\lambda^j y_j\right]\Bigg|_{\lambda=0}.\tag{3.5}$$

Substituting from Eqs. (3.4), (3.5) into (3.3) the result is

$$\mathcal{L}\{x_0\}=\frac{N_1}{S},\qquad \mathcal{L}\{y_0\}=\frac{N_2}{S},\qquad \mathcal{L}\{z_0\}=\frac{N_3}{S},\tag{3.6}$$

$$\mathcal{L}\{x_1\} = -\frac{\beta}{S^{\alpha_1}}\mathcal{L}\{A_0\}, \qquad \mathcal{L}\{y_1\} = \frac{\beta}{S^{\alpha_2}}\mathcal{L}\{A_0\} - \frac{\gamma}{S^{\alpha_2}}\mathcal{L}\{y_0\}, \quad \mathcal{L}\{z_1\} = \frac{\gamma}{S^{\alpha_3}}\mathcal{L}\{y_0\}, \tag{3.7}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\mathcal{L}\{x_{k+1}\} = -\frac{\beta}{S^{\alpha_1}}\mathcal{L}\{A_k\}, \quad \mathcal{L}\{y_{k+1}\} = \frac{\beta}{S^{\alpha_2}}\mathcal{L}\{A_k\} - \frac{\gamma}{S^{\alpha_2}}\mathcal{L}\{y_k\}, \ \mathcal{L}\{z_{k+1}\} = \frac{\gamma}{S^{\alpha_3}}\mathcal{L}\{y_k\}.$$

The aim is to study the mathematical behavior of the solution $x(t)$, $y(t)$, $z(t)$ for the different values of α. By Applying the inverse Laplace transform to both sides of Eqs.(3.6) the values of x_0, y_0, z_0 are obtained. Substituting these values of A_0, y_0 into Eqs.(3.7), the first component x_1, y_1, z_1 are obtained. The other terms of $x_2, x_3, ..., y_2, y_3, ...$ and $z_2, z_3, ...$ can be calculated recursively in a similar way and we can write the solution

$$x(t) = x_0 + x_1 + x_2 + ... \ , y(t) = y_0 + y_1 + y_2 + ... \ , z(t) = z_0 + z_1 + z_2 + ... \ . \tag{3.8}$$

4. Numerical results and discussion

The L-ADM provides an analytical approximate solution in terms of an infinite power series. For numerical results, the following values, for parameters, are considered [3]. The first few components of L-ADM solution $x(t)$, $y(t)$ and $z(t)$ are

Parameter	Description
$N_1 = 20$	initial population of $x(t)$, who are susceptible
$N_2 = 15$	initial population of $y(t)$, who are infective
$N_3 = 10$	initial population of $z(t)$, who are immumne
$\beta = 0.01$	rate of change of susceptible to infective population
$\gamma = 0.02$	rate of change of infectives to immumne population

Table 1. Parameters values.

calculated. We computed the first three terms of the L-ADM series solution for the system (1.2). We present two of them as follows:

$$x_1 = \frac{-3\ t^{\alpha_1}}{\Gamma(\alpha_1+1)}, \qquad y_1 = \frac{2.7\ t^{\alpha_2}}{\Gamma(\alpha_2+1)}, \qquad z_1 = \frac{0.3\ t^{\alpha_3}}{\Gamma(\alpha_3+1)},$$

$$x_2 = \frac{0.45\ t^{2\alpha_1}}{\Gamma(2\alpha_1+1)} - \frac{0.54\ t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}, \ y_2 = \frac{0.486\ t^{2\alpha_2}}{\Gamma(2\alpha_2+1)} - \frac{0.45\ t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}, \ z_2 = \frac{0.054\ t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)},$$

thus, the L-ADM series solution of the system (1.2) can be given by Eqs.3.8. At $\alpha_1 = \alpha_2 = \alpha_3 = 1$, we obtain three terms approximations which are the same solution obtained in [8] using L-ADM but in ordinary case as follows:

$$x(t) = 20 - 3\ t - 0.045\ t^2 + 0.02805\ t^3,$$
$$y(t) = 15 + 2.7\ t + 0.018\ t^2 - 0.02817\ t^3,$$
$$z(t) = 10 + 0.3\ t + 0.027\ t^2 + 0.00012\ t^3.$$

5. Conclusion

In the present work, we have considered a fractional version of the SIR model, describing the spread of an epidemic in a given population. The aim of this work is to use the Laplace-Adomian Decomposition method for obtaining the solution of the fractional epidemic model. The comparison for some different values of α has been obtained.

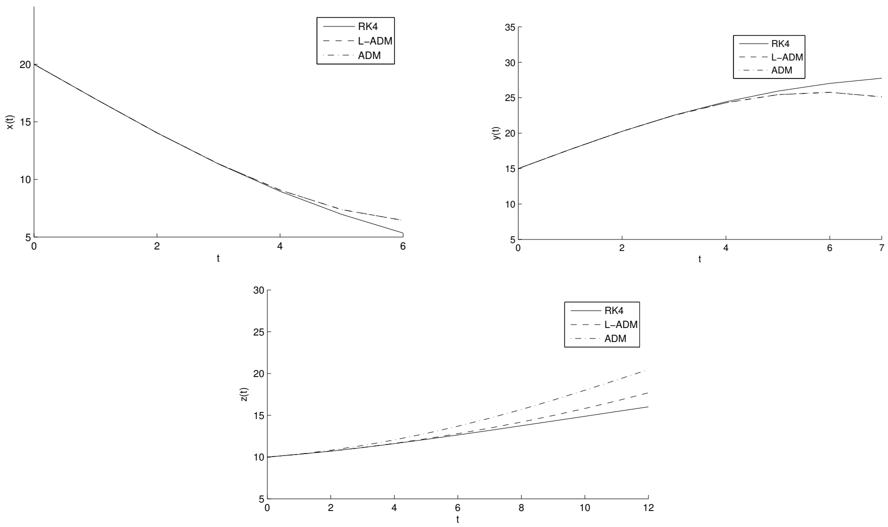

FIGURE 1. The numerical results for $x(t)$, $y(t)$, $z(t)$ at $\alpha_1 = \alpha_3 = \alpha_3 = 1$. The soultions of $x(t)$, $y(t)$ by using L-ADM and ADM completely overlap.

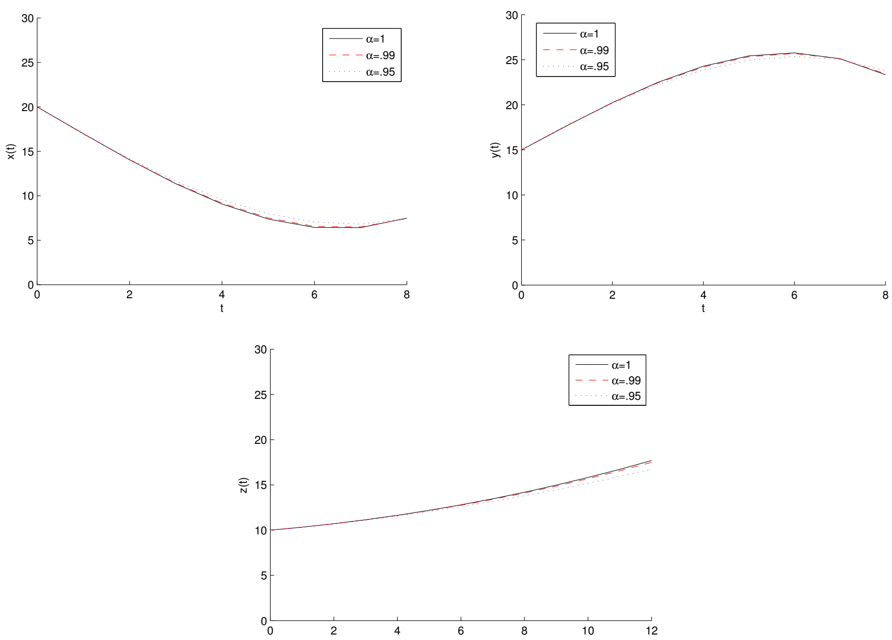

FIGURE 2. The numerical results for $x(t)$, $y(t)$, $z(t)$ at different values of α.

	$x(t)$					
t	$\alpha_1 = 1$ $\alpha_2 = 1$ $\alpha_3 = 1$	$\alpha_1 = 0.99$ $\alpha_2 = 0.99$ $\alpha_3 = 0.99$	$\alpha_1 = 0.95$ $\alpha_2 = 0.95$ $\alpha_3 = 0.95$	$\alpha_1 = 1$ $\alpha_2 = 0.99$ $\alpha_3 = 0.99$	$\alpha_1 = 1$ $\alpha_2 = 0.95$ $\alpha_3 = 0.95$	$\alpha_1 = 0.99$ $\alpha_2 = 0.99$ $\alpha_3 = 0.95$
0	20.0000	20.0000	20.0000	20.0000	20.0000	20.0000
1	16.9831	16.9704	16.9213	16.9807	16.9712	16.9704
2	14.0444	14.0616	14.1336	14.0419	14.0321	14.0616
3	11.3524	11.4109	11.6459	11.3540	11.3615	11.4109
4	9.07520	9.17302	9.56071	9.08472	9.12393	9.17302
5	7.38125	7.50352	7.98741	7.40056	7.47862	7.50352
6	6.43880	6.55779	7.03558	6.46736	6.58109	6.55779

TABLE 2. The number of suseptible individuals $x(t)$ at different values of α.

	$y(t)$					
t	$\alpha_1 = 1$ $\alpha_2 = 1$ $\alpha_3 = 1$	$\alpha_1 = 0.99$ $\alpha_2 = 0.99$ $\alpha_3 = 0.99$	$\alpha_1 = 0.95$ $\alpha_2 = 0.95$ $\alpha_3 = 0.95$	$\alpha_1 = 1$ $\alpha_2 = 0.99$ $\alpha_3 = 0.99$	$\alpha_1 = 1$ $\alpha_2 = 0.95$ $\alpha_3 = 0.95$	$\alpha_1 = 0.99$ $\alpha_2 = 0.99$ $\alpha_3 = 0.95$
0	15.0000	15.0000	15.0000	15.0000	15.0000	15.0000
1	17.6898	17.7007	17.7428	17.7031	17.7550	17.7007
2	20.2466	20.2306	20.1636	20.2329	20.1742	20.2306
3	22.5014	22.4498	22.2431	22.4438	22.2101	22.4498
4	24.2851	24.2029	23.8775	24.1748	23.7357	24.2029
5	25.4287	25.3336	24.9570	25.2646	24.6190	25.3336
6	25.7633	25.6861	25.3717	25.5515	24.7279	25.6861

TABLE 3. The number of infected individuals $y(t)$ at different values of α.

	$z(t)$					
t	$\alpha_1 = 1$ $\alpha_2 = 1$ $\alpha_3 = 1$	$\alpha_1 = 0.99$ $\alpha_2 = 0.99$ $\alpha_3 = 0.99$	$\alpha_1 = 0.95$ $\alpha_2 = 0.95$ $\alpha_3 = 0.95$	$\alpha_1 = 1$ $\alpha_2 = 0.99$ $\alpha_3 = 0.99$	$\alpha_1 = 1$ $\alpha_2 = 0.95$ $\alpha_3 = 0.95$	$\alpha_1 = 0.99$ $\alpha_2 = 0.99$ $\alpha_3 = 0.95$
0	10.0000	10.0000	10.0000	10.0000	10.0000	10.0000
1	10.3271	10.3289	10.3359	10.3289	10.3360	10.3348
2	10.7090	10.7078	10.7028	10.7079	10.7031	10.7019
3	11.1462	11.1393	11.1110	11.1394	11.1112	11.1129
4	11.6397	11.6241	11.5617	11.6240	11.5610	11.5701
5	12.1900	12.1628	12.0556	12.1622	12.0521	12.0744
6	12.7979	12.7561	12.5927	12.7544	12.5841	12.6265

TABLE 4. The number of isolated individuals $z(t)$ at different values of α.

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