

LYAPUNOV TYPE INEQUALITIES FOR FRACTIONAL STURM-LIOUVILLE PROBLEMS AND FRACTIONAL HAMILTONIAN SYSTEMS AND APPLICATIONS

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ABSTRACT. In this paper, we will introduce some new results about Lyapunov type inequalities by studying the given fractional order Sturm-Liouville problems and fractional order Hamiltonian systems. We state and prove Lyapunov type inequalities for fractional S-L problems equipped by fractional Riemann-Liouville derivatives. In this paper for establishing fractional Hamiltonian systems we will apply recently introduced so called fractional Conformable operators. At the end as applications of these inequalities, some interesting spectral properties for fractional order S-L problems together with stability of fractional Hamiltonian systems will be concluded.

keywords: Fractional derivatives and integrals, Sturm-Liouville problem, Hamiltonian system, Lyapunov type inequalities, Stability.

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1. INTRODUCTION

We begin with general description of main problems. A. M. Lyapunov during his study of general theory of stability of motion in 1892, introduced the stability criterion for second order differential equations, that in the case of periodic motion can be stated as follows:

The second order differential equation

$$y'' + p(t)y = 0, \quad t \in \mathbb{R}, \quad (1)$$

with ω -periodic coefficient $p(t)$ is stable in the sense that, all solutions of ODE (1) are bounded as $t \rightarrow \pm\infty$, provided that $p(t) \not\equiv 0$, $p(t) > 0$ for all $t \in \mathbb{R}$ and

$$\omega \int_0^\omega p(t)dt \leq 4. \quad (2)$$

The counter inequality of (2) is to be called Lyapunov inequality. For more details see [11],[12].

During the last decades, one can find considerable modifications of Lyapunov type

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inequalities related to ordinary differential equations such as stability and disconjugacy or oscillatory criterion for mentioned problems, while in the frame of theory of fractional calculus we can only find a few elementary research works about this inequalities. We are interested to give some new results about the Lyapunov type inequalities of fractional differential equations. An interested reader can find some recent studies about mentioned inequalities in [2],[4],[5],[8],[13]-[18].

G. Sh. Guseinov and *B. Kaymkçalan*, in [6] investigated the Lyapunov type inequalities for continuous linear Hamiltonian system

$$\begin{cases} x'(t) = a(t)x(t) + b(t)u(t), \\ u'(t) = -c(t)x(t) - a(t)u(t), \end{cases} \quad t \in \mathbb{R}, \tag{3}$$

and proved that if (x, u) is a nontrivial solution of Hamiltonian system (3), such that $b(t) \geq 0$ on \mathbb{R} and $x(\alpha) = 0 = x(\beta)$ and $x(t)$ is not vanish identically zero on $[\alpha, \beta]$, then the following Lyapunov type inequality satisfies

$$\int_{\alpha}^{\beta} |a(t)|dt + \sqrt{\int_{\alpha}^{\beta} b(t)dt \cdot \int_{\alpha}^{\beta} c_+(t)dt} \geq 2, \tag{4}$$

where $c_+(t) = \max\{c(t), 0\}$. This investigation makes the main motivation of our work.

Recently *Rui A. C. Ferreira* in [4], studied the fractional differential equation

$$\begin{cases} ({}_a D^{\alpha} y)(t) + q(t)y(t) = 0, & \alpha \in (1, 2], t \in (a, b), \\ y(a) = 0 = y(b), \end{cases} \tag{5}$$

where $q : [a, b] \rightarrow \mathbb{R}$ is a continuous function. The author using Green function technique obtained the following Lyapunov type inequality

$$\int_a^b |q(s)|ds > \Gamma(\alpha) \left(\frac{4}{b-a} \right)^{\alpha-1}.$$

Let us now consider the following fractional higher order linear Hamiltonian system:

$$D^{\alpha}y(t) = JH(t)y(t), \quad t \in \mathbb{R},$$

where

$$D^{\alpha}y(t) = \begin{pmatrix} \mathfrak{D}^{\mu}v(t) \\ \mathfrak{D}^{\nu}u(t) \end{pmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad H(t) = \begin{bmatrix} \gamma(t) & \alpha(t) \\ \alpha(t) & \beta(t) \end{bmatrix}.$$

Equivalently we consider the following system

$$\begin{cases} \mathfrak{D}^{\mu}u(t) = \alpha(t)u(t) + \beta(t)v(t), \\ \mathfrak{D}^{\nu}v(t) = -\gamma(t)u(t) - \alpha(t)v(t), \end{cases} \quad t \in \mathbb{R}, \tag{6}$$

where

$$\begin{cases} \mu, \nu \in (n, n + 1], n \in \mathbb{N} \cup \{0\}, \\ \alpha(t), \beta(t), \gamma(t) : \text{Real-valued piece-wise continuous functions on } \mathbb{R}, \\ \mathfrak{D}^{\rho} : \text{Relevant fractional derivative of order } \rho \text{ that will be defined later.} \end{cases} \tag{7}$$

Before beginning the main results, we state some classic and modified definitions and lemmas from fractional calculus.

Definition 1.1. [10] *The left and right sided Riemann-Liouville fractional integrals of order $\rho \geq 0$ for function $f \in L^1(a, b)$ are defined as below:*

$$\mathcal{I}_{a^+(b_-)}^\rho f(t) = \begin{cases} \mathcal{I}_{a^+}^\rho f(t) = \frac{1}{\Gamma(\rho)} \int_a^t (t-s)^{\rho-1} f(s) ds; & \rho > 0, \\ \mathcal{I}_{b_-}^\rho f(t) = \frac{1}{\Gamma(\rho)} \int_t^b (s-t)^{\rho-1} f(s) ds; & \rho > 0, \\ f(t) & ; \quad \rho = 0. \end{cases} \quad (8)$$

Definition 1.2. [10] *The left and right sided Riemann-Liouville fractional derivatives of order $\rho \geq 0$ for function $f \in L^1(a, b)$ is defined by:*

$$\mathcal{D}_{a^+(b_-)}^\rho f(t) = \begin{cases} \mathcal{D}_{a^+}^\rho f(t) = \frac{1}{\Gamma(n-\rho)} \left(\frac{d^n}{dt^n} \right) \int_a^t (t-s)^{n-\rho-1} f(s) ds; & \rho > 0, \\ \mathcal{D}_{b_-}^\rho f(t) = \frac{(-1)^n}{\Gamma(n-\rho)} \left(\frac{d^n}{dt^n} \right) \int_t^b (s-t)^{n-\rho-1} f(s) ds; & \rho > 0, \\ f(t) & ; \quad \rho = 0. \end{cases} \quad (9)$$

where $n = [\rho] + 1$.

Lemma 1.3.

$$\begin{aligned} \mathcal{D}_{a^+}^\rho \mathcal{I}_{a^+}^\rho u(t) &= u(t), & \mathcal{I}_{a^+}^\rho \mathcal{D}_{a^+}^\rho u(t) &= u(t) + \sum_{k=1}^n c_k (t-a)^{\rho-k}, \\ \mathcal{D}_{b_-}^\rho \mathcal{I}_{b_-}^\rho u(t) &= u(t), & \mathcal{I}_{b_-}^\rho \mathcal{D}_{b_-}^\rho u(t) &= u(t) + \sum_{k=1}^n d_k (b-t)^{\rho-k}, \end{aligned} \quad (10)$$

such that $\rho > 0$, $n = [\rho] + 1$.

As we know, all of the fractional order operators based on *Riemann – Liouville* operators do not satisfy in the so called Leibnitz rule. See [16]. But soon we will need to use those fractional derivatives that retain the classic form of first order Leibnitz rule for two functions $u(t), v(t)$, that is

$$(\mathfrak{D}^\rho uv)(t) = (\mathfrak{D}^\rho u)(t).v(t) + u(t).(\mathfrak{D}^\rho v)(t).$$

More recently *R. Khalil et al.* and *T. Abdeljawad* in [1],[9], introduced new definitions for fractional order operators by means of generalizing the limit approach of integer order differentiation operators, that retain interestingly some of important algebraic properties of fractional order differentiation. So we represent these so called fractional Conformable operators as follows.

Definition 1.4. [1] *The left and right sided fractional Conformable integrals of order $n < \rho \leq n + 1$, $n \in \mathbb{N} \cup \{0\}$, for function $f \in L^1(a, b)$ are defined as below:*

$$I^\rho f(t) = \begin{cases} I_{a^+}^\rho f(t) = \frac{1}{n!} \int_a^t (t-s)^n (s-a)^{\alpha-n-1} f(s) ds, \\ I_{b_-}^\rho f(t) = \frac{1}{n!} \int_t^b (s-t)^n (b-s)^{\alpha-n-1} f(s) ds. \end{cases} \quad (11)$$

Consequently the corresponding fractional operators are defined as follows.

Definition 1.5. [1],[9] *The left and right sided fractional Conformable derivatives of order $n < \rho \leq n + 1$, $n \in \mathbb{N} \cup \{0\}$, for n -differentiable function $f(t)$ on $t \in (a, b)$ is given by:*

$$T^\rho f(t) = \begin{cases} T_{a^+}^\rho f(t) = \lim_{\epsilon \rightarrow 0} \frac{f^{([\rho]-1)}(t + \epsilon(t - a)^{[\rho]-\rho}) - f^{([\rho]-1)}(t)}{\epsilon}, \\ T_{b^-}^\rho f(t) = \lim_{\epsilon \rightarrow 0} \frac{(-1)^{n+1} \{f^{([\rho]-1)}(t + \epsilon(b - t)^{[\rho]-\rho}) - f^{([\rho]-1)}(t)\}}{\epsilon}, \end{cases} \tag{12}$$

where $[\rho]$ is the smallest integer greater than or equal to ρ .

Some of interesting properties of newly defined fractional operators that will be needed in studying the fractional Hamiltonian systems are represented in the following lemma.

Lemma 1.6. [1],[9] *Fractional Conformable operators defined by (11),(12) have the following properties:*

(Q₁)

$$\begin{aligned} T_{a^+}^\alpha I_{a^+}^\alpha f(t) &= f(t), \quad I_{a^+}^\alpha T_{a^+}^\alpha f(t) = f(t) - \sum_{k=0}^n \frac{f^{(k)}(a)(t - a)^k}{k!}, \\ T_{b^-}^\alpha I_{b^-}^\alpha f(t) &= f(t), \quad I_{b^-}^\alpha T_{b^-}^\alpha f(t) = f(t) - \sum_{k=0}^n \frac{(-1)^k f^{(k)}(b)(b - t)^k}{k!}, \end{aligned} \tag{13}$$

provided that $n < \alpha \leq n + 1$, $n \in \mathbb{N} \cup \{0\}$ and $f(t)$ is n -differentiable.

(Q₂)

$$T^\alpha(uv)(t) = (T^\alpha u(t))v(t) + u(t)(T^\alpha v(t)), \tag{14}$$

where $n < \alpha \leq n + 1$, $n \in \mathbb{N} \cup \{0\}$ and $u(t), v(t)$ are n -differentiable.

(Q₃)

$$T^\alpha\left(\frac{u}{v}\right)(t) = \frac{(T^\alpha u(t))v(t) - u(t)(T^\alpha v(t))}{v^2(t)}, \tag{15}$$

such that $n < \alpha \leq n + 1$, $n \in \mathbb{N} \cup \{0\}$ and $u(t), v(t)$ are n -differentiable and $v(t)$ is nonzero for all t on its domain.

2. MAIN RESULTS

The organization of this section is as follows:

Step 1. Representing the Lyapunov type inequality for a certain fractional order Sturm-Liouville problem

$$\begin{cases} D_{a^+}^\alpha \left(p(t)u'(t) \right) + q(t)u(t) = 0, \quad 1 < \alpha \leq 2, \quad t \in \mathcal{J} = (a, b), \quad b \neq 0, \\ u(a) = u'(a) = 0, \quad u(b) = 0. \end{cases} \tag{16}$$

Step 2. Obtaining the Lyapunov type inequality for the coupled left(right) sided fractional Conformable Hamiltonian systems

$$\begin{cases} T_{a^+}^\nu u(t) = \underline{\alpha}(t)u(t) + \underline{\beta}(t)v(t), \\ T_{a^+}^\nu v(t) = -\underline{\gamma}(t)u(t) - \underline{\alpha}(t)v(t), \end{cases} \quad n < \nu \leq n + 1, \quad t \in \mathbb{R}, \tag{17}$$

and

$$\begin{cases} T_{b-}^{\nu} u(t) = \bar{\alpha}(t)u(t) + \bar{\beta}(t)v(t), \\ T_{b-}^{\nu} v(t) = -\bar{\gamma}(t)u(t) - \bar{\alpha}(t)v(t), \end{cases} \quad n < \nu \leq n+1, \quad t \in \mathbb{R}, \quad (18)$$

where $\underline{\omega}(t) = (t-a)\omega(t)$ and $\bar{\omega}(t) = (b-t)\omega(t)$.

Notation. $C_p(\mathcal{J}, \mathbb{R})$: Real valued piece-wise continuous functions on \mathcal{J} .

Lemma 2.1. Assume that $p(t), q(t) \in C_p(\mathcal{J}, \mathbb{R})$ and $p(t) > 0$, for $t \in \mathbb{R}$. Then for any nontrivial solution of the fractional S-L problem

$$\begin{cases} \mathcal{D}_{a+}^{\alpha} (p(t)u'(t)) + q(t)u(t) = 0, \quad 1 < \alpha \leq 2, \quad t \in \mathcal{J} = (a, b), \\ u(a) = u'(a) = 0 = u(b), \end{cases} \quad (19)$$

the following so called Lyapunov type inequality will be satisfied:

$$\int_a^b \int_a^b \left| \frac{q(s)}{p(w)} \right| ds dw > \frac{\Gamma(\alpha)}{2(b-a)^{\alpha-1}}. \quad (20)$$

Proof.

$$\mathcal{D}_{a+}^{\alpha} (p(t)u'(t)) = -q(t)u(t).$$

Properties (10) for Riemann-Liouville fractional derivatives imply that

$$u'(t) = c_1 \frac{(t-a)^{\alpha-1}}{p(t)} + c_2 \frac{(t-a)^{\alpha-2}}{p(t)} - \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)p(t)} q(s)u(s) ds.$$

So we have

$$\begin{aligned} u(t) = c_3 + c_1 \int_a^t \frac{(w-a)^{\alpha-1}}{p(w)} dw + c_2 \int_a^t \frac{(w-a)^{\alpha-2}}{p(w)} dw \\ - \int_a^t \int_a^w \frac{(w-s)^{\alpha-1}}{\Gamma(\alpha)p(w)} q(s)u(s) ds dw. \end{aligned}$$

Now implementing the boundary conditions, we conclude that

$$\begin{cases} u(a) = 0 \Rightarrow c_3 = 0, \\ u'(a) = 0 \Rightarrow c_2 = 0. \end{cases}$$

Thus the fractional order S-L problem (19) reduces to the integral equation

$$u(t) = c_1 \int_a^t \frac{(w-a)^{\alpha-1}}{p(w)} dw - \int_a^t \int_a^w \frac{(w-s)^{\alpha-1}}{\Gamma(\alpha)p(w)} q(s)u(s) ds dw. \quad (21)$$

Imposing the boundary condition $u(b) = 0$, we deduce that

$$c_1 = \frac{1}{\Gamma(\alpha)} \frac{\int_a^b \int_a^w \frac{(w-s)^{\alpha-1}}{p(w)} q(s)u(s) ds dw}{\int_a^b \frac{(w-a)^{\alpha-1}}{p(w)} dw}. \quad (22)$$

Finally substitution c_1 in (21), we find that

$$\begin{aligned}
 &u(t) \\
 &= \frac{1}{\Gamma(\alpha) \int_a^b \frac{(w-a)^{\alpha-1}}{p(w)} dw} \left\{ \int_a^t \frac{(w-a)^{\alpha-1}}{p(w)} dw \int_a^b \int_a^w \frac{(w-s)^{\alpha-1}}{p(w)} q(s)u(s) ds dw \right. \\
 &\quad \left. - \int_a^b \frac{(w-a)^{\alpha-1}}{p(w)} dw \int_a^t \int_a^w \frac{(w-s)^{\alpha-1}}{p(w)} q(s)u(s) ds dw \right\}.
 \end{aligned}
 \tag{23}$$

Now we define a suitable Banach space that will be required in the sequel, as below:

$$(\mathfrak{B}, \|\cdot\|_{\mathfrak{B}}), \quad \begin{cases} \mathfrak{B} = C[a, b], \\ \|u\|_{\mathfrak{B}} = \sup_{t \in [a, b]} |u(t)|, \quad u \in \mathfrak{B}. \end{cases}
 \tag{24}$$

It is easy to see that

$$\frac{\int_a^t \frac{(w-a)^{\alpha-1}}{p(w)} dw}{\int_a^b \frac{(w-a)^{\alpha-1}}{p(w)} dw} \leq 1.$$

Using triangular inequality and (23), imply that

$$\begin{aligned}
 |u(t)| \leq \frac{1}{\Gamma(\alpha)} \left\{ \int_a^t \int_a^w \frac{(w-s)^{\alpha-1}}{p(w)} |q(s)u(s)| ds dw \right. \\
 \left. + \int_a^b \int_a^w \frac{(w-s)^{\alpha-1}}{p(w)} |q(s)u(s)| ds dw \right\}.
 \end{aligned}$$

Taking sup-norm from both sides of the last inequality, implies that

$$\|u\|_{\mathfrak{B}} < \|u\|_{\mathfrak{B}} \frac{2(b-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b \int_a^b \left| \frac{q(s)}{p(w)} \right| ds dw.$$

Equivalently

$$\int_a^b \int_a^b \left| \frac{q(s)}{p(w)} \right| ds dw > \frac{\Gamma(\alpha)}{2(b-a)^{\alpha-1}}.$$

□

Remark 2.2. Taking $p(t) \equiv 1$, implies the corresponding Lyapunov inequality of the form

$$\int_a^b |q(s)| ds > \frac{\Gamma(\alpha)}{2(b-a)^\alpha}.$$

In what follows, we will study the fractional order Hamiltonian systems (17),(18) equipped with the fractional Conformable derivatives, and we will try to represent the corresponding Lyapunov type inequality for these systems.

Theorem 2.3. Assume that the fractional Conformable Hamiltonian systems (17), (18) have a real solution (u, v) such that $u(a) = u(b) = 0$ and u is not identically zero on $[a, b]$. Moreover, assume that $\beta(t) \geq 0$ and

$$u^{(k)}(a) = 0 = u^{(k)}(b), \quad k = 1, 2, \dots, n, \quad n \in \mathbb{N}, \quad n \geq 1.
 \tag{25}$$

Then the following Lyapunov type inequality holds

$$\int_a^b |\alpha(t)| dt + \sqrt{\int_a^b \beta(t) dt} \cdot \int_a^b \gamma_+(t) dt \geq 2 \frac{n!}{(b-a)^\nu}, \quad (26)$$

where $\gamma_+(t) = \max\{\gamma(t), 0\}$.

Proof. Let us recall once again the fractional Conformable left(right) sided Hamiltonian systems (17),(18) as follows, respectively

$$\begin{cases} T_{a^+}^\nu u(t) = \underline{\alpha}(t)u(t) + \underline{\beta}(t)v(t), \\ T_{a^+}^\nu v(t) = -\underline{\gamma}(t)u(t) - \underline{\alpha}(t)v(t), \end{cases} \quad (27)$$

and

$$\begin{cases} T_{b^-}^\nu u(t) = \bar{\alpha}(t)u(t) + \bar{\beta}(t)v(t), \\ T_{b^-}^\nu v(t) = -\bar{\gamma}(t)u(t) - \bar{\alpha}(t)v(t), \end{cases} \quad (28)$$

subjected to the conditions (7). Multiplying the first equation of (27) by $v(t)$ and second equation by $u(t)$ and adding both equations and doing similar operations on (28), imply that

$$\begin{aligned} T_{a^+}^\nu (u(t)v(t)) &= -\underline{\gamma}(t)u^2(t) + \underline{\beta}(t)v^2(t), \\ T_{b^-}^\nu (u(t)v(t)) &= -\bar{\gamma}(t)u^2(t) + \bar{\beta}(t)v^2(t). \end{aligned} \quad (29)$$

Left and right Conformable integration of both sides of the (29) yields

$$\begin{cases} I_{a^+}^\nu T_{a^+}^\nu (u(t)v(t)) = I_{a^+}^\nu \{-\underline{\gamma}(t)u^2(t) + \underline{\beta}(t)v^2(t)\}, \\ I_{b^-}^\nu T_{b^-}^\nu (u(t)v(t)) = I_{b^-}^\nu \{-\bar{\gamma}(t)u^2(t) + \bar{\beta}(t)v^2(t)\}. \end{cases} \quad (30)$$

Before continuing, let us describe the reason of defining $\underline{\omega}$, $\bar{\omega}$, $\omega \in \{\gamma, \beta\}$. As one can observe, independent from choosing order in fractional Conformable integrals, the statement $(s-a)^{\nu-n-1}$ permanently has non-positive power. So increasing the fractional order can not change the non-positivity of power of $(s-a)^{\nu-n-1}$. Thereby for solving this difficulty without changing the order, one can multiply $(s-a)^{\nu-n-1}$ by $\underline{\omega}$ for obtain positive power. In this case $(s-a)^{\nu-n-1}\underline{\omega}(s)$ will be transformed to the $(s-a)^{\nu-n}\omega(s)$. Thereafter we can find its non-zero upper bound. We may find Similar result for fractional right sided Conformable integrals. Now (13) together with assumptions (25) imply

$$\begin{aligned} u(t).v(t) &= \frac{1}{n!} \int_a^t (t-s)^n (s-a)^{\nu-n} \left\{ -\gamma(s)u^2(s) + \beta(s)v^2(s) \right\} ds, \\ u(t).v(t) &= \frac{1}{n!} \int_t^b (s-t)^n (b-s)^{\nu-n} \left\{ -\gamma(s)u^2(s) + \beta(s)v^2(s) \right\} ds. \end{aligned} \quad (31)$$

Now substitution t by b in the first equation of (31) and t by a in second one and then summing the resulting equations, one can see that subjected to $u(a) = u(b) = 0$, we have

$$\int_a^b \frac{(b-s)^n (s-a)^{\nu-n} + (b-s)^{\nu-n} (s-a)^n}{n!} \left\{ -\gamma(s)u^2(s) + \beta(s)v^2(s) \right\} ds = 0. \quad (32)$$

The equality (32) immediately yields

$$\gamma(s)u^2(s) = \beta(s)v^2(s). \quad (33)$$

On the other hand left sided fractional integrating of the first equation of (27) together with right sided fractional integrating of the first equation of (28), imply that

$$\begin{aligned}
 u(t) + \sum_{k=0}^n \frac{u^{(k)}(a)}{k!} (t-a)^k &= \int_a^t \frac{(t-s)^n (s-a)^{\nu-n}}{n!} \{ \alpha(s)u(s) + \beta(s)v(s) \} ds, \\
 u(t) + \sum_{k=0}^n \frac{(-1)^k u^{(k)}(b)}{k!} (b-t)^k &= \int_t^b \frac{(s-t)^n (b-s)^{(\nu-n)}}{n!} \{ \alpha(s)u(s) + \beta(s)v(s) \} ds.
 \end{aligned}
 \tag{34}$$

According to the assumption (25), we have

$$\begin{aligned}
 |u(t)| &\leq \frac{(b-a)^\nu}{n!} \left\{ \int_a^t |\alpha(s)||u(s)| ds + \int_a^t \beta(s)|v(s)| ds \right\}, \\
 |u(t)| &\leq \frac{(b-a)^\nu}{n!} \left\{ \int_t^b |\alpha(s)||u(s)| ds + \int_t^b \beta(s)|v(s)| ds \right\}.
 \end{aligned}
 \tag{35}$$

Now adding both equations (35), we deduce that

$$2|u(t)| \leq \frac{(b-a)^\nu}{n!} \left\{ \int_a^b |\alpha(s)||u(s)| ds + \int_a^b \beta(s)|v(s)| ds \right\}.
 \tag{36}$$

Applying the Cauchy-Schwartz inequality and (33), we have;

$$\begin{aligned}
 \int_a^b \beta(t)|v(t)| dt &\leq \sqrt{\int_a^b \beta(t) dt} \cdot \sqrt{\int_a^b \beta(t)v(t)^2 dt} \\
 &= \sqrt{\int_a^b \beta(t) dt} \cdot \sqrt{\int_a^b \gamma(t)u(t)^2 dt} \\
 &\leq \sqrt{\int_a^b \beta(t) dt} \cdot \sqrt{\int_a^b \gamma_+(t)u(t)^2 dt}.
 \end{aligned}
 \tag{37}$$

Therefore we conclude from (36) that

$$2\|u(t)\| \leq \frac{(b-a)^\nu}{n!} \left\{ \int_a^b |\alpha(s)||u(s)| ds + \sqrt{\int_a^b \beta(t) dt} \cdot \sqrt{\int_a^b \gamma_+(t)\|u(t)\|^2 dt} \right\},
 \tag{38}$$

where $\|\cdot\|$ is the standard sup-norm. Nontrivial nature of $u(t)$ implies that

$$\int_a^b |\alpha(t)| dt + \sqrt{\int_a^b \beta(t) dt} \cdot \int_a^b \gamma_+(t) dt \geq 2 \frac{n!}{(b-a)^\nu}.
 \tag{39}$$

This completes the proof. □

3. APPLICATIONS

As we know, there is no solid spectral theory for the fractional Sturm-Liouville problems. However Lyapunov type inequalities of fractional Sturm-Liouville problems give us many useful information about spectral properties of such problems. As an application for Lyapunov inequality (20), we represent the spreading of real eigenvalues of fractional S-L problem (19) or equivalently non-existence results for

real zeros of so called generalized Mittag-Leffler functions. To this aim, first we define the (generalized) Mittag-Leffler functions as follows.

Definition 3.1. *The generalized Mittag-Leffler function is defined by*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0, \quad z \in \mathbb{C}, \quad (40)$$

which is an analytic function on the whole complex plane.

Remark 3.2. *For $\beta = 1$, generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ reduces to the standard Mittag-Leffler function*

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \quad z \in \mathbb{C}. \quad (41)$$

Now Let us for simplicity consider the following substitutions for real valued continuous functions $p(t), q(t)$ in fractional S-L problem (19) as follows:

$$p(t) = b^{\alpha+1}, \quad q(t) = -\lambda.$$

So we have

$$\begin{cases} D_{a^+}^{\alpha+1} u(t) - \lambda b^{-(\alpha+1)} u(t) = 0, & \alpha \in (1, 2], \quad t \in (a, b), \quad b \neq 0, \\ u(a) = u'(a) = u(b) = 0. \end{cases} \quad (42)$$

According to [[10], Chapter 5, §5.2], the fundamental solutions of Fractional Sturm Liouville Problem (FSLP) (42) are as follows:

$$\begin{aligned} u_1(t) &= t^{\alpha} E_{\alpha+1, \alpha+1} \left(\lambda \left(\frac{t}{b} \right)^{\alpha+1} \right), \\ u_2(t) &= t^{\alpha-1} E_{\alpha+1, \alpha} \left(\lambda \left(\frac{t}{b} \right)^{\alpha+1} \right), \\ u_3(t) &= t^{\alpha-2} E_{\alpha+1, \alpha-1} \left(\lambda \left(\frac{t}{b} \right)^{\alpha+1} \right). \end{aligned}$$

Thus implementing the boundary condition $u_1(b) = 0$, we conclude that real eigenvalues of FSLP (42) are real solutions of

$$E_{\alpha+1, \alpha+1}(\lambda) = 0. \quad (43)$$

Therefore using Lemma 2.1, we deduce that $\lambda \in \mathbb{R}$ is an eigenvalue of FSLP (42) or equivalently $\lambda \in \mathbb{R}$ is a zero of (43), provided that

$$|\lambda| > \frac{\Gamma(\alpha)}{2} \frac{b^{\alpha+1}}{(b-a)^{\alpha+1}}. \quad (44)$$

So we can represent the following non-existence result for real eigenvalues of F(S-L)P (42).

Theorem 3.3. *Assume that $2 < \theta \leq 3$. Then the generalized Mittag-Leffler function $E_{\theta,\theta}(z)$ has no real zeros provided that*

$$|z| \leq \frac{\Gamma(\theta-1)}{2} \frac{b^{\theta}}{(b-a)^{\theta}}.$$

Remark 3.4. According to the Theorem 3.3, the FSLP (42) has no positive eigenvalue in

$$\left(0, \frac{\Gamma(\theta - 1)}{2} \frac{b^\theta}{(b - a)^\theta} \right].$$

As we stated in the introductory discussions, the Lyapunov type inequalities prepare an effective tool for fulfilment of stability in differential equations such as fractional Sturm-Liouville problems and fractional Hamiltonian systems. In the sequel as an another application for such inequalities we will obtain stability criterion for fractional Hamiltonian systems.

Definition 3.5. Fractional Hamiltonian system (6) is

- (D₁) stable if all solutions are bounded on \mathbb{R} .
- (D₂) conditionally stable if there exist a nontrivial solution which is bounded on \mathbb{R} .
- (D₃) unstable if all nontrivial solutions are unbounded on \mathbb{R} .

Theorem 3.6. Consider the linear fractional Hamiltonian systems (17),(18) where $\alpha(t), \beta(t), \gamma(t)$ are real valued piece-wise continuous functions defined on \mathbb{R} and periodic with a period $\omega > 0$, that is

$$\alpha(t + \omega) = \alpha(t), \quad \beta(t + \omega) = \beta(t), \quad \gamma(t + \omega) = \gamma(t), \quad t \in \mathbb{R}. \quad (45)$$

If the following conditions hold

$$(i) \beta(t) > 0, \quad \gamma(t) \geq 0, \quad \beta(t)\gamma(t) - \alpha^2(t) \geq 0, \quad t \in \mathbb{R}. \quad (46)$$

$$(ii) \beta(t)\gamma(t) - \alpha^2(t) \neq 0. \quad (47)$$

$$(iii) u^{(k)}(a) = 0 = u^{(k)}(b), \quad k = 1, \dots, n, \quad (48)$$

$$(iv) \int_0^\omega |\alpha(t)|dt + \sqrt{\int_0^\omega \beta(t)dt \cdot \int_0^\omega \gamma_+(t)dt} < \frac{2n!}{\omega^\nu}. \quad (49)$$

then the fractional Hamiltonian systems (17),(18) are stable.

Proof. Assume that fractional Hamiltonian systems (17),(18) are unstable on \mathbb{R} . So by means of Floquet theory, there exist non zero real constant ξ with $|\xi| \neq 1$ and nontrivial solution (u, v) of (17),(18) such that

$$u(t + \omega) = \xi u(t), \quad v(t + \omega) = \xi v(t), \quad t \in \mathbb{R}, \quad (50)$$

that implies the unboundedness of nontrivial solution (u, v) . Details can be found in [3],[7]. In order to apply the Theorem 2.3 we must prove that $u(t)$ has at least one zero in segment $[0, \omega]$. Assume that $u(t)$ has no zero in $[0, \omega]$. Hence according to (50), $u(t) \neq 0$ for all $t \in \mathbb{R}$.

Now multiplying the first equation of (27) by v and second one by u and then subtraction resulting equations and doing similar operations on the right Hamiltonian system (28), we get

$$T_{a+}^\nu \left(\frac{u}{v} \right) (t) = \frac{\underline{\gamma}(t)u^2(t) + 2\underline{\alpha}(t)u(t)v(t) + \underline{\beta}(t)v^2(t)}{v^2(t)}, \quad (51)$$

$$T_{b-}^\nu \left(\frac{u}{v} \right) (t) = \frac{\overline{\gamma}(t)u^2(t) + 2\overline{\alpha}(t)u(t)v(t) + \overline{\beta}(t)v^2(t)}{v^2(t)}. \quad (52)$$

Fractional left and right Conformable integrating of both sides of (51) and (52) respectively, and using assumption (iii), imply that

$$\begin{aligned} & \left(\frac{u}{v}\right)(t) - \left(\frac{u}{v}\right)(a) \\ &= \int_a^t \frac{(t-s)^n (s-a)^{\nu-n}}{n!} \left\{ \frac{\gamma(s)u^2(s) + 2\alpha(s)u(s)v(s) + \beta(s)v^2(s)}{v^2(s)} \right\} ds, \\ & \left(\frac{u}{v}\right)(t) - \left(\frac{u}{v}\right)(b) \\ &= \int_t^b \frac{(s-t)^n (b-s)^{\nu-n}}{n!} \left\{ \frac{\gamma(s)u^2(s) + 2\alpha(s)u(s)v(s) + \beta(s)v^2(s)}{v^2(s)} \right\} ds. \end{aligned} \quad (53)$$

If we substitute t by $a + \omega$ in first equation of (53) and t by a and b by $a + \omega$ in second one, (50) ensures that

$$\left(\frac{u}{v}\right)(a + \omega) - \left(\frac{u}{v}\right)(a) = \left(\frac{\xi u}{\xi v}\right)(a) - \left(\frac{u}{v}\right)(a) = 0. \quad (54)$$

Consequently we have

$$\begin{aligned} 0 &= \int_a^{a+\omega} \frac{(a+\omega-s)^n (s-a)^{\nu-n} + (a+\omega-s)^{\nu-n} (s-a)^n}{n!} \times \\ & \quad \times \left\{ \frac{\gamma(s)u^2(s) + 2\alpha(s)u(s)v(s) + \beta(s)v^2(s)}{v^2(s)} \right\} ds. \end{aligned} \quad (55)$$

Therefore it is clear that

$$\gamma(s)u^2(s) + 2\alpha(s)u(s)v(s) + \beta(s)v^2(s) = 0. \quad (56)$$

On the other hand we know that

$$\begin{aligned} & \gamma(s)u^2(s) + 2\alpha(s)u(s)v(s) + \beta(s)v^2(s) \\ &= \frac{1}{\beta(t)} \{ (\alpha(t)u(t) + \beta(t)v(t))^2 + (\beta(t)\gamma(t) - \alpha^2(t))u^2(t) \}. \end{aligned}$$

So according to the assumptions (i),(ii), equality (56) could not be true, since (u, v) is nontrivial. Hence $u(t)$ has at least two zeros, one zero a in $[0, \omega]$ and one another in $a + \omega$. Also it is not identically zero on $[a, a + \omega]$. Now applying Theorem 2.3 and taking $b = a + \omega$, we conclude that

$$\int_a^{a+\omega} |\alpha(t)| dt + \sqrt{\int_a^{a+\omega} \beta(t) dt \cdot \int_a^{a+\omega} \gamma_+(t) dt} \geq 2 \frac{n!}{\omega^\nu}. \quad (57)$$

It is clear that

$$\int_{t_0}^{t_0+\omega} f(t) dt = \int_0^\omega f(t) dt, \quad t_0 \in \mathbb{R}.$$

So (57) is equivalent to the following Lyapunov type inequality

$$\int_0^\omega |\alpha(t)| dt + \sqrt{\int_0^\omega \beta(t) dt \cdot \int_0^\omega \gamma_+(t) dt} \geq 2 \frac{n!}{\omega^\nu}, \quad (58)$$

that contradicts with assumption (iv). Thus has been proved that the fractional Hamiltonian systems (17),(18) are stable on \mathbb{R} . \square

4. CONCLUDING REMARKS

In this paper we studied the fractional order Sturm-Liouville problems and fractional linear Hamiltonian systems for obtaining the corresponding Lyapunov type inequalities. Since the standard fractional operators do not preserve the first order Leibnitz rule, so we could not apply these operators for studying the fractional order Hamiltonian systems. That is why we began our research process intentionally with Riemann-Liouville fractional operators and as a result of necessity of using different class of fractional order operators from viewpoint of their algebraic properties, we applied fractional Conformable operators for Hamiltonian systems and successfully refined fractional order stability result corresponding to ordinary one described in [6].

At the end, we point out that we used the technique $\underline{\omega}(t) = (t - a)\omega(t)$ and $\bar{\omega}(t) = (b - t)\omega(t)$ in fractional Conformable Hamiltonian systems (17),(18) only for keeping the boundedness of appearing integral operators.

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