Journal of Fractional Calculus and Applications
Vol. 7(1) Jan. 2016, pp. 147-175.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

ON THE PROBABILISTIC APPROACH TO THE SOLUTION OF GENERALIZED FRACTIONAL DIFFERENTIAL EQUATIONS OF CAPUTO AND RIEMANN-LIOUVILLE TYPE

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Abstract. This paper provides a probabilistic approach to solve linear equations involving Caputo and Riemann-Liouville type derivatives. Using the probabilistic interpretation of these operators as the generators of interrupted Feller processes, we obtain well-posedness results and explicit solutions (in terms of the transition densities of the underlying stochastic processes). The problems studied here include fractional linear differential equations, well analyzed in the literature, as well as their far reaching extensions.

1. Introduction

The theory of fractional differential equations is a valuable tool for modeling a variety of physical phenomena arising in different fields of science. Their numerous applications include areas such as engineering, physics, biophysics, continuum and statistical mechanics, finance, control processing, econophysics, probability and statistics, and so on (see, e.g., [3], [19], [32], [36], and references therein). Fractional ordinary differential equations (FODE's) and fractional partial differential equations (FPDE's) have been used as more accurate models to describe, e.g., relaxation phenomena, viscoelastic systems, anomalous diffusions, Lévy flights and continuous time random walks (CTRW's) (see, e.g.,[2], [20]-[21], [25], [27], [29], [36]).

To solve this type of equations different analytical and numerical methods have been investigated. Analytical methodologies include *the Laplace, the Mellin* and *the Fourier transform* techniques [7], [19], [31]-[33], and the *operational calculus* method [16]-[17], [24]. Regarding the numerical approaches, we can mention the *fractional difference method*, the *quadrature formula approach*, the *predictor-corrector approach* as well as some numerical approximations using the *short memory principle* amongst others (see, e.g., [4]-[7], [9], [32] and references therein). In the present article we focus on a probabilistic approach to solve linear differential equations involving natural extensions (from a probabilistic point of view) of the Caputo and Riemann-Liouville (RL) derivatives of order $\beta \in (0, 1)$.

2010 *Mathematics Subject Classification.* 34A08, 60H30, 35S15, 34A05.
Key words and phrases. Caputo derivative, Riemann-Liouville derivative, Generalized fractional operators, Mittag-Leffler functions, Feller process, β-stable subordinator, Stopping time.
Submitted Jun. 27, 2015.

The use of probability theory to solve classical differential equations is an effective approach to obtain their solutions by relating them with boundary value problems of diffusion processes. In the fractional framework, connections between probability and FPDE's have been also analyzed in the literature [14], [20]-[21], [29], [30], [35]. For instance, the probabilistic interpretations of the Green (or fundamental) solution to the *time-fractional diffusion equation* and *the time-space fractional diffusion equation* are already known (see references above).

In this paper we employ similar probabilistic arguments (transforming the original problem into a Dirichlet type problem) to study linear equations involving a general class of Caputo and Riemann-Liouville type operators. These operators can be thought of as the extensions of the classical Caputo and RL fractional derivatives, respectively. As was shown in [22], they can be obtained as the generators of Markov processes interrupted on an attempt to cross a boundary point. The Caputo and RL derivatives of order $\beta \in (0,1)$ are particular cases arising by stopping and killing a β-stable subordinator, respectively. This fact allows one to solve fractional equations as particular cases of more general equations involving $D-$ and D_*-operators of the type $-D_{a+}^{(\nu)}$ and $-D_{a+*}^{(\nu)}$, respectively (see definitions later). The problems we are addressing here are the following:

(i) the linear equation with the Caputo type operator:

$$-D_{a+*}^{(\nu)}u(t) = \lambda u(t) - g(t), \quad t \in (a,b], \quad u(a) = u_a, \tag{1}$$

for a given $\lambda \geq 0$, a bounded function g and $u_a \in \mathbb{R}$. Since there is a relationship between Caputo and RL type operators (similar to that between the classical Caputo and RL derivatives), we also studied the problem with the RL type operator:

$$-D_{a+}^{(\nu)}u(t) = \lambda u(t) - g(t), \quad t \in (a,b], \quad u(a) = 0, \tag{2}$$

(ii) the generalized *mixed fractional linear equation*

$$-\sum_{i=1}^{d} \tilde{D}^{(\nu_i)}\, u(t_1,\dots,t_d) = \lambda u(t_1,\dots,t_d) - g(t_1,\dots,t_d), \tag{3}$$

with some prescribed boundary condition, where $-\tilde{D}^{(\nu_i)}$ denotes either the RL type operator $-{}_{t_i}D_{a_i+}^{(\nu_i)}$ or the Caputo type operator $-{}_{t_i}D_{a_i+*}^{(\nu_i)}$. The left subscript t_i indicates that the operator is acting on the variable t_i.

Fractional linear differential equations with Caputo derivatives of order $\beta \in (0,1)$ are particular cases of equation (1). They have been extensively investigated by means of the Laplace transform method. Hence, it is known that [7], [32]-[33]

$$D_{a+*}^{\beta}u(t) = \lambda u(t) + g(t), \quad u(a) = u_a, \quad \lambda \in \mathbb{R}, \tag{4}$$

for $\beta \in (0,1)$ and a given continuous function g on $[a,b]$, has the unique solution

$$u(t) = u_a E_\beta\left[-\lambda(t-a)^\beta\right] + \int_a^t g(r)(t-r)^{\beta-1}E_{\beta,\beta}\left(-\lambda(t-r)^\beta\right)dr, \tag{5}$$

where E_β and $E_{\beta,\beta}$ denote the *Mittag-Leffler functions* (see definitions later). This solution can be written in terms of $\beta-$stable densities by means of the integral representation of the Mittag-Leffler functions given in (49). The probabilistic approach introduced here gives this expression directly once one writes down the expectations involved in the general stochastic representation (47). On the other hand, using the results obtained here and the uniqueness of solutions, we obtain a pure probabilistic proof of the well-known equality in (49).

Apart from the classical Caputo derivatives, operators $-D^{(\nu)}_{a+*}$ include, as simple particular cases, the multi-term fractional derivatives $\sum_{i=1}^d \omega_i(t) D^{\beta_i}_{a_i+*} u(t)$ with non-negative functions ω_i. Hence, as another example of (1), our approach also applies to the *multi-term fractional equation*

$$\sum_{i=0}^{k} \omega_i(t) D^{\beta_i}_{a+*} u(t) = -\lambda u(t) + g(t), \qquad \beta_i \in (0,1),\ t \in (a,b], \tag{6}$$

with some given functions g and ω_i for $i \in \{1,\dots,k\}$. The explicit solution when the functions ω_i are constants and (6) is a *commensurate equation* (i.e., the quotients β_i/β_j are rational numbers for all i,j), has been analyzed by the reduction of the original problem to either a single- or multi-order fractional differential equation system (see, e.g., [7], [9], and references therein). An approximation for its solution has been also studied, e.g., in [7]. Our approach encompasses not only the commensurate case with constant coefficients ω_i but also the more general case with non constant coefficients $\omega_i(\cdot)$ and, even more generally, functions $\beta_i(t)$.

The fractional counterpart of equation (3) is the *mixed* fractional equation

$$-\ {}_{t_1}D^{\beta}_{0+}\, u(t_1,t_2) -\ {}_{t_2}D^{\alpha}_{0+*}\, u(t_1,t_2) = \lambda u(t_1,t_2) - g(t_1,t_2), \quad \beta,\alpha \in (0,1), \tag{7}$$

subject to some boundary condition, where $\lambda \geq 0$ and g is a given function on $[0,b_1] \times [0,b_2]$. The probabilistic approach presented here provides the explicit solution in terms of $\beta-$ and $\alpha-$stable densities. To our knowledge this type of mixed fractional equations with Caputo and RL derivatives of order in $(0,1)$ has not been explored explicitly in the literature. The same arguments also apply to the well-posedness for the $d-$dimensional case

$$-\sum_{i=1}^{d} \tilde{D}^{\beta_i}\, u(t_1,\dots,t_d) = \lambda u(t_1,\dots,t_d) - g(t_1,\dots,t_d), \tag{8}$$

with $\tilde{D}^{\beta_i}$ being either the RL or the Caputo derivative.

The probabilistic ideas behind the solution to (1) and (3) can be used to solve the linear equation with non-constant coefficients, as well as more general FPDE's involving operators of the type $-{}_tD^{(\nu)}_{a+*} - A_{\mathbf{x}}$, where $-A_{\mathbf{x}}$ denotes the generator of a Feller process on $\mathbb{R}^d$ acting on the variable $\mathbf{x} \in \mathbb{R}^d$. Some specific cases include the widely studied time-fractional diffusion equation (see, e.g., [2], [10], [28] and references above)

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} = C \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 < \beta \leq 1, \quad C > 0,$$

where (x,t) denotes the space-time variables. These cases will be addressed in a forthcoming paper in preparation.

It is worth mentioning that equations involving D_*- and $D-$operators do not usually have solutions in the domain of the generators $-D^{(\nu)}_{a+*}$ and $-D^{(\nu)}_{a+}$, respectively. The existence of such solutions is restricted to a specific value in the boundary condition. Thus, as usual in classical stochastic analysis, by introducing the concept of a *generalized solution* we are able to study the well-posedness (in a generalized sense) for these equations. To illustrate this concept, consider the very-well known ordinary differential equation (ODE)

$$u'(t) = -\lambda u(t) + g(t), \quad t \in (0, b], \quad u(0) = u_0,$$

where $b > 0$, $g \in C[0, b]$ and $\lambda > 0$, whose solution is

$$u(t) = u_0 e^{\lambda t} + \int_0^t exp\{\lambda(t-r)\} g(r) dr, \quad t \in (0, b]. \tag{9}$$

Probabilistically, this problem can be thought of as the boundary value problem associated with the deterministic linear motion on $(-\infty, b]$ which is stopped at reaching the boundary point $t = 0$. In this case, the semigroup $\{S_s\}_{s\geq 0}$ of the deterministic process is given by $S_s f(t) = f(t-s)$ for any $t \in (-\infty, b]$ and $f \in C(-\infty, b]$, whilst the semigroup $\{S_s^{0+*}\}_{s\geq 0}$ of the stopped process corresponds to $S_s^{0+*} f(t) = f(\max\{0, t-s\})$ for any $t \in [0, b]$ and $f \in C[0, b]$. Hence, the resolvent operator of the semigroup S_s^{0+*} provides equation (9) as the unique *solution in the domain of the generator* (the space $C^1[0, b]$) if, and only if, $u_0 = \frac{1}{\lambda} g(0)$. Otherwise this solution is (in our terminology) only a *generalized solution* as it can be obtained as a limit of solutions in the domain of the generator. Moreover, in this case the generalized solution is also a *classical (smooth) solution* lying on $C[0, b] \cap C^1(0, b]$ instead of $C^1[0, b]$. Similar situations occur when considering fractional differential equations. We will see that the solutions found in the literature are usually solutions in the generalized sense, as they usually do not belong to the domain of $-D^\beta_{a+*}$ or $-D^\beta_{a+}$ as generators of Feller processes.

The main contribution of this work relies on providing well-posedness results and explicit integral representations of solutions to linear equations with Caputo and RL type operators. We pay attention to the existence of two types of solutions: *solutions in the domain of the generator* and *generalized solutions*. The latter concept defined for rather general, even not continuous, functions g in (1). Moreover, all solutions are given in terms of expectations of functionals of Markov processes. From the point of view of numerical analysis, this representation can be exploited to obtain numerical solutions to a variety of problems by performing Monte Carlo techniques. Simulation methods have been effectively used for classical differential equations and, in recent years, different methods for evaluating path functionals of Lévy processes have been actively researched (see, e.g., [12]-[13], [23]). Finally, by using the monotonicity of the underlying processes, we obtain explicit solutions in terms of the transition densities of the Markov processes involved.

The paper is organized as follows. The next section provides some standard notation and gives a quick summary about Caputo and RL derivatives as well as $\beta-$stable subordinators. Section 3 introduces the definition of the generalized RL and Caputo type operators, $-D^{(\nu)}_{a+}$ and $-D^{(\nu)}_{a+*}$, respectively. In Section 4 we study important properties of the underlying stochastic processes associated with the generalized fractional operators. Then, the probabilistic solution to the linear equation (1) is addressed in Section 5. This section concludes with the application of these results to deduce the known solutions to fractional linear equations with the classical Caputo derivatives D^β_{a+*}, for $\beta \in (0, 1)$. The last section presents the well-posedness for the mixed equations (3) and (7).

2. Preliminaries

2.1. **Notation.** Let $\mathbb{N}$, $\mathbb{C}$ and $\mathbb{R}^d$ be the set of positive integers, the complex space and the d-dimensional Euclidean space, respectively. For any interval A, the standard notation $B(A)$, $C(A)$, $C_b(A)$ and $C^1(A)$ denotes the set of bounded Borel measurable functions, continuous functions, bounded continuous functions and continuously differentiable functions defined on A, respectively. If A is closed, $C^1(A)$ refers to the space of continuously differentiable functions up to the boundary. Notation $||\cdot||$ stands for the sup-norm $||f|| = \sup_{x\in A}|f(x)|$ for $f \in B(A)$. If $A = [a, b]$, notation $C_a[a, b]$ and $C^1_a[a, b]$ denote the space of continuos functions vanishing at a and the space $C_a[a, b] \cap C^1[a, b]$, respectively.

As usual, E_β refers to the *Mittag-Leffler function* of order $\beta > 0$ defined by

$$E_\beta(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\beta + 1)}, \quad z \in \mathbb{C},$$

whilst E_{β_1,β_2} means the *two-parameter Mittag-Leffler function*:

$$E_{\beta_1,\beta_2}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j\beta_1 + \beta_2)}, \quad z \in \mathbb{C}, \;\; \beta_1, \beta_2 > 0.$$

For properties of these functions see, e.g., [7], [32]-[33].

Letters t and r are mainly used as space variables, and the letter s is used as a time variable. Bold letters, e.g., $\mathbf{t}$ and $\mathbf{a}$, denote elements in $\mathbb{R}^d$ for $d \geq 2$, and bold capital letters stand for $\mathbb{R}^d-$valued stochastic processes, e.g., $\mathbf{T} = \{\mathbf{T}(s) : s \geq 0\}$. Letters $\mathbf{P}$ and $\mathbf{E}$ are reserved for the probability and the mathematical expectation, respectively. Notation $W_\beta(\sigma, \gamma)$ means a $\beta-$stable random variable (r.v.) with scaling parameter σ, skewness parameter γ and location parameter zero. Its density function is denoted by $w_\beta(\cdot; \sigma, \gamma)$. Finally, for a given Feller semigroup $\{S_t\}_{t\geq 0}$ on $C_b(S)$, its *resolvent operator* at $\lambda > 0$ is denoted by R_λ and is defined as the Bochner integral (see e.g., [11])

$$R_\lambda g := \int_0^\infty e^{-\lambda s} S_s g \, ds, \quad g \in C_b(S). \tag{10}$$

By taking $\lambda = 0$ in (10), one obtains the *potential operator* which is denoted by $R_0 g$ (whenever it exists). Additional superscripts shall be used to identify different resolvents and potential operators.

2.2. **Fractional differential operators.** This section provides a quick summary of basic results concerning the classical Riemann-Liouville and Caputo fractional operators. For a detailed treatment refer, e.g., to [32]-[33] and references therein.

Classical *fractional differential operators* are defined in terms of both the standard differential operator (hereafter denoted by D^m, $m \in \mathbb{N}$) and the *Riemann-Liouville integral fractional operator* I^α_{a+} for any $a \in \mathbb{R} \cup \{-\infty\}$, defined by

$$I^\alpha_{a+} h(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} h(s) ds, \quad \alpha > 0, \, t > a.$$

For convention, I^0_{0+} refers to the identity operator.

The *left-sided Riemann-Liouville (RL) operator of order* $\beta > 0$ (shortly the RL derivative) is defined as the left-inverse of the corresponding integral operator ([7],

[33]). To be precise, if $\beta \in \mathbb{R}^+$ and $m = \lceil \beta \rceil$ ($\lceil \cdot \rceil$ denoting the ceiling function), then the RL derivative D^{β}_{a+} is defined by

$$D^{\beta}_{a+}h(t) := D^m I^{m-\beta}_{a+} h(t), \quad \beta > 0,\ \beta \notin \mathbb{N},\ t > a. \tag{11}$$

A sufficient condition for D^{β}_{a+} to be well-defined is to assume that $f \in A^m[a, \infty)$, i.e., its derivatives of order $m - 1$ are absolutely continuous (see, e.g., [7]).

An alternative fractional differential operator is the *left-sided Caputo operator* D^{β}_{a+*} (shortly the Caputo derivative):

$$D^{\beta}_{a+*}h(t) := I^{m-\beta}_{a+} D^m h(t), \quad \beta > 0,\ \beta \notin \mathbb{N}, \quad t > a, \tag{12}$$

for which h requires the absolute integrability of its derivatives of order $m = \lceil \beta \rceil$.

It can be proved (see, e.g., [7]) that both operators are related by the equality

$$D^{\beta}_{a+*}h(t) = D^{\beta}_{a+}[h - T_{m-1}[h; a]], \tag{13}$$

where $T_{m-1}[h; a]$ denotes the Taylor expansion of order $m - 1$, centered at a, for the function h. Hence, in general

$$D^{\beta}_{a+}h(t) := D^m I^{m-\beta}_{a+} h(t) \quad \neq \quad I^{m-\beta} D^m h(t) =: D^{\beta}_{a+*}h(t),$$

unless the function $h(t)$ along with its first $m - 1$ derivatives vanish at $a+$ (or as $t \to -\infty$ whenever $a = -\infty$).

Remark 2.1. The left-sided derivatives have a direct counterpart to the right-sided versions (see previous references for details). However, we work everywhere in this paper only with the left-sided operators and their generalizations. The right-sided version of these results is a straightforward modification.

2.3. **Special case:** $\beta \in (0, 1)$**.** In this paper we are mostly interested in fractional derivatives of order $\beta \in (0, 1)$. In this case, equations (11) and (12) become

$$D^{\beta}_{a+}h(t) = \frac{1}{\Gamma(\beta)} \frac{d}{dt} \left(\int_a^t (t - r)^{-\beta} h(r) dr \right), \quad t > a,$$

and

$$D^{\beta}_{a+*}h(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t - r)^{-\beta} h'(r) dr, \quad t > a,$$

respectively. Further, for smooth enough functions h (e.g., h in the Schwartz space), one obtains

$$D^{\beta}_{a+}h(t) = \frac{1}{\Gamma(-\beta)} \int_0^{t-a} \frac{h(t - r) - h(t)}{r^{1+\beta}} dr + \frac{h(t)}{\Gamma(1 - \beta)(t - a)^{\beta}}, \quad t > a, \tag{14}$$

and

$$D^{\beta}_{a+*}h(t) = \frac{1}{\Gamma(-\beta)} \int_0^{t-a} \frac{h(t - r) - h(t)}{r^{1+\beta}} dr + \frac{h(t) - h(a)}{\Gamma(1 - \beta)(t - a)^{\beta}}, \quad t > a, \tag{15}$$

for details see, e.g., Appendix in [22].

Thus, for these values of β the relationship between the Caputo and the RL derivates in (13) translates to

$$D^{\beta}_{a+*}h(t) = D^{\beta}_{a+}[\, h - h(a)](t) = D^{\beta}_{a+}h(t) - \frac{h(a)}{\Gamma(1 - \beta)(t - a)^{\beta}}, \quad t > a.$$

For smooth bounded integrable functions or functions that vanish at $t = a$ (or as $t \to -\infty$ whenever $a = -\infty$), the previous equality implies that the Caputo derivative and the RL derivative coincide. Its common value for $a=-\infty$, denoted by d^β/dt^β, is sometimes called *the generator form of the fractional derivative of order* $\beta\in(0,1)$, [29]:

$$\frac{d^\beta}{dt^\beta}h(t):=D^\beta_{-\infty+}h(t)=D^\beta_{-\infty+*}h(t)=\frac{1}{\Gamma(-\beta)}\int_0^\infty\frac{h(t-r)-h(t)}{r^{1+\beta}}dr. \tag{16}$$

2.4. **Stable subordinators.** Hereafter, we will always assume the existence of a probability space $(\Omega,\mathcal{G},\mathbb{P})$ such that all the stochastic processes of our interest are defined on it. Notation $\mathcal{F}^X_s$ means the completed natural filtration generated by a process $X=\{X(s)\}_{s\ge0}$, i.e. $\mathcal{F}^X_s:=\sigma(X_r\,:\,0\le r\le s)$.

A β-stable subordinator, for $\beta\in(0,1)$, is a real-valued stable Lévy process $T^\beta=\{T^\beta(s)\,:\,s\ge0\}$ started at 0 almost surely (a.s.) with independent increments, $T^\beta(s)-T^\beta(r)$ for any $0\le r<s$, having the same distribution as the r.v. $W_\beta((s-r)^{1/\beta},1)$, a totally skewed positive β-stable r.v. with scale parameter $\sigma=(s-r)^{1/\beta}$, see, e.g., [1], [34].

This process has nondecreasing sample paths a.s. and is time-homogeneous with respect to its natural filtration. Further, since the β-stable processes are self-similar with index $1/\beta$, the process $\{c^{1/\beta}T^\beta(s)\,:\,s\ge0\}$ has the same distribution as the process $\{T^\beta(cs)\,:\,s\ge0\}$ for any positive constant c. Consequently, the transition probabilities $p^\beta_s(t,E):=\mathbf{P}[T^\beta(s)\in E|T^\beta(0)=t]$ for any $E\in\mathcal{B}(\mathbb{R})$ (the Borel sets of $\mathbb{R}$) satisfy

$$p^\beta_s(t,E)=s^{-1/\beta}\int_E w_\beta(s^{-1/\beta}(r-t);1,1)dr,$$

where $w_\beta(\cdot;1,1)$ is the density of a standard $\beta-$stable r.v. $W_\beta(1,1)$. The density in this case equals

$$w_\beta(x;1,1)=\frac{1}{\pi}\Re\int_0^\infty\exp\left\{-iux-u^\beta\exp\left(-i\frac{\pi}{2}\beta\right)\right\}du,$$

where $\Re(z)$ means the real part of $z\in\mathbb{C}$ (see Theorem 2.2.1 in [37]).

The infinitesimal generator A^β of a β-stable subordinator is the generator of a jump-type Markov process of the form

$$A^\beta h(t)=\int_0^\infty(h(t+r)-h(t))\nu_\beta(dr),\quad h\in\mathfrak{D}_\beta, \tag{17}$$

with a domain $\mathfrak{D}_\beta$ and with the jump intensity given by the Lévy measure ν supported in $\mathbb{R}_+$:

$$\nu_\beta(dr)=\frac{\beta}{\Gamma(1-\beta)r^{1+\beta}}dr=-\frac{1}{\Gamma(-\beta)r^{1+\beta}}dr. \tag{18}$$

The last equality holds due to the identity $\Gamma(x)=(x-1)\Gamma(x-1)$.

We say that the process $T^{+\beta}=\left\{T^{+\beta}(s)\,:\,s\ge0\right\}$ is an *inverted* $\beta-$*stable subordinator* if $-T^{+\beta}$ is a $\beta-$stable subordinator with $\beta\in(0,1)$. Thus, $T^{+\beta}$ is a Markov process with non increasing sample paths a.s. and with the generator

$$A^{+\beta}h(t)=\int_0^\infty\left(h(t-r)-h(t)\right)\nu_\beta(dr).$$

Notice that the relation

$$w_\beta(-t;\sigma,1)=w_\beta(t;\sigma,-1),$$

implies that $T^{+\beta}(s)-T^{+\beta}(r)$ has the same distribution as the r.v. $W_\beta\left((s-r)^{1/\beta},-1\right)$. Hence, the transition probabilities $p_s^{+\beta}(t,E) := \mathbf{P}[T^{+\beta}(s)\in E|T^{+\beta}(0)=t]$ are given by

$$p_s^{+\beta}(t,E) = s^{-1/\beta}\int_E w_\beta(s^{-1/\beta}(t-r);1,1)dr, \quad E\in\mathcal{B}(\mathbb{R}). \tag{19}$$

3. Generalized fractional operators

Let T^β be a $\beta-$stable subordinator with the generator A^β and let $T^{+\beta}$ be the corresponding inverted $\beta-$stable subordinator with the generator $A^{+\beta}$. Then the operator $-d^\beta/dt^\beta$ in (16) coincides with $A^{+\beta}$, i.e.,

$$-\frac{d^\beta}{dt^\beta}h(t) = D^\beta_{-\infty+}h(t) = D^\beta_{-\infty+*}h(t) = A^{+\beta}h(t).$$

As was shown in [22], an analogous probabilistic interpretation of the Caputo operators D^β_{a+*} (resp. the RL operators D^β_{a+}), for any $a\in\mathbb{R}$ and $\beta\in(0,1)$, can be obtained by interrupting (resp. killing) $T^{+\beta}$ on an attempt to cross the boundary point $t=a$. Moreover, this interruption procedure naturally yields an extension of the Caputo and the RL fractional derivatives.

Namely, let $-D_+^{(\nu)}$ be the generator of a decreasing Feller process with values on the interval $(-\infty,b]$ and given by

$$-D_+^{(\nu)}h(t) = \int_0^\infty \left(h(t-r)-h(r)\right)\nu(t,r)dr, \quad t\le b, \tag{20}$$

where $\nu(t,r)$ satisfies the condition:

(H0) the function $\nu(t,r)$ is continuous as a function of two variables and continuously differentiable in the first variable. Furthermore,

$$\sup_t \int r\nu(t,r)dr<\infty, \quad \sup_t\int r\Big|\frac{\partial}{\partial t}\nu(t,r)\Big|dr<\infty,$$

and

$$\lim_{\delta\to0}\sup_t\int_{|r|\le\delta} r\nu(t,r)dr = 0.$$

Then, by Theorem 4.1 in [22], the process interrupted (and forced to land exactly at $t=a$) on the first attempt to cross the barrier point $t=a$ is a Feller process on $[a,b]$ and has the generator

$$-D_{a+*}^{(\nu)}h(t) = \int_0^{t-a}(h(t-r)-h(t))\nu(t,r)dr + (h(a)-h(t))\int_{t-a}^\infty \nu(t,r)dr, \tag{21}$$

with the invariant core $C^1[a,b]$ and a domain denoted by $\mathfrak{D}^{(\nu)}_{a+*}$. Note that, since the process is decreasing, the interruption procedure effectively means stopping the process at the boundary.

Moreover, if the process is also killed at the barrier point $t=a$ (meaning analytically to set $h(a)=0$) then the corresponding Feller sub-Markov process on $(a,b]$ has the generator

$$-D_{a+}^{(\nu)}h(t) := \int_0^{t-a}(h(t-r)-h(t))\nu(t,r)dr - h(t)\int_{t-a}^\infty \nu(t,r)dr, \tag{22}$$

with the invariant core $C^1_a[a,b]$ and a domain $\mathfrak{D}^{(\nu)}_{a+}$.

Definition 3.1. Let ν be a function satisfying condition (H0). The operator in (21) is called *the D_*-operator defined by ν* and is denoted by $-D_{a+*}^{(\nu)}$ (the sign $-$ is introduced to comply with the standard notation of fractional derivatives). Similarly, the operator in (22) is called *the D-operator defined by ν* and is denoted by $-D_{a+}^{(\nu)}$.

Remark 3.1. Since we are interested in the solutions to equations on finite intervals $[a, b]$, we applied the results in [22] to the case of stochastic processes taking values on $(-\infty, b]$.

By the standard theory of Feller processes, the domain of the generators $-D_{a+*}^{(\nu)}$ and $-D_{a+}^{(\nu)}$ coincides with the image of their corresponding *resolvent operators*, $R_\lambda^{a+*(\nu)}$ and $R_\lambda^{a+(\nu)}$ (for any $\lambda > 0$), respectively. Namely,

$$\mathfrak{D}_{a+*}^{(\nu)} = \left\{ u_g : u_g(t) = R_\lambda^{a+*(\nu)} g(t), \;\; g \in C[a, b] \right\},$$

and

$$\mathfrak{D}_{a+}^{(\nu)} = \left\{ u_g : u_g(t) = R_\lambda^{a+(\nu)} g(t), \; g \in C_a[a, b] \right\}.$$

Moreover, the images of the resolvent operators are independent of λ (for details see, e.g., [8]).

3.1. **Classical fractional derivatives.** For any $\beta \in (0, 1)$, the fractional operators D_{a+*}^{β} and D_{a+}^{β} are obtained as particular cases of D_*- and D-operators, respectively. Namely, on smooth enough functions h,

$$\text{if} \quad \nu(t, r) = -\frac{1}{\Gamma(-\beta) r^{1+\beta}}, \quad \beta \in (0, 1) \quad \text{then} \quad \begin{array}{rcl} -D_{a+*}^{(\nu)} h(t) & = & -D_{a+*}^{\beta} h(t), \\ -D_{a+}^{(\nu)} h(t) & = & -D_{a+}^{\beta} h(t). \end{array}$$

Thus, $-D_{a+*}^{\beta}$ is the generator of a Feller process on $[a, b]$ which is obtained by stopping an inverted β-stable subordinator $T^{+\beta}$ on an attempt to cross the boundary point $t = a$. The RL derivative $-D_{a+}^{\beta}$ is the generator obtained by killing $T^{+\beta}$ at the boundary, see [22].

Remark 3.2. The probabilistic interpretation of fractional derivatives of order $\beta \in (1, 2)$ was also analyzed in [22] but this case is not discussed here.

3.2. **Fractional derivatives of position-dependent order.** For a given function $\beta : \mathbb{R} \to (0, 1)$, define

$$\nu(t, r) = -\frac{1}{\Gamma(-\beta(t)) r^{1+\beta(t)}}. \tag{23}$$

Lemma 3.1. If $\beta : \mathbb{R} \to (0, 1)$ is a continuously differentiable function with values on a compact subset of $(0, 1)$, then the function defined in (23) satisfies condition (H0).

Proof. Follows by the smoothness of the function β in a compact set of $(0, 1)$. $\square$

Lemma 3.1 allows us to define D_*−operators via the function (23). They are denoted by $-D_{a+*}^{(\nu)} \equiv -D_{a+*}^{\beta(t)}$ and can be seen as generators of inverted *stable-like processes* [21] (with the jump density (23)) which are stopped on the attempt to cross the boundary point $t = a$. These operators are referred to as *Caputo-type operators of position dependent order*. Analogously, we define the RL type operators $-D_{a+}^{(\nu)} \equiv -D_{a+}^{\beta(t)}$.

Remark 3.3. Note that in the previous case the 'order' of the derivative depends only on t. Some other extensions can be made by taking the function ν depending on external variables. This case (which we shall analyze in detail in a forthcoming paper) allows us to deal with operators of the form

$$\Big(-\,{}_tD_{a+*}^{\beta(t,\mathbf{x})} - A_{\mathbf{x}}^{(t)}\Big)\, h(t,\mathbf{x}), \quad t \geq a,\ \mathbf{x} \in \mathbb{R}^d,$$

where $-\,{}_tD_{a+*}^{\beta(t,\mathbf{x})}$ denotes the Caputo type derivative acting on the variable t and depending on the variable $\mathbf{x}$ as a parameter; and $-\,A_{\mathbf{x}}^{(t)}$ denotes the generator of a Feller process acting on the variable $\mathbf{x}$ and depending on the variable t as a parameter.

3.3. **Multi-term fractional derivatives.** Other particular cases of the generalized fractional derivatives include the *multi-term fractional operators*:

$$-\ D_{a+*}^{(\nu)} h(t) = -\sum_{i=1}^{d} \omega_i(t) D_{a+*}^{\beta_i} h(t),$$

with ω_i and ν such that $\omega_i(\cdot) \geq 0$ for $i \in \{1,\ldots,d\}$ and

$$\nu(t,r) = -\sum_{i=1}^{d} \omega_i(t) \frac{1}{\Gamma(-\beta_i) r^{1+\beta_i}}.$$

Even more generally, they include the case

$$-\ D_{a+*}^{(\nu)} h(t) = -\int_{-\infty}^{\infty} \omega(s,t) D_{a+*}^{\beta(s,t)} h(t)\, \mu(ds), \tag{24}$$

with

$$\nu(t,r) = -\int_{-\infty}^{\infty} \omega(s,t) \frac{ds}{\Gamma(-\beta(s,t)) r^{1+\beta(s,t)}},$$

satisfying condition $(H0)$. Particular cases of (24) have been studied, e.g., in [26, 15].

4. Properties of the underlying stochastic processes

In this section we study some facts about the underlying stochastic processes generated by the operators $-D_{a+*}^{(\nu)}$ and $-D_{a+}^{(\nu)}$. These results are needed to obtain the explicit solutions to the linear equations involving D_*- and $D-$operators.

For a given function ν satisfying condition (H0) and for $t \in (a,b]$, the notation

$$T_t^{+(\nu)},\ -D^{+(\nu)},\ T_t^{a+*(\nu)},\ -\,D_{a+*}^{(\nu)},\ \ T_t^{a+(\nu)} \quad \text{and}\ -\,D_{a+}^{(\nu)},$$

means the following: $T_t^{+(\nu)} = \{T_t^{+(\nu)}(s) \,:\, s \geq 0\}$ is the decreasing Feller process (started at t) generated by $-D^{+(\nu)}$ as given in (20); $T_t^{a+*(\nu)} = \{T_t^{a+*(\nu)}(s) \,:\, s \geq 0\}$ stands for the Feller process generated by $-\,D_{a+*}^{(\nu)}$ with the invariant core $C^1[a,b]$; and $T_t^{a+(\nu)} = \{T_t^{a+(\nu)}(s) \,:\, s \geq 0\}$ denotes the Feller sub-Markov process generated by $-D_{a+}^{(\nu)}$ with the invariant core $C_a[a,b]$.

For $t \in [a,b]$, notation $\tau_a^{t,(\nu)}$ refers to the first time the process $T_t^{+(\nu)}$ (or the process $T_t^{a+*(\nu)}$) leaves $(a,b]$, i.e.

$$\tau_a^{t,(\nu)} := \inf\Big\{ s \geq 0 \,:\, T_t^{+(\nu)}(s) \notin (a,b] \Big\} = \inf\Big\{ s \geq 0 \,:\, T_t^{a+*(\nu)}(s) \notin (a,b] \Big\},$$

and, of course, $\tau_a^{a,(\nu)} = 0$. Note that $\tau_a^{t,(\nu)}$ is a stopping time with respect to $\mathcal{F}_s^{T_t^{+(\nu)}}$. Further, $\tau_a^{t,(\nu)}$ is also the first exit time from $(a,b]$ of the killed process $T_t^{a+(\nu)}$.

Let $p_s^{+(\nu)}(t,E)$ be the transition probabilities of $T_t^{+(\nu)}$ from t to $E \in \mathcal{B}(-\infty,b]$ during the interval $[0,s]$, that is, $p_s^{+(\nu)}(t,E) = \mathbf{P}\left[T_t^{+(\nu)}(s) \in E \,|\, T_t^{+(\nu)}(0) = t\right]$. If $p_s^{a+*(\nu)}(t,E)$ denotes the corresponding transition probabilities of $T_t^{a+*(\nu)}$, then

$$p_s^{a+*(\nu)}(r,E) = \begin{cases} p_s^{+(\nu)}(r,E), & E \in \mathcal{B}(a,b] \\ p_s^{+(\nu)}(r,(-\infty,a]), & E = \{a\}. \end{cases} \qquad r \in (a,b].$$

Since the process is stopped at a,

$$p_s^{a+*(\nu)}(t,[a,t]) = p_s^{+(\nu)}(t,(a,t]) + p_s^{a+*(\nu)}(t,\{a\}) = 1,$$

and $p_s^{a+*(\nu)}(a,E) = 1$ whenever $E \in \mathcal{B}[a,b]$ and $a \in E$. Furthermore, if $p_s^{a+(\nu)}(r,E)$ denotes the transition probabilities of $T_t^{a+(\nu)}$, then $p_s^{a+(\nu)}(r,E) = p_s^{+(\nu)}(r,E)$ for all $r \in (a,t]$ and $E \in \mathcal{B}(a,b]$. Moreover,

$$p_s^{a+(\nu)}(t,(a,t]) = p_s^{+(\nu)}(t,(a,t]) = 1 - p_s^{+(\nu)}(t,(-\infty,a]) \leq 1.$$

The previous implies that

$$p_s^{+(\nu)}(r,E) = p_s^{a+*(\nu)}(r,E) = p_s^{a+(\nu)}(r,E), \quad r \in (a,t],$$

on Borel sets E of $(a,b]$.

Sometimes we will use the following additional assumptions concerning the function ν and the transition probabilities of the underlying process $T^{+(\nu)}$:

(H1) There exist $\epsilon > 0$ and $\delta > 0$ such that $\nu(t,r) \geq \delta > 0$ for all t and all $|r| < \epsilon$.

(H2) The transition probabilities of the process $T^{+(\nu)}$ are absolutely continuous with respect to the Lebesgue measure (the transition densities are denoted by $p_s^{+(\nu)}(r,y)$).

(H3) The transition density function $p_s^{+(\nu)}(r,y)$ is continuously differentiable in the variable s.

Remark 4.1. There exist several criteria in terms of ν that ensure the validity of $(H2)$ and $(H3)$, see, e.g., [18].

Lemma 4.1. Suppose the conditions (H0)-(H1) hold for a function ν. Then, the stopping time $\tau_a^{t,(\nu)}$ is finite a.s. and $\mathbf{E}\left[\tau_a^{t,(\nu)}\right] < +\infty$ uniformly on $t \in (a,b]$. Further, the point a is regular in expectation for both operators $-D_{a+*}^{(\nu)}$ and $-D_{a+}^{(\nu)}$, i.e. $\mathbf{E}\left[\tau_a^{t,(\nu)}\right] \to 0$ as $t \downarrow a$.

Proof. The result follows by comparing the process $T_t^{+(\nu)}$ with a compound Poisson process with Lévy kernel $\nu(r) = \delta \mathbf{1}_{\{[0,\epsilon]\}}(r)$ for which the result holds. □

Observe now that, since $T_t^{a+*(\nu)}$ is a decreasing process, the equivalence between the events $\left\{\tau_a^{t,(\nu)} > s\right\}$ and $\left\{T_t^{a+*(\nu)}(s) > a\right\}$ for $t \in (a,b]$ and all $s > 0$ implies

$$\mathbf{P}\left[\tau_a^{t,(\nu)} > s\right] = \mathbf{P}\left[T_t^{a+*(\nu)}(s) > a\right] = \int_{(a,t]} p_s^{a+*(\nu)}(t,r)dr = 1 - \int_{-\infty}^{a} p_s^{+(\nu)}(t,r)dr,$$

yielding the following result.

Proposition 4.1. Suppose the conditions (H0)-(H3) hold. Then, the probability law of $\tau_a^{t,(\nu)}$, denoted by $\mu_a^{t,(\nu)}(ds)$, is absolutely continuous with respect to Lebesgue measure for $t \in (a,b]$ and its density $\mu_a^{t,(\nu)}(s)$ is given by

$$\mu_a^{t,(\nu)}(s) = \frac{\partial}{\partial s}\int_{-\infty}^{a} p_s^{+(\nu)}(t,r)dr = -\frac{\partial}{\partial s}\int_a^t p_s^{+(\nu)}(t,r)dr. \tag{25}$$

We also need the joint distribution of $T_t^{a+*(\nu)}(s)$ and $\tau_a^{t,(\nu)}$ for any $s \geq 0$. Notice that for any $a \leq r < t$,

$$\mathbf{P}\left[T_t^{a+*(\nu)}(s) > r,\, \tau_a^{t,(\nu)} > \xi\right] = \mathbf{P}\left[T_t^{a+*(\nu)}(s) > r,\, T_t^{a+*(\nu)}(\xi) > a\right].$$

Moreover, $\xi \leq s$ implies

$$\mathbf{P}\left[T_t^{a+*(\nu)}(s) > r,\, T_t^{a+*(\nu)}(\xi) > a\right] = \mathbf{P}\left[T_t^{a+*(\nu)}(s) > r\right],$$

whilst for $s < \xi$,

$$\begin{aligned}\mathbf{P}\left[T_t^{a+*(\nu)}(s) > r,\, T_t^{a+*(\nu)}(\xi) > a\right] &= \int_r^t p_s^{a+*(\nu)}(t,w)\left(\int_a^w p_{\xi-s}^{a+*(\nu)}(w,y)dy\right)dw \\ &= \int_r^t p_s^{+(\nu)}(t,w)\left(1 - \int_{-\infty}^a p_{\xi-s}^{+(\nu)}(w,y)dy\right)dw.\end{aligned}$$

Therefore, defining

$$\varphi_{s,a}^{t,(\nu)}(r,\xi) := \frac{\partial^2}{\partial\xi\partial r}\mathbf{P}\left[T_t^{a+*(\nu)}(s) \leq r,\, \tau_a^{t,(\nu)} \leq \xi\right],$$

yields the next result.

Proposition 4.2. Suppose the conditions (H0)-(H3) hold. Then, for any $s \geq 0$ and $t \in (a,b]$, the joint distribution of the pair $\left(T_t^{a+*(\nu)}(s), \tau_a^{t,(\nu)}\right)$, denoted by $\varphi_{s,a}^{t,(\nu)}(dr, d\xi)$, has the density $\varphi_{s,a}^{t,(\nu)}(r,\xi)$ given by

$$\varphi_{s,a}^{t,(\nu)}(r,\xi) = \mathbf{1}_{\{s<\xi\}} p_s^{+(\nu)}(t,r)\frac{\partial}{\partial\xi}\int_{-\infty}^a p_{\xi-s}^{+(\nu)}(r,y)dy, \qquad a \leq r < t. \tag{26}$$

Remark 4.2. Since the processes $T_t^{a+(\nu)}$, $T_t^{a+*(\nu)}$ and $T_t^{+(\nu)}$ coincide before the first exit time $\tau_a^{t,(\nu)}$, the equation (26) provides the joint density of $(T_t^{a+(\nu)}(s), \tau_a^{t,(\nu)})$ and $(T_t^{a+(\nu)}(s), \tau_a^{t,(\nu)})$ for any $s \geq 0$ and for $s < \xi$, respectively.

By definition of the generator of a Feller process (see, e.g., [11], [21]), if $S_s^{a+*(\nu)}$ is the semigroup of the stopped process $T_t^{a+*(\nu)}$, then $u \in \mathfrak{D}_{a+*}^{(\nu)}$ if, and only if,

$$-D_{a+*}^{(\nu)}u = \lim_{s\downarrow 0}\frac{S_s^{a+*(\nu)}u - u}{s},$$

where the limit is in the sense of the norm in $C[a,b]$. Analogously, if $S_s^{a+(\nu)}$ is the semigroup of the killed process $T_t^{a+(\nu)}$, then $u \in \mathfrak{D}_{a+}^{(\nu)}$ if, and only if,

$$-D_{a+}^{(\nu)}u = \lim_{s\downarrow 0}\frac{S_s^{a+(\nu)}u - u}{s},$$

where the limit is in the sense of the norm in $C_a[a,b]$.

Let us now introduce an operator which plays an important role to characterize the domain of the generators $-D_{a+*}^{(\nu)}$ and $-D_{a+}^{(\nu)}$.

For any $\lambda \geq 0$, define

$$M_{a,\lambda}^{+(\nu)} g(t) := \mathbf{E}\left[\int_0^{\tau_a^{t,(\nu)}} e^{-\lambda s} g\left(T_t^{+(\nu)}(s)\right) ds\right], \quad t \in (a,b], \tag{27}$$

for (non constant) functions $g \in B[a,b]$, and

$$M_{a,\lambda}^{+(\nu)} 1(t) := \mathbf{E}\left[\int_0^{\tau_a^{t,(\nu)}} e^{-\lambda s} ds\right], \quad t \in [a,b], \tag{28}$$

when $g(t) \equiv 1(t)$ (the constant function 1). Then

$$M_{a,\lambda}^{+(\nu)} \cdot 1(t) = \frac{1}{\lambda}\left(1 - \mathbf{E}\left[e^{-\lambda \tau_a^{t,(\nu)}}\right]\right), \tag{29}$$

implying

$$\mathbf{E}\left[e^{-\lambda \tau_a^{t,(\nu)}}\right] = 1 - \lambda M_{a,\lambda}^{+(\nu)} \cdot 1(t).$$

Further,

$$M_{a,\lambda}^{+(\nu)} c = c M_{a,\lambda}^{+(\nu)} \cdot 1(t), \quad t \in [a,b],$$

for any constant function equals to c (we shall use it mainly for the constant $g(a)$). Note that the stochastic continuity of the process $T_t^{a+*(\nu)}$ implies that $M_{a,\lambda}^{(\nu)} g(\cdot)$ is continuous on $(a,b]$. Moreover,

$$|M_{a,\lambda}^{+(\nu)} g(t)| \leq ||g|| \sup_{t\in[a,b]} \mathbf{E}\left[\tau_a^{t,(\nu)}\right].$$

Lemma 4.2. Suppose that ν satisfies conditions (H0)-(H3). Then

$$\mathbf{E}\left[e^{-\lambda \tau_a^{t,(\nu)}}\right] = \int_0^\infty e^{-\lambda s}\left(\frac{\partial}{\partial s}\int_{-\infty}^a p_s^{+(\nu)}(t,r)dr\right) ds, \quad t \in (a,b]; \tag{30}$$

and for any $g \in B[a,b]$

$$M_{a,\lambda}^{+(\nu)} g(t) = \int_0^{t-a} g(t-r)\int_0^\infty e^{-\lambda s} p_s^{+(\nu)}(t,t-r)\, ds\, dr, \quad t \in (a,b]. \tag{31}$$

Proof. Equality (30) follows directly by using the density function $\mu_a^{t,(\nu)}$ of the r.v. $\tau_a^{t,(\nu)}$ as given in (25). To prove (31), observe that Fubini's theorem allows one to rewrite $M_{a,\lambda}^{+(\nu)} g(t)$ as

$$M_{a,\lambda}^{+(\nu)} g(t) = \int_0^\infty e^{-\lambda s} \mathbf{E}\left[\mathbf{1}_{\{\tau_a^{t,(\nu)} > s\}} g\left(T_t^{+(\nu)}(s)\right)\right] ds.$$

Using (26), i.e., the joint density $\varphi^{t,(\nu)}_{s,a}(r,\xi)$ of the process $\left(T^{+(\nu)}_t(s), \tau^{t,(\nu)}_a\right)$ for $s<\xi$, yields

$$\begin{aligned}
M^{+(\nu)}_{a,\lambda} g(t) &= \int_0^\infty e^{-\lambda s}\left[\int_a^t \int_0^\infty \mathbf{1}_{\{\xi>s\}} g\left(r\right) \varphi^{t,(\nu)}_{s,a}(r,\xi)\, d\xi\, dr\right] ds \\
&= \int_0^\infty e^{-\lambda s}\int_a^t g(r) p^{+(\nu)}_s(t,r)\int_s^\infty \left(\frac{\partial}{\partial \xi}\int_{-\infty}^a p^{+(\nu)}_{\xi-s}(r,y)dy\right) d\xi\, dr\, ds \\
&= \int_0^\infty e^{-\lambda s}\int_a^t g(r) p^{+(\nu)}_s(t,r)\int_0^\infty \left(\frac{\partial}{\partial \gamma}\int_{-\infty}^a p^{+(\nu)}_{\gamma}(r,y)dy\right) d\gamma\, dr\, ds \\
&= \int_0^\infty e^{-\lambda s}\int_a^t g(r) p^{+(\nu)}_s(t,r)\int_0^\infty \mu^{r,(\nu)}_a(\gamma) d\gamma\, dr\, ds \\
&= \int_0^\infty e^{-\lambda s}\int_a^t g(r) p^{+(\nu)}_s(t,r)\, dr\, ds,
\end{aligned}$$

where the last equality holds as $\mu^{r,(\nu)}_a$ is the density function of the r.v. $\tau^{r,(\nu)}_a$. The result follows then by another interchange in the order of integration and by a change of variable. □

Remark 4.3. Equality (30) can be written as

$$\mathbf{E}\left[e^{-\lambda \tau^{t,(\nu)}_a}\right] = \lambda \int_0^\infty e^{-\lambda s}\left(\int_{-\infty}^a p^{+(\nu)}_s(t,r)dr\right) ds, \quad t\in(a,b], \tag{32}$$

which follows by integration by parts.

Let us now define the space of functions

$$\mathfrak{M}^{+(\nu)}_{\mathfrak{a},\lambda} := \left\{\ u:\ u(t) = cM^{+(\nu)}_{a,\lambda}\cdot 1(t) + d;\ \ t\in[a,b],\ c,d\in\mathbb{R}\right\}. \tag{33}$$

Lemma 4.3. Let ν be a function satisfying conditions (H0)-(H1). If $\lambda>0$, then

$$\mathfrak{D}^{(\nu)}_{a+*} = \left\{u_g:\ u_g(t) = g(a)\frac{1}{\lambda}\left(1-\lambda M^{+(\nu)}_{a,\lambda}\cdot 1(t)\right) + M^{+(\nu)}_{a,\lambda} g(t), \quad g\in C[a,b]\right\},$$

and

$$\mathfrak{D}^{(\nu)}_{a+} = \left\{w_g:\ w_g(t) = M^{+(\nu)}_{a,\lambda} g(t), \quad g\in C_a[a,b]\right\}.$$

Further, if ν also satisfies (H2)-(H3), then equalities (30) and (31) give explicit expressions for $\left(1-\lambda M^{+(\nu)}_{a,\lambda}\cdot 1(t)\right)$ and $M^{+(\nu)}_{a,\lambda} g(t)$, respectively.

Proof. Let us take any $u\in\mathfrak{D}^{(\nu)}_{a+*}$. Since $-D^{(\nu)}_{a+*}$ is the generator of a Feller process on $C[a,b]$, Theorem 1.1 in [8] implies the existence of a function $g\in C[a,b]$ such that $u = R^{a+*(\nu)}_\lambda g$. By definition of the resolvent and by Fubini's theorem

$$u(t) = \mathbf{E}\left[\int_0^\infty e^{-\lambda s} g(T^{a+*(\nu)}_t(s))ds\right] = \mathbf{E}\left[\left(\int_0^{\tau^{t,(\nu)}_a} + \int_{\tau^{t,(\nu)}_a}^\infty\right) e^{-\lambda s} g(T^{a+*(\nu)}_t(s))ds\right],$$

where $\mathbf{E}\left[\tau^{t,(\nu)}_a\right] < +\infty$ by Lemma 4.1.

Since the process is stopped at time $\tau^{t,(\nu)}_a$, it holds $g\left(T^{a+*(\nu)}_t(s)\right) = g(a)$ for all $s\geq \tau^{t,(\nu)}_a$. Moreover, before time $\tau^{t,(\nu)}_a$ the processes $T^{a+*(\nu)}_t$ and $T^{+(\nu)}_t$ coincide.

Therefore,

$$
\begin{aligned}
u(t) &= g(a)\mathbf{E}\left[\int_{\tau_a^{t,(\nu)}}^{\infty} e^{-\lambda s} ds\right] + \mathbf{E}\left[\int_0^{\tau_a^{t,(\nu)}} e^{-\lambda s} g(T_t^{+(\nu)}(s)) ds\right] \\
&= g(a)\left\{\frac{1}{\lambda} - \mathbf{E}\left[\int_0^{\tau_a^{t,(\nu)}} e^{-\lambda s} ds\right]\right\} + \mathbf{E}\left[\int_0^{\tau_a^{t,(\nu)}} e^{-\lambda s} g(T_t^{+(\nu)}(s)) ds\right] \\
&= g(a)\frac{1}{\lambda}\left(1 - \lambda M_{a,\lambda}^{+(\nu)} \cdot 1(t)\right) + M_{a,\lambda}^{+(\nu)} g(t), \qquad (34)
\end{aligned}
$$

as required. The characterization of the domain $\mathfrak{D}_{a+}^{(\nu)}$ is similar to the previous case. Take any $w \in \mathfrak{D}_{a+}^{(\nu)}$, then there exists a function $g \in C_a[a,b]$ such that $w = R_\lambda^{a+(\nu)} g$. Hence, a similar procedure yields (34) which implies (since $g(a) = 0$)

$$R_\lambda^{a+(\nu)} g(t) = M_{a,\lambda}^{+(\nu)} g(t).$$

Finally, observe that under assumptions (H2)-(H3), Lemma 4.2 holds. □

Let us now see how the resolvents (and hence the domains) of the stopped and killed processes are related to.

Lemma 4.4. Let ν be a function satisfying condition (H0). Suppose $\lambda > 0$ and $g \in C[a,b]$. Define $\tilde{g}(t) = g(t) - g(a)$, then

$$R_\lambda^{a+(\nu)} \tilde{g}(t) = R_\lambda^{a+*(\nu)} \tilde{g}(t) = R_\lambda^{a+*(\nu)} g(t) - g(a) R_\lambda^{a+*(\nu)} \cdot 1(t),$$

and

$$R_\lambda^{a+(\nu)} \tilde{g}(t) = M_{a,\lambda}^{+(\nu)} g(t) - g(a) M_{a,\lambda}^{+(\nu)} \cdot 1(t). \qquad (35)$$

In particular, $R_\lambda^{a+(\nu)} \tilde{g}(t)$ belongs to both domains $\mathfrak{D}_{a+*}^{(\nu)}$ and $\mathfrak{D}_{a+}^{(\nu)}$.

Proof. Follows directly from the linearity of the operators $R_\lambda^{a+*(\nu)}$ and $M_{a,\lambda}^{(\nu)}$, and by using that $\tilde{g}(a) = 0$. □

Remark 4.4. Let us stress that Lemma 4.4 implies that $M_{a,\lambda}^{+(\nu)} g$ coincides with the resolvent $R_\lambda^{a+(\nu)} g$ only when the function $g(a) = 0$. Hence, only in this case $M_{a,\lambda}^{+(\nu)} g$ belongs to the domain of both generators $-D_{a+*}^{(\nu)}$ and $-D_{a+}^{(\nu)}$.

It is worth noting that Theorem 1.1 in [8] also guarantees that for $g \in C[a,b]$ and $\lambda > 0$, the function $u_g := R_\lambda^{a+*(\nu)} g$ is the unique *solution in the domain of the generator* to

$$-D_{a+*}^{(\nu)} u(t) = \lambda u(t) - g(t), \quad t \in [a,b]. \qquad (36)$$

Similarly, if $g \in C_a[a,b]$, then $w = R_\lambda^{a+(\nu)} g$ is the unique *solution in the domain of the generator* to

$$-D_{a+}^{(\nu)} w(t) = \lambda w(t) - g(t), \quad t \in [a,b].$$

5. Equations involving $D-$ and D_*- operators

The probabilistic representation of solutions (in the generalized sense) to linear equations involving D- and D_*-operators are studied in this section.

5.1. **Linear equations involving D-operators.** Consider the problem of finding a continuous function u on $[a,b]$ satisfying

$$-D_{a+}^{(\nu)}w(t) = \lambda w(t) - g(t), \quad t \in [a,b], \qquad w(a) = w_a, \tag{37}$$

for $\lambda \geq 0$, $g \in B[a,b]$ and $w_a = 0$. Hereafter, we refer to (37) as the RL type problem $(-D_{a+}^{(\nu)}, \lambda, g, w_a)$, wherein we always assume $w_a = 0$. Similar notation is used in the linear equation with the corresponding Caputo type operator: $(-D_{a+*}^{(\nu)}, \lambda, g, w_a)$ for any $w_a \in \mathbb{R}$.

Definition 5.1. Let $g \in B[a,b]$ and $\lambda \geq 0$. A function $w \in C_a[a,b]$ is said to solve the RL type problem $(-D_{a+}^{(\nu)}, \lambda, g, 0)$ as

(i) a *solution in the domain of the generator* if w satisfies (37) and belongs to the domain of the generator $-D_{a+}^{(\nu)}$;
(ii) a *generalized solution* if for all sequence of functions $g_n \in C_a[a,b]$ such that $\sup_n ||g_n|| < \infty$ uniformly on n, and $\lim_{n\to\infty} g_n \to g$ a.e., it holds that $w(t) = \lim_{n\to\infty} w_n(t)$ for all $t \in [a,b]$, where w_n is the solution (in the domain of the generator) to the RL problem $(-D_{a+}^{(\nu)}, \lambda, g_n, 0)$.

Remark 5.1. From this definition it follows that if there exists a generalized solution, then this is unique.

Definition 5.2. The RL type equation (37) is *well-posed in the generalized sense* if it has a unique generalized solution.

Well-posedness result for the RL type linear equation.

Theorem 5.1. (**Case** $\lambda > 0$) Let ν be a function satisfying conditions (H0)-(H1) and assume $\lambda > 0$.

(i) If $g \in C_a[a,b]$, then the linear problem $(-D_{a+}^{(\nu)}, \lambda, g, 0)$ has a unique solution in the domain of the generator given by $w = R_\lambda^{a+(\nu)} g$ (the resolvent operator at λ).
(ii) For any $g \in B[a,b]$, the linear equation $(-D_{a+}^{(\nu)}, \lambda, g, 0)$ is well-posed in the generalized sense and the solution admits the stochastic representation

$$w(t) = \mathbf{E}\left[\int_0^{\tau_a^{t,(\nu)}} e^{-\lambda s} g\left(T_t^{+(\nu)}(s) ds\right)\right]. \tag{38}$$

Moreover, if additionally ν satisfies conditions (H2)-(H3), then

$$w(t) = \int_0^{t-a} g(t-r) \int_0^\infty e^{-\lambda s} p_s^{+(\nu)}(t, t-r)\, ds\; dr. \tag{39}$$

(iii) If $g \in C[a,b]$, then the solution to (37) belongs to $\mathfrak{D}_{a+}^{(\nu)} \oplus \mathfrak{M}_{a,\lambda}^{+(\nu)}$, the direct sum of the domain of the generator $-D_{a+}^{(\nu)}$ and the space defined in (33).

Proof. (i) Take $g \in C_a[a,b]$. Using the conditions $g(a) = 0$, $w(a) = 0$ and $\lambda > 0$, Theorem 1.1 in [8] implies directly that $w(t) = R_\lambda^{a+(\nu)} g(t)$ is the unique solution to (37) belonging to the domain of the generator. Moreover, Lemma 4.3 implies

$$w(t) = M_{a,\lambda}^{+(\nu)} g(t) = \mathbf{E}\left[\int_0^{\tau_a^{t,(\nu)}} e^{-\lambda s} g\left(T_t^{+(\nu)}(s)\right) ds\right].$$

(ii) Let us now take any function $g \in B[a,b]$. Since g does not necessarily belong to $C_a[a,b]$, the resolvent operator no longer provides a solution to (37). However, using Definition 5.1 we will see that there exists a unique *generalized solution*. To do this, take any sequence $g_n \in C_a[a,b]$ such that $\lim_{n\to\infty} g_n = g$ a.e. and $\sup_n ||g_n|| < \infty$ uniformly on n. The procedure consists in finding the generalized solution as a limit of solutions to the equations

$$-D_{a+}^{(\nu)} w_n(t) = \lambda w_n(t) - g_n(t), \quad t \in (a,b], \qquad w_n(a) = 0.$$

Since each $g_n \in C_a[a,b]$, the previous case guarantees the existence of a unique solution $w_n \in \mathfrak{D}_{a+}^{(\nu)}$ given by

$$w_n(t) = \mathbf{E}\left[\int_0^{\tau_a^{t,(\nu)}} e^{-\lambda s} g_n\left(T_t^{+(\nu)}(s)\right) ds\right].$$

Using that $||g_n||$ is uniformly bounded, the dominated convergence theorem (DCT) implies

$$\lim_{n\to\infty} w_n(t) = \mathbb{E}\left[\int_0^{\tau_a^{t,(\nu)}} e^{-\lambda s} g\left(T_t^{+(\nu)}(s)\right) ds\right] =: w(t).$$

Observe that the continuity of $w(t)$ on $(a,b]$ is a consequence of the continuity of the mapping $t \to \mathbf{E}\left[g\left(T_t^{+(\nu)}(s)\right)\right]$. The continuity at $t=a$ holds by the regularity in expectation of $\tau_a^{t,(\nu)}$ (a consequence of assumption (H1)). Therefore, $w \in C_a[a,b]$ is a generalized solution to the linear equation (37). Finally, the representation in (39) follows directly from Lemma 4.2.

(iii) To prove that $w \in \mathfrak{D}_{a+}^{(\nu)} \oplus \mathfrak{M}_{a,\lambda}^{(\nu)}$ whenever $g \in C[a,b]$, we use the equality (35) in Lemma 4.4 to obtain

$$M_{a,\lambda}^{+(\nu)} g(t) = R_\lambda^{a+(\nu)} \hat{g}(t) + g(a) M_{a,\lambda}^{+(\nu)} \cdot 1(t),$$

where $\hat{g}(t) = g(t) - g(a)$, implying the result. □

Theorem 5.2. (Case $\lambda = 0$) Theorem 5.1 with $\lambda = 0$ is valid for the equation

$$-D_{a+}^{(\nu)} w(t) = -g(t), \quad t \in (a,b]; \qquad w(a) = 0. \tag{40}$$

Proof. It follows from the same arguments used for $\lambda > 0$, so that we skip the details. Let us just notice that, since

$$|R_0^{a+(\nu)} g(t)| \le \mathbf{E}\left[\int_0^{\tau_a^{t,(\nu)}} |g(T_t^{a+(\nu)}(s))| ds\right] \le ||g|| \sup_{t\in(a,b]} \mathbf{E}\left[\tau_a^{t,(\nu)}\right],$$

Lemma 4.1 implies that the *potential operator* corresponding to $T_t^{a+(\nu)}$ is bounded. Thus, Theorem 1.1' in [8] ensures that the proof in 5.1 remains true for $\lambda > 0$ if one replaces the resolvent operator $R_\lambda^{a+(\nu)}$ by the potential operator $R_0^{a+(\nu)}$. Also observe that $w \in \mathfrak{D}_{a+}^{(\nu)} \oplus \mathfrak{M}_{a,0}^{+(\nu)}$ whenever $g \in C[a,b]$ as w can be written as

$$w(t) = R_0^{a+(\nu)} \tilde{g}(t) + g(a) \mathbf{E}\left[\int_0^{\tau_a^{t,(\nu)}} ds\right],$$

where $\tilde{g}(t) := g(t) - g(a)$, for all $t \in [a,b]$. □

5.2. **Linear equations involving D_*-operators.** Let $a \in \mathbb{R}$ and $\lambda \geq 0$. Consider the problem of finding a function $u \in C[a,b]$ solving

$$-D_{a+*}^{(\nu)}u(t) = \lambda u(t) - g(t), \quad t \in (a,b], \quad u(a) = u_a, \tag{41}$$

for a given function g on $[a,b]$.

If u belongs to the domain of the generator $-D_{a+*}^{(\nu)}$, then the linear equation (41) can be written in terms of the RL type operator $D_{a+}^{(\nu)}$ as follows. Define $w(t) := u(t) - u_a$ for all $t \in [a,b]$, then $-D_{a+*}^{(\nu)}w(t) = -D_{a+*}^{(\nu)}u(t)$ as $-D_{a+*}^{(\nu)}u_a = 0$. Setting $\tilde{g}(t) := g(t) - \lambda u_a$, it follows that

$$-D_{a+}^{(\nu)}w(t) = \lambda w(t) - \tilde{g}(t), \quad t \in (a,b] \quad w(a) = 0. \tag{42}$$

Hence, $u(t) = w(t) + u_a$ is a solution to the original problem if, and only if, w solves (42). The previous leads to the following definition.

Definition 5.3. Let $g \in B[a,b]$ and $\lambda \geq 0$. A function $u \in C[a,b]$ is said to solve the Caputo type problem $(-D_{a+*}^{(\nu)}, \lambda, g, u_a)$ as

(i) a *solution in the domain of the generator* if w satisfies (41) and belongs to the domain of the generator $-D_{a+*}^{(\nu)}$;

(ii) a *generalized solution* if $u(t) = u_a + w(t)$ for all $t \in [a,b]$, where w is the (possibly generalized) solution to the RL type problem $(-D_{a+}^{(\nu)}, \lambda, g - \lambda u_a, 0)$.

Remark 5.2. Definition 5.3, (ii), is given in terms of the RL type solution, but it can also be written in terms of the approximation of solutions belonging to the domain of the corresponding generator.

Definition 5.4. The Caputo type equation (41) is *well-posed in the generalized sense* if it has a unique generalized solution and it depends continuously on the initial condition.

Well-posedness result for the Caputo type linear equation.

Theorem 5.3. (Case $\lambda > 0$) Let ν be a function satisfying conditions (H0)-(H1) and suppose $\lambda > 0$.

(i) If $g \in C[a,b]$ and $g(a) = \lambda u_a$, then the linear equation $(-D_{a+*}^{(\nu)}, \lambda, g, u_a)$ has a unique solution in the domain of the generator given by $u = R_\lambda^{a+*(\nu)} g$ (the resolvent operator at λ).

(ii) For any $g \in B[a,b]$ and $u_a \in \mathbb{R}$, the linear equation $(-D_{a+*}^{(\nu)}, \lambda, g, u_a)$ is well-posed in the generalized sense and the solution admits the stochastic representation

$$u(t) = u_a \mathbf{E}\left[e^{-\lambda \tau_a^{t,(\nu)}}\right] + \mathbf{E}\left[\int_0^{\tau_a^{t,(\nu)}} e^{-\lambda s} g\left(T_t^{+(\nu)}(s) ds\right)\right]. \tag{43}$$

Moreover, if additionally ν satisfies conditions (H2)-(H3), then

$$u(t) = u_a \int_0^\infty e^{-\lambda s} \mu_a^{t,(\nu)}(s) ds + \int_0^{t-a} g(t-r) \int_0^\infty e^{-\lambda s} p_s^{+(\nu)}(t, t-r)\, ds\, dr, \tag{44}$$

where $\mu_a^{t,(\nu)}(s)$ denotes the density function of the r.v. $\tau_a^{t,(\nu)}$.

(iii) If $g \in C[a,b]$, then the solution to (41) belongs to $\mathfrak{D}_{a+*}^{(\nu)} \oplus \mathfrak{M}_{a,\lambda}^{(\nu)}$, the direct sum of the domain of the generator $-D_{a+*}^{(\nu)}$ and the space defined in (33).

Proof. (i) Since $-D_{a+*}^{(\nu)}$ is the generator of a Feller process on $C[a,b]$, for any $g \in C[a,b]$ the function $u(t) = R_\lambda^{a+*(\nu)} g(t)$ is the unique solution to (36) which belongs to the domain of the generator. A simple calculation shows that $u(a) = R_\lambda^{a+*(\nu)} g(a) = g(a)/\lambda$. Hence, condition $g(a) = \lambda u_a$ ensures that u satisfies the boundary condition in (41), as required.

(ii) By Definition 5.3, u is the generalized solution if $u(t) = w(t) + u_a$, where w is the solution to the RL type problem $(-D_{a+}^{(\nu)}, \lambda, g(t) - \lambda u_a, 0)$ whose well-posedness is guaranteed by Theorem 5.1.

Moreover, the equality (38) and the linearity of $M_{a,\lambda}^{+(\nu)}$ yield

$$w(t) = M_{a,\lambda}^{+(\nu)} g(t) - \lambda u_a M_{a,\lambda}^{+(\nu)} \cdot 1(t) = M_{a,\lambda}^{+(\nu)} g(t) - u_a \left(1 - \mathbf{E}\left[e^{-\lambda \tau_a^{t,(\nu)}}\right]\right),$$

where the last equality holds due to equation (29). Thus, (43) is obtained by plugging the previous expression into $u(t) = w(t) + u_a$. This representation implies directly the continuity on the initial condition u_a, as required for the well-posedness. Finally, the explicit solution (44) follows from Lemma 4.2.

(iii) Assume now that $g \in C[a,b]$. Since $u(t) = M_{a,\lambda}^{+(\nu)} g(t) - \lambda u_a M_{a,\lambda}^{(\nu)} \cdot 1(t) + u_a$, by linearity one can rewrite it as

$$\begin{aligned} u(t) &= M_{a,\lambda}^{+(\nu)} [g - g(a) + g(a)](t) - \lambda u_a M_{a,\lambda}^{+(\nu)} \cdot 1(t) + u_a \\ &= R_\lambda^{a+*(\nu)} [g - g(a)](t) + [g(a) - \lambda u_a] M_{a,\lambda}^{+(\nu)} \cdot 1(t) + u_a. \end{aligned}$$

We then conclude that $u \in \mathfrak{D}_{a+*}^{(\nu)} \oplus \mathfrak{M}_{a,\lambda}^{+(\nu)}$ since

$$R_\lambda^{a+*(\nu)} [g - g(a)] \ \in \ \mathfrak{D}_{a+*}^{(\nu)},$$

and

$$[g(a) + \lambda u_a] M_{a,\lambda}^{+(\nu)} \cdot 1(t) + u_a \ \in \ \mathfrak{M}_{a,\lambda}^{+(\nu)}.$$

□

Theorem 5.4. (**Case** $\lambda = 0$) All assertions in Theorem 5.3 with $\lambda = 0$ hold for the equation

$$-D_{a+*}^{(\nu)} u(t) = -g(t), \quad t \in (a,b], \qquad u(a) = u_a. \tag{45}$$

Proof. Since the problem (45) rewrites as

$$-D_{a+}^{(\nu)} w(t) = \tilde{g}(t), \quad t \in (a,b], \qquad w(a) = 0, \tag{46}$$

with $w(t) = u(t) - u_a$ and $\tilde{g}(t) = g(t)$, Theorem 5.2 gives the potential operator $R_0^{a+(\nu)} \tilde{g}(t)$ as the solution to (46) for any $\tilde{g} \in C_a[a,b]$. Hence, the unique generalized solution to (45) is given by $u(t) = u_a + \lim_{n\to\infty} R_0^{a+(\nu)} \tilde{g}_n(t)$ for any sequence $\tilde{g}_n$ as in Definition 5.1. Consequently, the same arguments used for $\lambda > 0$ remain valid. □

5.3. **Fractional linear equations involving Caputo derivatives.** Since Caputo derivatives are particular cases of the D_*−operators, the solution to fractional linear equations with the Caputo derivative D_{a+*}^{β}, for $\beta \in (0,1)$, is obtained by a direct application of the previous results. Namely, Theorem 5.3 implies that the problem

$$D_{a+*}^{\beta} u(t) = -\lambda u(t) + g(t), \quad t \in (a,b], \quad u(a) = u_a \in \mathbb{R},$$

for a given $g \in B[a,b]$ and $\lambda > 0$, has a unique generalized solution given by

$$u(t) = u_a \mathbf{E}\left[e^{-\lambda \tau_a^{t,\beta}}\right] + \mathbf{E}\left[\int_0^{\tau_a^{t,\beta}} e^{-\lambda s} g(T_t^{+\beta}(s))ds\right], \tag{47}$$

where $T_t^{+\beta}$ is the inverted $\beta-$stable subordinator started at t. Moreover, substituting (19) into formula (44) yields

$$\begin{aligned} u(t) = \; & u_a \frac{1}{\beta}(t-a)\int_0^\infty e^{-\lambda s}\left(s^{-\frac{1}{\beta}-1} w_\beta\left((t-a)s^{-1/\beta};1,1\right)\right) ds + \\ & + \int_0^{t-a} g(t-r)\left(r^{\beta-1}\int_0^\infty \exp\left\{-\lambda s r^\beta\right\} s^{-1/\beta} w_\beta(s^{-1/\beta};1,1)ds\right) dr. \end{aligned} \tag{48}$$

Further, if $g(a) = \lambda u_a$ and $g \in C[a,b]$, then u belongs to the domain of the generator $-D_{a+*}^{(\nu)}$.

Corollary 5.1. Let $t \in (a,b]$ and $\lambda > 0$. Then the Laplace transform of the first exit time from $(a,b]$ for the inverted $\beta-$stable subordinator started at t is given by

$$\mathbf{E}[e^{-\lambda \tau_a^{t,\beta}}] = E_\beta(-\lambda(t-a)^\beta),$$

with E_β denoting the Mittag-Leffler function (see Preliminaries). Further,

$$E_\beta(-\lambda(t-a)^\beta) = \frac{1}{\beta}(t-a)\int_0^\infty \exp(-\lambda s)\, s^{-\frac{1}{\beta}-1} w_\beta\left((t-a)s^{-1/\beta};1,1\right) ds.$$

Proof. By uniqueness, it follows as a consequence of formulas (5) and (48) with $g \equiv 0$ and $u_a = 1$. □

Remark 5.3. Alternatively, Corollary 5.1 can also be obtained by using the identity (see [37], Theorem 2.10.2)

$$\beta E_\beta(-u) = \int_0^\infty \exp(-uy) y^{-1-1/\beta} w_\beta(y^{-1/\beta};1,1)dy. \tag{49}$$

Moreover, this identity also shows that (48) coincides with the well known solution given in (5).

6. Mixed linear equations

In this section we study linear equations involving both the RL type and the Caputo type operators. The general setting will be explained first in $\mathbb{R}^d$, and then we shall restrict ourselves to the simplest 2-dimensional case. This is done to avoid cumbersome calculations which nevertheless can be extended straightforward from the simple case analyzed here.

Let $\mathbf{a} = (a_1,\ldots,a_d), \mathbf{b} = (b_1,\ldots,b_d) \in \mathbb{R}^d$ such that $\mathbf{a} < \mathbf{b}$. The Euclidean space $\mathbb{R}^d$ is assumed to be equipped with its natural partial order, the Pareto order, i.e. $\mathbf{a} < \mathbf{b}$ means $a_i < b_i$ for all $i \in \{1,\ldots,d\}$. Notation $[\mathbf{a},\mathbf{b}]$ denotes the cartesian product $[a_1,b_1]\times\cdots\times[a_d,b_d]$ and $\mathbf{t} \in [\mathbf{a},\mathbf{b}]$ means $t_i \in [a_i,b_1]$ for all $i \in \{1,\ldots,d\}$. Let us denote by $B[\mathbf{a},\mathbf{b}]$ and $C[\mathbf{a},\mathbf{b}]$ the space of bounded Borel measurable functions and continuous functions on $[\mathbf{a},\mathbf{b}]$, respectively, and by $C^1[\mathbf{a},\mathbf{b}]$ the space of continuous functions on $[\mathbf{a},\mathbf{b}]$ with continuous first order partial derivatives up to the boundary on $[\mathbf{a},\mathbf{b}]$. Similar notation is used for $(\mathbf{a},\mathbf{b}]$ and $(-\infty,\mathbf{b}]$.

Notation $\mathbf{r}^{a_i}$ means a vector $\mathbf{r} = (r_1, \ldots, r_d) \in [\mathbf{a}, \mathbf{b}]$ having $r_i = a_i$ as its ith-coordinate. Since all the processes considered here have decreasing sample paths, we are only interested in the boundary of $(\mathbf{a}, \mathbf{b}]$ given by $\mathbf{r}^{a_i}$ for all $i \in \{1, \ldots, d\}$. This subset is denoted by

$$\partial_{\mathbf{a}}(\mathbf{a}, \mathbf{b}] := \bigcup_{i=1}^{d} \left\{ \mathbf{r} \in [\mathbf{a}, \mathbf{b}] \, : \, \mathbf{r} = \mathbf{r}^{a_i} \right\}.$$

The space of continuous functions on $[\mathbf{a}, \mathbf{b}]$ vanishing at the boundary $\partial_{\mathbf{a}}(\mathbf{a}, \mathbf{b}]$ is denoted by $C_{\mathbf{a}}[\mathbf{a}, \mathbf{b}]$.

For $i \in \{1, \ldots, d\}$, let ν_i be a function satisfying conditions (H0)-(H1) and let $\mathbf{t} \in (\mathbf{a}, \mathbf{b}]$. The operator $-{}_{t_i}D^{(\nu_i)}_{a_i+}$ represents the D-operator defined by ν_i acting (independently of the other operators) on the variable t_i. For notational convenience set $\nu = (\nu_1, \ldots, \nu_d)$ and define the *mixed* $D-$*operator* associated with the vector ν by

$$-\, \mathbf{D}^{(\nu)}_{\mathbf{a}+} := -\sum_{i=1}^{d} \; {}_{t_i}D^{(\nu_i)}_{a_i+}, \tag{50}$$

Hence, the operator $-\, \mathbf{D}^{(\nu)}_{\mathbf{a}+}$ is a sum of RL type operators each one acting on a different variable. Analogously, we define the mixed operators $-\, \mathbf{D}^{+(\nu)}$ and $-\, \mathbf{D}^{(\nu)}_{\mathbf{a}+*}$ by using $D^{+(\nu_i)}$ and ${}_{t_i}D^{(\nu_i)}_{a_i+*}$, respectively.

Consider the RL type linear equation

$$\begin{aligned} -\, \mathbf{D}^{(\nu)}_{\mathbf{a}+} w(\mathbf{t}) &= \lambda w(\mathbf{t}) - g(\mathbf{t}), && \mathbf{t} \in (\mathbf{a}, \mathbf{b}], \\ w(\mathbf{t}) &= 0, && \mathbf{t} \in \partial_{\mathbf{a}}(\mathbf{a}, \mathbf{b}], \end{aligned} \tag{51}$$

for a given function $g \in B[\mathbf{a}, \mathbf{b}]$ and $\lambda \geq 0$.

The operator $-\mathbf{D}^{(\nu)}_{\mathbf{a}+}$ can be thought of as the generator of a Feller process on $(\mathbf{a}, \mathbf{b}]$ obtained by killing the process generated by $-\, \mathbf{D}^{+(\nu)}$ on an attempt to cross the boundary $\partial_{\mathbf{a}}(\mathbf{a}, \mathbf{b}]$. The killed process (started at $\mathbf{t}$) is denoted by $\mathbf{T}^{\mathbf{a}+(\nu)}_{\mathbf{t}} = \left\{ \mathbf{T}^{\mathbf{a}+(\nu)}_{\mathbf{t}}(s) : s \geq 0 \right\}$.

Due to the independence, the Lie-Trotter theorem [11, 21] implies that

$$\mathbf{T}^{\mathbf{a}+(\nu)}_{\mathbf{t}}(s) = \left(T^{a_1+(\nu_1)}_{t_1}(s), \ldots, T^{a_d+(\nu_d)}_{t_d}(s) \right),$$

wherein each coordinate $T^{a_i+(\nu_i)}_{t_i}$ is an independent process generated by $-{}_{t_i}D^{(\nu_i)}_{a_i+}$. Let $\mathbf{S}^{\mathbf{a}+(\nu)}_{s}$ denote the semigroup of the process $\mathbf{T}^{\mathbf{a}+(\nu)}_{\mathbf{t}}$ and let $\hat{\mathfrak{D}}^{(\nu)}_{\mathbf{a}+}$ be the domain of its generator. Then, $u \in \hat{\mathfrak{D}}^{(\nu)}_{\mathbf{a}+}$ if, and only if, the limit

$$-\, \mathbf{D}^{(\nu)}_{\mathbf{a}+} u(\mathbf{t}) = \lim_{s \to 0} \frac{\mathbf{S}^{\mathbf{a}+(\nu)}_{s} u(\mathbf{t}) - u(\mathbf{t})}{s},$$

exists in the norm of $C_{\mathbf{a}}[\mathbf{a}, \mathbf{b}]$.

To solve (51), let us introduce some definitions which extend those used in the one-dimensional case.

Definition 6.1. Let $g \in B[\mathbf{a}, \mathbf{b}]$, and $\lambda \geq 0$. A function $w \in C_{\mathbf{a}}[\mathbf{a}, \mathbf{b}]$ is said to solve the RL type problem $(-\, \mathbf{D}^{(\nu)}_{\mathbf{a}+}, \lambda, g, 0)$ as

(i) a *solution in the domain of the generator* if w satisfies (51) and belongs to $\hat{\mathfrak{D}}^{(\nu)}_{\mathbf{a}+}$;

(ii) a *generalized solution* if for all sequence of functions $g_n \in C_{\mathbf{a}}[\mathbf{a}, \mathbf{b}]$ such that $\sup_n ||g_n|| < \infty$ uniformly on n, and $g_n \to g$ a.e., it holds that $w(\mathbf{t}) = \lim_{n\to\infty} w_n(\mathbf{t})$ for all $\mathbf{t} \in [\mathbf{a}, \mathbf{b}]$, where w_n is the solution (in the domain of the generator) to the RL type problem $(-\mathbf{D}^{(\nu)}_{\mathbf{a}+}, \lambda, g_n, 0)$.

Remark 6.1. By definition, if there exists a generalized solution, then this is unique.

For the sake of transparency, hereafter we restrict ourselves to the analysis for $d = 2$ and $\mathbf{a} = \mathbf{0}$. Namely, let $\mathbf{t} = (t_1, t_2)$ and $\mathbf{b} = (b_1, b_2)$ in $\mathbb{R}^2$ with $\mathbf{t} \in [\mathbf{0}, \mathbf{b}]$. Consider the equation

$$\begin{aligned} -\ {}_{t_1}D^{(\nu_1)}_{0+} w(t_1, t_2) -\ {}_{t_2}D^{(\nu_2)}_{0+} w(t_1, t_2) &= \lambda w(t_1, t_2) - g(t_1, t_2), \\ w(0, t_2) = w(t_1, 0) &= 0, \end{aligned}$$

where $t_i \in (0, b_i]$ for $i \in \{1, 2\}$.

Let $p^{+(\nu_i)}_s(t_i, r)$ (resp. $p^{0+(\nu_i)}_s(t_i, r)$) denote the transition density function of the process $T^{+(\nu_i)}_{t_i}$ (resp. $T^{0+(\nu_i)}_{t_i}$) during the interval $[0, s]$. If $\tau^{t_i,(\nu_i)}_0$ is the first exit time from $(0, b_i]$ of the process $T^{+(\nu_i)}_{t_i}$ (started at t_i), then the first exit time from $(\mathbf{0}, \mathbf{b}] = (0, b_1] \times (0, b_2]$ of the process $\mathbf{T}^{+(\nu)}_{\mathbf{t}}$, denoted by $\tau^{\mathbf{t},(\nu)}_{\mathbf{0}}$, equals

$$\tau^{\mathbf{t},(\nu)}_{\mathbf{0}} = \min \left\{ \ \tau^{t_i, \nu_i}_0 \ : \ i \in \{1, 2\} \right\}.$$

Due to the independence between the coordinates of the process $\mathbf{T}^{+(\nu)}_{\mathbf{t}}$, its transition density function, denoted by $\mathbf{p}^{+(\nu)}_s(\mathbf{t}, \mathbf{r})$, satisfies

$$\mathbf{p}^{+(\nu)}_s(\mathbf{t}, \mathbf{r}) = \prod_{i=1}^{2} p^{+(\nu_i)}_s(t_i, r_i),$$

yielding the following result.

Lemma 6.1. Let $\mathbf{t} = (t_1, t_2) \in (0, b_1] \times (0, b_2]$. Suppose (H0)-(H1) hold for both functions ν_1 and ν_2. Then,

(i) The boundary points $(0, t_2) \in \mathbb{R}^2$ for all $t_2 \in [0, b_2)$, and $(t_1, 0) \in \mathbb{R}^2$ for all $t_1 \in [0, b_1)$, are regular in expectation for both operators $-\mathbf{D}^{(\nu)}_{\mathbf{0}+}$ and $-\mathbf{D}^{(\nu)}_{\mathbf{0}+*}$. Moreover, $\mathbf{E}\left[\tau^{\mathbf{t},(\nu)}_{\mathbf{0}}\right] < +\infty$ uniformly on $\mathbf{t}$.

(ii) If additionally each ν_i satisfies assumptions (H2)-(H3) and $\mu^{\mathbf{t},(\nu)}_{\mathbf{0}}(ds)$ denotes the probability law of $\tau^{\mathbf{t},(\nu)}_{\mathbf{0}}$, then its density function $\mu^{\mathbf{t},(\nu)}_{\mathbf{0}}(s)$ is given by

$$\mu^{\mathbf{t},(\nu)}_{\mathbf{0}}(s) \ = \ \mu^{t_1,(\nu_1)}_0(s) \int_0^{t_2} p^{+(\nu_2)}_s(t_2, r) + \mu^{t_2,(\nu_2)}_0(s) \int_0^{t_1} p^{+(\nu_1)}_s(t_1, r), \quad s \geq 0.$$

(iii) Further, assuming again that each ν_i also satisfies (H2)-(H3), the joint distribution $\varphi^{\mathbf{t},(\nu)}_{s,\mathbf{a}}(d\mathbf{r}, d\xi)$ of the pair $(\mathbf{T}^{\mathbf{0}+(\nu)}_{\mathbf{t}}(s), \tau^{\mathbf{t},(\nu)}_{\mathbf{0}})$ has the density

$$\begin{aligned} \varphi^{\mathbf{t},(\nu)}_{s,\mathbf{0}}(\mathbf{r}, \xi) \ &= \varphi^{t_2,(\nu_2)}_{s,0}(r_2, \xi) p^{+(\nu_1)}_s(t_1, r_1) \int_0^{r_1} p^{+(\nu_1)}_{\xi-s}(r_1, y) dy \ + \\ &+ \varphi^{t_1,(\nu_1)}_{s,0}(r_1, \xi) p^{+(\nu_2)}_s(t_2, r_2) \int_0^{r_2} p^{+(\nu_2)}_{\xi-s}(r_2, y) dy, \end{aligned}$$

for $0 \leq s < \xi$ and $\mathbf{r} = (r_1, r_2) \in (\mathbf{0}, \mathbf{t}]$.

Proof. (i) The regularity in expectation of the boundary $\partial_{\mathbf{0}}(\mathbf{0},\mathbf{b}]$ is a consequence of assumption (H1) and the Lyapunov method (same reasoning as in [22]) applied to the Lyapunov function

$$h_\omega(t_1,t_2)=t_1^{\omega_1}t_2^{\omega_2},\quad \omega_1,\omega_2\in(0,1).$$

The finite expectation of $\tau_{\mathbf{0}}^{\mathbf{t},(\nu)}$ is a consequence of the finite expectation of each $\tau_0^{t_i,(\nu_i)}$.

(ii) This is a generalization of Proposition 4.1 and follows directly by differentiating

$$\mathbf{P}\left[\tau_{\mathbf{0}}^{\mathbf{t},(\nu)}>s\right]=\mathbf{P}\left[\tau_0^{t_1,(\nu_1)}>s\right]\mathbf{P}\left[\tau_0^{t_2,(\nu_2)}>s\right],$$

with respect to s. Notice the use of the independence assumption in the previous equality.

(iii) This is a generalization of Proposition 4.2 and is obtained by differentiating

$$\mathbf{P}\left[\mathbf{T}_{\mathbf{t}}^{\mathbf{0}+(\nu)}(s)>\mathbf{r},\tau_{\mathbf{0}}^{\mathbf{t},(\nu)}>\xi\right]=\prod_{i=1}^{2}\mathbf{P}\left[T_{t_i}^{0+(\nu_i)}(s)>r_i,T_{t_i}^{0+(\nu_i)}(\xi)>0\right],$$

with respect to r_1,r_2 and ξ. □

Let us now generalize the definitions given in (27) and (28). For $\lambda\geq 0$ and $g\in B[\mathbf{0},\mathbf{b}]$ define

$$\mathbf{M}_{\mathbf{0},\lambda}^{+(\nu)}g(\mathbf{t}):=\mathbf{E}\left[\int_0^{\tau_{\mathbf{0}}^{\mathbf{t},(\nu)}}e^{-\lambda s}g(\mathbf{T}_{\mathbf{t}}^{+(\nu)}(s))ds\right],\quad \mathbf{t}\in(\mathbf{0},\mathbf{b}],$$

and

$$\mathbf{M}_{\mathbf{0},\lambda}^{+(\nu)}1(\mathbf{t}):=\mathbf{E}\left[\int_0^{\tau_{\mathbf{0}}^{\mathbf{t},(\nu)}}e^{-\lambda s}ds\right],\quad \mathbf{t}\in[\mathbf{0},\mathbf{b}].$$

Note that $\mathbf{M}_{\mathbf{0},\lambda}^{+(\nu)}g(\cdot)$ is continuous on $(\mathbf{0},\mathbf{b}]$ and $|\mathbf{M}_{\mathbf{0},\lambda}^{+(\nu)}g(\mathbf{t})|\leq||g||\sup_{\mathbf{t}\in[\mathbf{0},\mathbf{b}]}\mathbf{E}\left[\tau_{\mathbf{0}}^{\mathbf{t},(\nu)}\right]$. Moreover,

$$\mathbf{M}_{\mathbf{0},\lambda}^{+(\nu)}\cdot 1(\mathbf{t})=\frac{1}{\lambda}\left(1-\mathbf{E}\left[e^{-\lambda\tau_{\mathbf{0}}^{\mathbf{t},(\nu)}}\right]\right),$$

implying

$$\mathbf{E}\left[e^{-\lambda\tau_{\mathbf{0}}^{\mathbf{t},(\nu)}}\right]=1-\lambda\mathbf{M}_{\mathbf{0},\lambda}^{+(\nu)}\cdot 1(\mathbf{t}),$$

and yielding the next generalization of Lemma 4.2.

Lemma 6.2. Let $\mathbf{t}=(t_1,t_2)\in(\mathbf{0},\mathbf{b}]$ and $\lambda>0$. Suppose that ν_i satisfies conditions (H0)-(H1) for $i\in\{1,2\}$. Then,

$$\mathbf{E}\left[e^{-\lambda\tau_{\mathbf{0}}^{\mathbf{t},(\nu)}}\right]=\mathbf{E}\left[e^{-\lambda\tau_0^{t_1,(\nu_1)}}1_{\left\{\tau_0^{t_1,(\nu_1)}<\tau_0^{t_2,(\nu_2)}\right\}}\right]+\mathbf{E}\left[e^{-\lambda\tau_0^{t_2,(\nu_2)}}1_{\left\{\tau_0^{t_2,(\nu_2)}<\tau_0^{t_1,(\nu_1)}\right\}}\right].$$

If additionally ν_i satisfies (H2)-(H3) for $i\in\{1,2\}$, then

$$\mathbf{E}\left[e^{-\lambda\tau_{\mathbf{0}}^{\mathbf{t},(\nu)}}\right]=\int_0^\infty e^{-\lambda s}\left(\mu_0^{t_1,(\nu_1)}(s)\int_0^{t_2}p_s^{+(\nu_2)}(t_2,r)\right)ds+ \\ +\int_0^\infty e^{-\lambda s}\left(\mu_0^{t_2,(\nu_2)}(s)\int_0^{t_1}p_s^{+(\nu_1)}(t_1,r)\right)ds.$$

Further,

$$\mathbf{M}^{+(\nu)}_{\mathbf{0},\lambda} g(\mathbf{t}) = \int_0^{t_1}\int_0^{t_2} g(t_1-r_1,\, t_2-r_2)\int_0^{\infty} e^{-\lambda s} p_s^{+(\nu_1)}(t_1, t_1-r_1) p_s^{+(\nu_2)}(t_2, t_2-r_2)\, ds\, dr_2\, dr_1. \tag{52}$$

Proof. Similar to the proof of Lemma 4.2 but using the density function of the r.v. $\tau_{\mathbf{0}}^{\mathbf{t},(\nu)}$ and the joint distribution of the pair $(\mathbf{T}^{\mathbf{0}+(\nu)}_{\mathbf{t}}(s), \tau_{\mathbf{0}}^{\mathbf{t},(\nu)})$ both given in Lemma 6.1. □

Well-posedness result for the RL type linear equation.

Theorem 6.1. (**Case** $\lambda > 0$) Let $\nu = (\nu_1, \nu_2)$ be a vector such that each ν_i is a function satisfying conditions (H0)-(H1). Suppose $\lambda > 0$ and $\mathbf{t} \in [\mathbf{0}, \mathbf{b}]$ with $\mathbf{t} = (t_1, t_2)$ and $[\mathbf{0}, \mathbf{b}] = [0, b_1] \times [0, b_2]$.

(i) If $g \in C_{\mathbf{0}}[\mathbf{0}, \mathbf{b}]$, then the equation $(-\ {}_{t_1}D^{(\nu_1)}_{0+} -\ {}_{t_2}D^{(\nu_2)}_{0+}, \lambda, g, 0)$ has a unique solution in the domain of the generator given by $w = \mathbf{R}^{\mathbf{0}+(\nu)}_{\lambda} g$, the resolvent operator of the process $\mathbf{T}^{\mathbf{0}+(\nu)}_{\mathbf{t}}$.

(ii) For any $g \in B[\mathbf{0}, \mathbf{b}]$, the mixed equation $(-\ {}_{t_1}D^{(\nu_1)}_{0+} -\ {}_{t_2}D^{(\nu_2)}_{0+}, \lambda, g, 0)$ is well-posed in the generalized sense and the solution admits the stochastic representation

$$w(t_1, t_2) = \mathbf{E}\left[\int_0^{\tau_{\mathbf{0}}^{(t_1,t_2),(\nu)}} e^{-\lambda s} g\left(T^{0+(\nu_1)}_{t_1}(s)\ ,\ T^{0+(\nu_2)}_{t_2}(s)\right) ds\right]. \tag{53}$$

Moreover, if additionally ν_i satisfies conditions (H2)-(H3) for $i \in \{1, 2\}$, then $w(t_1, t_2)$ takes the explicit form in (52).

Proof. (i) Follows from Theorem 1.1 in [8] as in the one-dimensional case.

(ii) If $g \in B[\mathbf{0}, \mathbf{b}]$, the solution is obtained as a limit of solutions $\mathbf{R}^{\mathbf{0}+(\nu)}_{\lambda} g_n(\mathbf{t})$ in the domain of the generator $-\mathbf{D}^{(\nu)}_{\mathbf{0}+}$, where the sequence of functions $\{g_n\}_{n\geq 1}$ satisfies the conditions of Definition 6.1. Finally, Lemma 6.2 provides the explicit representation of the solution w in terms of transition densities. □

Theorem 6.2. (**Case** $\lambda = 0$) All assertions in Theorem 6.1 are valid for $\lambda = 0$.

Proof. The arguments in the proof of Theorem 6.1 remain valid for the case $\lambda = 0$ replacing the resolvent $\mathbf{R}^{\mathbf{0}+(\nu)}_{\lambda}$ by the corresponding potential operator $\mathbf{R}^{\mathbf{0}+(\nu)}_{0}$. □

Finally, we analyze the mixed linear equation which involves both the RL type and the Caputo type operator:

$$\begin{aligned} -\ {}_{t_1}D^{(\nu_1)}_{0+} u(t_1, t_2) -\ {}_{t_2}D^{(\nu_2)}_{0+*} u(t_1, t_2) &= \lambda u(t_1, t_2) - g(t_1, t_2), && (t_1, t_2) \in (0, b_1] \times (0, b_2], \\ u(0, t_2) &= 0, && t_2 \in [0, b_2] \\ u(t_1, 0) &= \phi(t_1) && t_1 \in (0, b_1], \end{aligned} \tag{54}$$

for a given function $\phi \in C_0[0, b_1]$. This equation will be referred to as the *mixed linear problem* $(-\ {}_{t_1}D^{(\nu_1)}_{0+} -\ {}_{t_2}D^{(\nu_2)}_{0+*}, \lambda, g, \phi)$.

Denote by $\mathbf{T}^{\mathbf{0}+(\nu)*}_{\mathbf{t}} := \left(T^{0+(\nu_1)}_{t_1}, T^{0+*(\nu_2)}_{t_2}\right)$ the Feller process (with values on $(0, b_1] \times [0, b_2]$) generated by the operator $-\ {}_{t_1}D^{(\nu_1)}_{0+} -\ {}_{t_2}D^{(\nu_2)}_{0+*}$. This process is obtained from a process $\mathbf{T}^{+(\nu)}_{\mathbf{t}} := \left(T^{+(\nu_1)}_{t_1}\ ,\ T^{+(\nu_2)}_{t_2}\right)$ by either killing it whether the first coordinate attempt to cross the boundary point $t_1 = 0$, or by stopping it if the second coordinate does the same with the boundary point $t_2 = 0$. As before, $\tau_{\mathbf{0}}^{\mathbf{t},(\nu)}$ denotes the first exit time from $(0, b_1] \times (0, b_2]$.

In order to solve the mixed equation (54), we rewrite it as a linear equation involving only RL type operators. Namely, let $\psi \in C([0, t_1] \times [0, t_2])$ be a function satisfying the boundary conditions in (54). Define $w(\mathbf{t}) := u(\mathbf{t}) - \psi(\mathbf{t})$ for any $\mathbf{t} = (t_1, t_2) \in [\mathbf{0}, \mathbf{b}]$. Observe that, by definition, w vanishes at the boundary $\partial_{\mathbf{0}}[\mathbf{0}, \mathbf{b}]$.

If u and ψ belong to the domain of the generator $-\ {}_{t_1}D_{0+}^{(\nu_1)} -\ {}_{t_2}D_{0+*}^{(\nu_2)}$, then

$$\begin{aligned}\left(-\ {}_{t_1}D_{0+}^{(\nu_1)} -\ {}_{t_2}D_{0+*}^{(\nu_2)}\right) w &= \left(-\ {}_{t_1}D_{0+}^{(\nu_1)} -\ {}_{t_2}D_{0+*}^{(\nu_2)}\right) u + \left(\ {}_{t_1}D_{0+}^{(\nu_1)} +\ {}_{t_2}D_{0+*}^{(\nu_2)}\right) \psi \\ &= \lambda u - \left[g - \left(\ {}_{t_1}D_{0+}^{(\nu_1)} +\ {}_{t_2}D_{0+*}^{(\nu_2)}\right) \psi \right] =: \lambda w - \tilde{g},\end{aligned}$$

with $\tilde{g} := g - \lambda\psi -\ {}_{t_1}D_{0+}^{(\nu_1)}\psi -\ {}_{t_2}D_{0+*}^{(\nu_2)}\psi$. Due to the properties satisfied by ψ, the function w satisfies $-\ {}_{t_2}D_{0+*}^{(\nu_2)}w = -\ {}_{t_2}D_{0+}^{(\nu_2)}w = 0$ at the boundary $\partial_{\mathbf{0}}(\mathbf{0}, \mathbf{b}]$. Consequently, the solution u to (54) can be written as $u = w + \psi$, where w is the solution to the corresponding RL type equation. This leads us to the next definition.

Definition 6.2. Let $g \in B[\mathbf{0}, \mathbf{b}]$, $\lambda \geq 0$, and $\phi \in C_0[0, b_1]$. A function $u \in C[\mathbf{0}, \mathbf{b}]$ is said to solve the mixed linear problem $(-\ {}_{t_1}D_{0+}^{(\nu_1)} -\ {}_{t_2}D_{0+*}^{(\nu_2)}, \lambda, g, \phi)$ as

(i) a *solution in the domain of the generator* if u satisfies (54) and belongs to the domain of the generator $-\ {}_{t_1}D_{0+}^{(\nu_1)} -\ {}_{t_2}D_{0+*}^{(\nu_2)}$;

(ii) a *generalized solution* if for any function ψ in the domain of $-\ {}_{t_1}D_{0+}^{(\nu_1)} -\ {}_{t_2}D_{0+*}^{(\nu_2)}$ such that $\psi(0, \cdot) = 0$ and $\psi(\cdot, 0) = \phi(\cdot)$, then $u = \omega + \psi$, where ω is a solution (possibly generalized) to the RL type problem

$$(-\ {}_{t_1}D_{0+}^{(\nu_1)} -\ {}_{t_2}D_{0+}^{(\nu_2)}, \lambda, \tilde{g}, 0),$$

with $\tilde{g} := g - \lambda\psi -\ {}_{t_1}D_{0+}^{(\nu_1)}\psi -\ {}_{t_2}D_{0+*}^{(\nu_2)}\psi$.

Remark 6.2. By definition, it seems that a generalized solution depends on the function ψ, the next result shows that this solution is actually independent of ψ.

Remark 6.3. The concept of a generalized solution can also be given as a limit of solutions in the domain of the corresponding generator.

Well-posedness result for the mixed linear equation.

Theorem 6.3. (**Case** $\lambda > 0$) Let $\nu = (\nu_1, \nu_2)$ such that each ν_i is a function satisfying conditions (H0)-(H1). Suppose $\lambda > 0$ and $\phi \in C_0[0, b_1]$.

(i) If $g \in C[\mathbf{0}, \mathbf{b}]$ satisfies $g(0, \cdot) \equiv 0$ and $g(\cdot, 0) = \lambda\phi(\cdot)$, then the mixed equation $(-\ {}_{t_1}D_{0+}^{(\nu_1)} -\ {}_{t_2}D_{0+*}^{(\nu_2)}, \lambda, g, \phi)$ has a unique solution in the domain of the generator given by $u = \mathfrak{R}_{\lambda}^{\mathbf{0}+(\nu)*}g$, the resolvent operator of the process $\left(T_{t_1}^{0+(\nu_1)},\ T_{t_2}^{0+*(\nu_2)}\right)$.

(ii) For any $g \in B[\mathbf{0}, \mathbf{b}]$, the mixed linear equation $(-\ {}_{t_1}D_{0+}^{(\nu_1)} -\ {}_{t_2}D_{0+*}^{(\nu_2)}, \lambda, g, \phi)$ is well-posed in the generalized sense and the solution admits the stochastic representation

$$u(t_1,t_2) = \mathbf{E}\left[e^{-\lambda\tau_0^{t_2,(\nu_2)}}\phi\left(T_{t_1}^{0+(\nu_1)}\left(\tau_0^{t_2,(\nu_2)}\right)\right)\ 1_{\left\{\tau_0^{t_2,(\nu_2)}<\tau_0^{t_1,(\nu_1)}\right\}}\right] + \mathbf{E}\left[\int_0^{\tau_{\mathbf{0}}^{\mathbf{t},(\nu)}} e^{-\lambda s} g\left(T_{t_1}^{0+(\nu_1)}(s),\ T_{t_2}^{0+*(\nu_2)}(s)\right)\right]. \tag{55}$$

Moreover, if additionally each ν_i for $i \in \{1,2\}$, satisfies conditions (H2)-(H3), then the solution rewrites as

$$u(t_1,t_2) \ = \ \int_0^{t_1}\phi(t_1-r)\int_0^{\infty} e^{-\lambda s}\mu_0^{t_2,(\nu_2)}(s)p_s^{+(\nu_1)}(t_1,t_1-r_1)\ ds\ dr + + \int_0^{t_1}\int_0^{t_2} g(t_1-r_1,t_2-r_2)\int_0^{\infty} e^{-\lambda s}p_s^{+(\nu_1)}(t_1,t_1-r_1)p_s^{+(\nu_2)}(t_2,t_2-r_2)\,ds\ dr_2\ dr_1. \tag{56}$$

Proof. (i) As before, we apply Theorem 1.1 in [8] to the generator $-\ {}_{t_1}D_{0+}^{(\nu_1)} - {}_{t_2}D_{0+*}^{(\nu_2)}$. Therefore, if g is a continuous function on $[0,b_1]\times[0,b_2]$ such that $g(0,\cdot\,)\equiv 0$, then the function $u(t_1,t_2) = \mathfrak{R}_\lambda^{\mathbf{0}+(\nu)*}g(t_1,t_2)$ solves the equation (54) without any boundary condition. Further, a simple calculation shows that

$$u(t_1,0) = \mathfrak{R}_\lambda^{\mathbf{0}+(\nu)*}g(t_1,0) = g(t_1,0)/\lambda,$$

which implies that, under condition $g(\,\cdot\,,0) = \lambda\phi(\cdot)$, the function u solves the problem (54).

(ii) In case of general $g \in B[\mathbf{0},\mathbf{b}]$, take a function ψ satisfying the conditions of Definition 6.2 and set $w := u - \psi$. Since w vanishes at the boundary $\partial_{\mathbf{0}}(\mathbf{0},\mathbf{b}]$, Theorem 6.1 yields

$$w(t) = \mathbf{E}\left[\int_0^{\tau_{\mathbf{0}}^{(t_1,t_2),(\nu)}} e^{-\lambda s}\tilde{g}\left(T_{t_1}^{0+(\nu_1)}(s)\ ,\ T_{t_2}^{0+(\nu_2)}(s)\right)ds\right],$$

with $\tilde{g} = g - \lambda\psi - (\ {}_{t_1}D_{0+}^{(\nu_1)} + \ {}_{t_2}D_{0+*}^{(\nu_2)})\,\psi$. Hence $w(t) = I - II$, where

$$I - II := \mathbf{E}\left[\int_0^{\tau_{\mathbf{0}}^{(t_1,t_2),(\nu)}} e^{-\lambda s} g\left(T_{t_1}^{0+(\nu_1)}(s)\ ,\ T_{t_2}^{0+(\nu_2)}(s)\right)ds\right] - \mathbf{E}\left[\int_0^{\tau_{\mathbf{0}}^{(t_1,t_2),(\nu)}} e^{-\lambda s}(\lambda + \ {}_{t_1}D_{0+}^{(\nu_1)} + \ {}_{t_2}D_{0+*}^{(\nu_2)})\,\psi\left(T_{t_1}^{0+(\nu_1)}(s)\ ,\ T_{t_2}^{0+(\nu_2)}(s)\right)ds\right].$$

Using that ψ belongs to the domain of the generator $-\ {}_{t_1}D_{0+}^{(\nu_1)} - \ {}_{t_2}D_{0+*}^{(\nu_2)}$, Doob's stopping theorem, applied to the martingale

$$e^{-\lambda r}\psi\left(T_{t_1}^{0+(\nu_1)}(r)\ ,\ T_{t_2}^{0+*(\nu_2)}(r)\right) + + \int_0^{r} e^{-\lambda s}\left(\lambda + \ {}_{t_1}D_{0}^{(\nu_1)} + \ {}_{t_2}D_{0+*}^{(\nu_2)}\right)\psi\left(T_{t_1}^{0+(\nu_1)}(s)\ ,\ T_{t_2}^{0+*(\nu_2)}(s)\right)ds \tag{57}$$

and the stopping time $\tau_{\mathbf{0}}^{(t_1,t_2),(\nu)}$, implies

$$II \ = \ \psi(t_1,t_2) \ - \ \mathbf{E}\left[e^{-\lambda\tau_{\mathbf{0}}^{(t_1,t_2),(\nu)}}\psi\left(T_{t_1}^{0+(\nu_1)}(\tau_{\mathbf{0}}^{(t_1,t_2),(\nu)})\ ,\ T_{t_2}^{0+(\nu_2)}(\tau_{\mathbf{0}}^{(t_1,t_2),(\nu)})\right)\right],$$

which in turn yields (55) as $u = w + \psi$ and $\psi(\cdot,0) = \phi(\cdot)$.

Finally, the second term in (56) is a consequence of Lemma 6.2, whilst the first term is obtained by conditioning first on $\tau_0^{t_2,(\nu_2)}$ and then by using the joint density of the pair $(T_{t_1}^{0+(\nu)}(s), \tau_0^{t_1,(\nu_1)})$. □

Theorem 6.4. (Case $\lambda = 0$) All the assertions in Theorem 6.3 are valid for the case $\lambda = 0$.

Proof. For functions $g \in C_{\mathbf{0}}[\mathbf{0}, \mathbf{b}]$, the arguments in the proof of Theorem 6.3 remain valid using the potential operator $\mathfrak{R}_0^{\mathbf{0}+(\nu)*}$ instead of the resolvent operator $\mathfrak{R}_\lambda^{\mathbf{0}+(\nu)*}$. In case of general $g \in B[\mathbf{0}, \mathbf{b}]$, the martingale (57) should be replaced by the corresponding martingale with $\lambda = 0$. □

Remark 6.4. As an application of Theorem 6.3, one obtains that for $\mathbf{t} = (t_1, t_2)$ the function

$$u(\mathbf{t}) = \frac{1}{\alpha} t_2 \int_0^{t_1} \phi(t_1 - r) \int_0^\infty e^{-\lambda s}\ s^{-\frac{1}{\alpha}-\frac{1}{\beta}-1} w_\alpha\left(t_2 s^{-1/\alpha}; 1, 1\right) w_\beta\left(r_1 s^{-1/\beta}; 1, 1\right)\ ds\ dr +$$
$$+ \int_0^{t_1}\int_0^{t_2} g(t_1 - r_1, t_2 - r_2) \int_0^\infty e^{-\lambda s} s^{-\frac{1}{\beta}-\frac{1}{\alpha}} w_\beta\left(r_1 s^{-1/\beta}; 1, 1\right) w_\alpha\left(r_2 s^{-1/\alpha}; 1, 1\right) ds\, dr_2\, dr_1$$

is the generalized solution to the *mixed fractional linear equation*

$$\begin{array}{rcll} -\ {}_{t_1}D_{0+}^{\beta} u(t_1, t_2) -\ {}_{t_2}D_{0+*}^{\alpha} u(t_1, t_2) &=& \lambda u(t_1, t_2) - g(t_1, t_2), & (t_1, t_2) \in (\mathbf{0}, \mathbf{b}], \\ u(0, t_2) &=& 0, & t_2 \in [0, b_2], \\ u(t_1, 0) &=& \phi(t_1) & t_1 \in (0, b_1], \end{array}$$

for a given function $\phi \in C_0[0, b_1]$ and $\beta, \alpha \in (0, 1)$. Let us recall that $-\ {}_{t_1}D_{0+}^{\beta}$ and $-\ {}_{t_2}D_{0+*}^{\alpha}$ stand for the classical RL and Caputo derivatives of order β and α, respectively; and w_β and w_α denote $\beta-$ and $\alpha-$stable densities, respectively (see Preliminaries).

The solution u belongs to the domain of the generator only when $g \in C[\mathbf{0}, \mathbf{b}]$, $g(\cdot, 0) = \lambda\phi(\cdot)$ and $g(0, \cdot) \equiv 0$.

7. Acknowledgements

The first author is supported by Chancellor International Scholarship through the University of Warwick, UK.

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