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A FRACTIONAL TRAPEZOIDAL RULE TYPE DIFFERENCE SCHEME FOR FRACTIONAL ORDER INTEGRO-DIFFERENTIAL EQUATION

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Abstract. A fractional trapezoidal rule type difference scheme for fractional order integro-differential equation is considered. The equation is discretized in time by means of a method based on the trapezoidal rule: while the time derivative is approximated by the standard trapezoidal rule, the integral term is discretized by means of a fractional quadrature rule constructed again from the trapezoidal rule. The solvability, stability and L^2-norm convergence are proved. The convergence order is second order both in temporal and spatial directions. Furthermore, a spatial compact scheme, based on the fractional trapezoidal rule type difference scheme, is also proposed and the similar results are derived. The convergence order is second for time and fourth for space. Preliminary numerical experiment confirms our theoretical results.

1. Introduction

Consider a fractional trapezoidal rule (FTR) type difference scheme for the fractional order integro-differential equation [1, 3, 5, 11, 16, 17, 24]

$$u_t(x,t) - I^{(\alpha-1)}u_{xx}(x,t) = f(x,t),\ 0 < x < L,\ 0 < t \le T, \tag{1.1}$$

where the α-th integral $I^{(\alpha)}\varphi(t)$ is defined by the Riemann-Liouville operator (see [18]) as

$$I^{(\alpha)}\varphi(t) = \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)ds, \qquad t > 0.$$

for $1 < \alpha < 2$, with the boundary conditions

$$u(0,t) = u(L,t) = 0, \qquad 0 < t \le T, \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x), \qquad 0 \le x \le L. \tag{1.3}$$

Equation (1.1) can be found in the modelling of wave propagation involving viscoelastic forces, heat conduction in materials with memory [7, 8, 18].

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Currently, various algorithms are designed for the integro-differential equations of fractional order. Chen, Thomee and Wahlbin [1] used backward Euler scheme in time, piecewise linear finite element method in space, the integral term by means of product integration, and gave the regularity and error boundness of the solution. Mclean, Thomee [16] employed backward Euler, Crank-Nicolson and second order BDF scheme, Galerkin finite element method for spatial variables and gave the regularity, stability and error estimate. Xu [25] considered backward Euler and Crank-Nicolson scheme, with one and second order convolution quadrature to the integral term, respectively, and drove long time error boundness with weights. Lin and Xu [10] proposed an effective numerical method based on a finite difference scheme in time and Legendre spectral methods in space. Many researchers developed finite difference methods for it, e.g., It was analyzed by Lopez-Marcos [11], Tang [24] and Chen [2], using a backward-Euler scheme, a Crank-Nicolson scheme and a second order backward differentiation formula (BDF) scheme, respectively. The stability and convergence are obtained. Sun and Wu [20] derived a fully discrete difference scheme for the fractional wave equation and proved the difference scheme is convergent in maximum norm. Zhang, Sun and Wu [22] constructed and analyzed the Crank-Nicolson-type difference scheme for the subdiffusion equation with a Riemann-Liouville fractional derivative. Chen et al. [3], Zhang et al. [22, 23] concentrated on the compact difference scheme for promoting the spatial accuracy. The advantage of the compact difference scheme is high accuracy in spatial direction only and the coefficient matrix of the linear system of equations of the unknowns is tridiagonal and can be easily solved by the Thomas algorithm.

The main purpose of this paper is to construct a fractional trapezoidal rule type difference scheme for the fractional order integro-differential equations. The equations are discretized in time by means of a method based on the trapezoidal rule: while the time derivative is approximated by the standard trapezoidal rule, the integral term is discreted by means of a fractional quadrature rule constructed again from the trapezoidal rule. The solvability, stability and L^2-norm convergence are proved. The convergence order is second order both in temporal and spatial directions. Furthermore, a spatial compact scheme, based on the fractional trapezoidal rule type difference scheme, is also proposed and the similar results are derived. The convergence order is second for time and fourth for space.

An overview of the paper follows. In section 2, the FTR type difference scheme is derived. Section 3 is devoted to the analysis of the unique solvability, stability and L^2-norm convergence of the scheme. The compact difference scheme and the unique solvability, stability and L^2-norm convergence are presented in Section 4. In section 5, we will give a numerical example that is in total agreement with our analysis. The article ends with a brief conclusions section.

2. Derivation of the difference scheme

For the finite difference approximation, let J and N be two positive integers, $h = L/J$ and $k = T/N$. The domain $[0, L] \times (0, T]$ is covered by $\Omega_h \times \Omega_k$, where $\Omega_h = \{x_j | x_j = jh,\ 0 \le j \le J\}$ and $\Omega_k = \{t_n | t_n = nk,\ 1 \le n \le N\}$. In addition, denote $t_{n-\frac{1}{2}} = (n - \frac{1}{2})k$.

For any grid function $\mathbf{w} = \{w_j^k | 0 \le j \le J,\ 1 \le n \le N\}$ defined on $\Omega_h \times \Omega_k$, let us introduce the following notations:

$$\begin{aligned} \delta_x w_{j-\frac{1}{2}}^n &= \tfrac{1}{h}(w_j^n - w_{j-1}^n), \qquad & \delta_x^2 w_j^n &= \tfrac{1}{h}(\delta_x w_{j+\frac{1}{2}}^n - \delta_x w_{j-\frac{1}{2}}^n), \\ w_j^{n-\frac{1}{2}} &= \tfrac{1}{2}(w_j^n + w_j^{n-1}), \qquad & \delta_t w_j^{n-\frac{1}{2}} &= \tfrac{1}{k}(w_j^n - w_j^{n-1}). \end{aligned} \tag{2.1}$$

To approximate the continuous convolution integral $I^{(\alpha-1)}\varphi(t_n)$, we will introduce the following fractional quadrature rule (FQR) [5, 9, 12, 13]

$$q_n(\varphi) = k^\alpha \sum_{p=0}^{n} \beta_p \varphi^{n-p} + \rho_{n0}\varphi^0, \tag{2.2}$$

where the quadrature weights β_p are determined by their generating power series

$$K[\delta(z)] = [\delta(z)]^{1-\alpha} = [\tfrac{1}{2}\tfrac{1+z}{1-z}]^{\alpha-1} = \sum_{p=0}^{\infty} \beta_p z^p. \tag{2.3}$$

Here $K(s)$ denotes the Laplace transform of the convolution kernel and $\delta(z)$ is a rational function. For concreteness, in this paper we consider the trapezoidal rule, for which

$$\delta(z) = \tfrac{2(1-z)}{1+z}. \tag{2.4}$$

To approximate the integral formally to second order and we take the correction quadrature weights ρ_{n0} so that the quadrature formula becomes exact for constant polynomial, namely

$$k^\alpha \sum_{p=0}^{n} \beta_p + \rho_{n0} = \tfrac{1}{\Gamma(\alpha-1)} \int_0^{t_n} (t_n - s)^{\alpha-2} ds = \tfrac{1}{\Gamma(\alpha)} t_n^{\alpha-1},$$

that is

$$k^\alpha \sum_{p=0}^{n} \beta_p + \rho_{n0} = \tfrac{1}{\Gamma(\alpha)} t_n^{\alpha-1}. \tag{2.5}$$

We will give the quadrature error of $E(\varphi)(t_n) = I^{(\alpha-1)}\varphi(t_n) - q_n(\varphi)$, where $q_n(\varphi)$ is defined in (2.2).

Lemma 2.1 [5]. Let φ be continuous and such that $\varphi_{tt} \in \mathbf{B}_\delta(0,T]$ for some $0 \le \delta < 1$, then there exists $C = C(\alpha, \delta)$ such that

$$|E(\varphi)(t_n)| \le C(t_n^{\alpha-2}|\varphi_t(0)| + t_n^{\alpha-\delta-1}|\varphi_{tt}|_\delta)k^2, \qquad n \ge 1,$$

where $\mathbf{B}_\delta(0,T] = \{f \,|\, |f|_\delta = \sup\limits_{0<t\le T} t^\delta |f(t)| < +\infty\}$.

Using Taylor expansion with integral remainder, we have the following lemma.

Lemma 2.2 [22]. Suppose $u(x,t) \in C_{x,t}^{4,2}([0,L]\times(0,T])$. It holds that

$$\tfrac{\partial^2 u}{\partial x^2}(x_j, t_n) = \delta_x^2 U_j^n - \tfrac{h^2}{6}\int_0^1 [\tfrac{\partial^4 u}{\partial x^4}(x_j + sh, t_n) + \tfrac{\partial^4 u}{\partial x^4}(x_j - sh, t_n)](1-s)^3 ds.$$

The boundness of $(R_1)_j^n = I^{(\alpha-1)} u_{xx}(x_j, t_n) - q_n(\delta_x^2 u_j^n)$ will be given as follow.

Lemma 2.3. Suppose $u(x,t) \in C_{x,t}^{4,2}([0,L]\times(0,T])$, then it holds that

$$|\,(R_1)_j^n\,| = |I^{(\alpha-1)} u_{xx}(x_j,t_n) - q_n(\delta_x^2 u_j^n)| \le C(h^2 + k^2), \qquad 1 \le n \le N.$$

Proof. Let

$$v(x,t) = \tfrac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} u(x,s) ds, \qquad V_j^n = v(x_j, t_n).$$

Then

$$I^{(\alpha-1)} \tfrac{\partial^2 u}{\partial x^2} = \tfrac{\partial^2 v}{\partial x^2}.$$

Utilizing Taylor expansion with integral remainder, we have

$$\frac{\partial^2 v}{\partial x^2}(x_j,t_n) = \delta_x^2 V_j^n - \frac{h^2}{6}\int_0^1 [\frac{\partial^4 v}{\partial x^4}(x_j+sh,t_n) + \frac{\partial^4 v}{\partial x^4}(x_j-sh,t_n)](1-s)^3 ds.$$

From Lemma 2.2 and triangle inequality, we have

$$\begin{aligned}
|\ (R_1)_j^n\ | &= |I^{(\alpha-1)}u_{xx}(x_j,t_n) - q_n(\delta_x^2 u_j^n)| \\
&= |I^{(\alpha-1)}u_{xx}(x_j,t_n) - q_n(u_{xx}(x_j,\cdot)) + q_n(u_{xx}(x_j,\cdot)) - q_n(\delta_x^2 u_j^n)| \\
&\le |I^{(\alpha-1)}u_{xx}(x_j,t_n) - q_n(u_{xx}(x_j,\cdot))| + |q_n(u_{xx}(x_j,\cdot)) - q_n(\delta_x^2 u_j^n)| \\
&\le q_n(1)\frac{h^2}{6}|\int_0^1 [\frac{\partial^4 u}{\partial x^4}(x_j+sh,t_n) + \frac{\partial^4 u}{\partial x^4}(x_j-sh,t_n)](1-s)^3 ds| \\
&\quad + C(t_n^{\alpha-2}|u_{xxt}(0)| + t_n^{\alpha-\delta-1}|u_{xxtt}|_\delta)k^2.
\end{aligned}$$

We finish the proof.

Lemma 2.4 [22]. Let $y \in C^3[t_{n-1},t_n]$. It holds that

$$\begin{aligned}
&\frac{1}{2}[y'(t_n) + y'(t_{n-1})] - \frac{1}{k}[y(t_n) - y(t_{n-1})] \\
&\qquad = \frac{k^2}{16}\int_0^1 [y^{(3)}(t_{n-\frac{1}{2}} + \frac{sk}{2}) + y^{(3)}(t_{n-\frac{1}{2}} - \frac{sk}{2})](1-s^2) ds.
\end{aligned}$$

Define the grid function

$$U_j^n = u(x_j,t_n), \qquad 0 \le j \le J, \qquad 0 \le n \le N.$$

We now derive the fractional trapezoidal rule type difference scheme for the problem (1.1)-(1.3).

Considering the equation (1.1) at the point (x_j,t_n)

$$u_t(x_j,t_n) - I^{(\alpha-1)}u_{xx}(x_j,t_n) = f(x_j,t_n),\ 1 \le j \le J-1,\ 1 \le n \le N. \tag{2.6}$$

Then it holds

$$\begin{aligned}
&\frac{1}{2}[u_t(x_j,t_n) + u_t(x_j,t_{n-1})] - \frac{1}{2}[I^{(\alpha-1)}u_{xx}(x_j,t_n) + I^{(\alpha-1)}u_{xx}(x_j,t_{n-1})] \\
&\quad = \frac{1}{2}[f(x_j,t_n) + f(x_j,t_{n-1})], \qquad 1 \le j \le J-1, \qquad 1 \le n \le N.
\end{aligned} \tag{2.7}$$

It follows from Lemma 2.3 and Lemma 2.4 that

$$\begin{aligned}
&\frac{1}{2}[I^{(\alpha-1)}u_{xx}(x_j,t_n) + I^{(\alpha-1)}u_{xx}(x_j,t_{n-1})] = \frac{1}{2}[q_n(\delta_x^2 U_j) + q_{n-1}(\delta_x^2 U_j)] \\
&\qquad + \frac{1}{2}[(R_1)_j^n + (R_1)_j^{n-1}].
\end{aligned} \tag{2.8}$$

and

$$\frac{1}{2}[u_t(x_j,t_n) + u_t(x_j,t_{n-1})] = \delta_t U_j^{n-\frac{1}{2}} + (R_2)_j^{n-\frac{1}{2}}, \tag{2.9}$$

where

$$(R_2)_j^{n-\frac{1}{2}} = \frac{k^2}{16}\int_0^1 [\frac{\partial^3 u}{\partial t^3}(x_j,t_{n-\frac{1}{2}} + \frac{sk}{2}) + \frac{\partial^3 u}{\partial t^3}(x_j,t_{n-\frac{1}{2}} - \frac{sk}{2})](1-s^2) ds.$$

Using the notations

$$U_j^{\frac{1}{2}} = \frac{1}{2}U_j^1, \qquad \rho_n = \frac{1}{2}[\rho_{n0} + \rho_{n-1,0}], \qquad 1 \le n \le N.$$

We have

$$\begin{aligned}
&\frac{1}{2}[q_n(\delta_x^2 U_j) + q_{n-1}(\delta_x^2 U_j)] \\
&\quad = \frac{1}{2}[k^\alpha \sum_{p=0}^{n} \beta_p \delta_x^2 U_j^{n-p} + \rho_{n0}\delta_x^2 U_j^0 + k^\alpha \sum_{p=0}^{n-1} \beta_p \delta_x^2 U_j^{n-1-p} + \rho_{n-1,0}\delta_x^2 U_j^0] \\
&\quad = k^\alpha \sum_{p=0}^{n} \beta_p \delta_x^2 U_j^{n-p-\frac{1}{2}} + \rho_n \delta_x^2 U_j^0.
\end{aligned} \tag{2.10}$$

Substituting (2.8)-(2.10) into (2.7), we have

$$\begin{aligned}&\delta_t U_j^{n-\frac{1}{2}} - (k^\alpha \sum_{p=0}^{n} \beta_p \delta_x^2 U_j^{n-p-\frac{1}{2}} + \rho_n \delta_x^2 U_j^0) = f_j^{n-\frac{1}{2}} + R_j^{n-\frac{1}{2}},\\ &\qquad 1 \le j \le J-1, \qquad 1 \le n \le N,\end{aligned} \tag{2.11}$$

where $f_j^{n-\frac{1}{2}} = \frac{1}{2}[f(x_j, t_n) + f(x_j, t_{n-1})]$ and

$$R_j^{n-\frac{1}{2}} = (R_1)_j^{n-\frac{1}{2}} - (R_2)_j^{n-\frac{1}{2}}, \qquad 1 \le j \le J-1, \qquad 1 \le n \le N. \tag{2.12}$$

Therefore, there exists a constant C_R independent of h and k such that

$$|R_j^{n-\frac{1}{2}}| \le C_R(k^2 + h^2), \qquad 1 \le j \le J-1, \qquad 1 \le n \le N. \tag{2.13}$$

In addition, the initial and boundary value conditions can be written as

$$U_0^n = U_J^n = 0, \qquad 1 \le n \le N. \tag{2.14}$$

$$U_j^0 = u_0(x_j), \qquad 0 \le j \le J. \tag{2.15}$$

Omitting the small term in equation (2.11), and replacing the function U_j^n with its numerical approximation u_j^n, we get the following difference scheme

$$\delta_t u_j^{n-\frac{1}{2}} - (k^\alpha \sum_{p=0}^{n} \beta_p \delta_x^2 u_j^{n-p-\frac{1}{2}} + \rho_n \delta_x^2 u_j^0) = f_j^{n-\frac{1}{2}},\ 1 \le j \le J-1,\ 1 \le n \le N. \tag{2.16}$$

$$u_0^n = u_J^n = 0, \qquad 1 \le n \le N. \tag{2.17}$$

$$u_j^0 = u_0(x_j), \qquad 0 \le j \le J. \tag{2.18}$$

3. Analysis of the difference scheme

3.1. Solvability

Let $r = \frac{k^{\alpha+1}}{h^2}\beta_0$. The difference scheme (2.16)-(2.18) can be written as the following matrix form

$$A\mathbf{u}^n = \mathbf{u}^{n-1} + \frac{r}{2}B\mathbf{u}^{n-1} + \frac{k^{\alpha+1}}{h^2}\sum_{p=1}^{n} \beta_p B\mathbf{u}^{n-p-\frac{1}{2}} + \frac{k}{h^2}\rho_n B\mathbf{u}^0 + k\mathbf{f}^{n-\frac{1}{2}}, 1 \le n \le N, \tag{3.1}$$

where

$$A = \begin{bmatrix} 1+r & -r/2 & & & \\ -r/2 & 1+r & -r/2 & & \\ & \ddots & \ddots & \ddots & \\ & & -r/2 & 1+r & -r/2 \\ & & & -r/2 & 1+r \end{bmatrix}_{(J-1)\times(J-1)}, \tag{3.2}$$

$$B = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}_{(J-1)\times(J-1)} \tag{3.3}$$

and

$$\mathbf{f}^{n-\frac{1}{2}} = \begin{bmatrix} f_1^{n-\frac{1}{2}} + \frac{k^\alpha}{2h^2}\beta_n u_0^0 + \frac{1}{h^2}\rho_n u_0^0 \\ f_2^{n-\frac{1}{2}} \\ \vdots \\ f_{J-2}^{n-\frac{1}{2}} \\ f_{J-1}^{n-\frac{1}{2}} + \frac{k^\alpha}{2h^2}\beta_n u_J^0 + \frac{1}{h^2}\rho_n u_J^0 \end{bmatrix}_{(J-1)\times 1}. \tag{3.4}$$

It is easy to see that matrix A is tri-diagonal and strictly diagonally dominant, thus $\mathbf{u}^n$ can be obtained from (3.1). This can be written in the following result.

Theorem 3.1. The difference scheme (2.16)-(2.18) is uniquely solvable.

3.2. Stability

We first introduce some notations and lemmas which will be used in the stability and convergence analysis.

Let $V_h = \{v | v = (v_0, v_1, \cdots, v_J),\ v_0 = v_J = 0\}$. For any $v, w \in V_h$, we define the discrete inner product, L^2-norm, H^1- semi-norm and maximum norm as follows:

$$(v, w) = h\sum_{j=1}^{J-1} v_j w_j\ , \qquad \|v\| = \sqrt{(v, v)}\ ,$$

$$\|\delta_x v\| = \sqrt{h\sum_{j=0}^{J-1}(\delta_x v_{j+\frac{1}{2}})^2}\ , \qquad \|v\|_\infty = \max_{0\le j\le J}|v_j|\ .$$

Lemma 3.2. [2, 3]. For any grid function $v \in V_h$ and $0 \le n, m \le N$, then

$$|(\delta_x^2 v^n, v^m)| \le \tfrac{4}{h^2}\|v^n\|\|v^m\|.$$

Lemma 3.3. [2, 3, 11]. For any grid functions $v, w \in V_h$, we have

$$(\delta_x^2 v, w) = -\sum_{j=0}^{J-1} h(\delta_x v_{j+\frac{1}{2}})(\delta_x w_{j+\frac{1}{2}}) = -(\delta_x v, \delta_x w).$$

We will give a general result on the nonnegative character of certain real quadratic form with convolution structure. In order to treat more general choices, we say that q_n is β_0-positive [16] if

$$Q_N(\Phi) = k\sum_{n=0}^{N} q_n(\varphi)\varphi^n \ge -\beta_0(\varphi^0)^2, \qquad \forall\ N \ge 1, \qquad \Phi = (\varphi^0, \cdots, \varphi^N)^T.$$

Lemma 3.4. [11, 25]. If $\{a_0, a_1, \cdots, a_n, \cdots\}$ is a real-valued sequence such that $\hat{a}(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $D = \{z \in \mathcal{C} : |z| \le 1\}$, then for any positive integer N and for any $(U^0, U^1, \cdots, U^N) \in \mathbb{R}^{N+1}$,

$$\sum_{n=0}^{N}\left(\sum_{p=0}^{n} a_p U^{n-p}\right)U^n \ge 0, \tag{3.5}$$

if and only if

$$\Re\hat{a}(z) \ge 0, \qquad \text{for} \qquad z \in D.$$

where $\Re$ denotes the real part of a complex number.

Lemma 3.5. We have $\Re[\delta(z)] = \Re[\frac{2(1-z)}{1+z}] > 0$, when $|z| \le 1, z \ne 1$.

Proof: With $z = \xi + i\eta$, we have, for $\xi^2 + \eta^2 \le 1, \xi \ne 1$,

$$\Re[\delta(z)] = \Re[2\tfrac{1-\xi-i\eta}{1+\xi+i\eta}] = 2\tfrac{1-\xi^2-\eta^2}{(1+\xi)^2+\eta^2} > 0. \tag{3.6}$$

It is easy to know the convolution kernel $\beta(t) = \frac{1}{\Gamma(\alpha-1)} t^{\alpha-2}$ is positive type, viz. $\Re K[\beta(s)] \geq 0$, when $\Re(s) \geq 0$. From Lemma 3.5, we have, for $|z| < 1$, $\Re K[\beta(\frac{2(1-z)}{1+z})] = \Re(\frac{2(1-z)}{1+z}) > 0$, the generating function (2.3) satisfies the conditions of Lemma 3.4.

We now prove that the difference scheme (2.16)-(2.18) is stable to the initial value and the inhomogeneous term.

Theorem 3.6. Suppose $\{u_j^n | 0 \leq j \leq J,\ 1 \leq n \leq N\}$ is the solution of the difference scheme (2.16)-(2.18), then it holds that

$$\|u^n\| \leq C(T)\|u^0\| + 2\sum_{l=1}^{n} k\|f^{l-\frac{1}{2}}\|. \tag{3.7}$$

Proof. Taking the inner product of (2.16) with $2u^{n-\frac{1}{2}}$, we have

$$\begin{aligned} 2(\delta_t u^{n-\frac{1}{2}}, u^{n-\frac{1}{2}}) &= 2k^\alpha \sum_{p=0}^{n} \beta_p (\delta_x^2 u^{n-p-\frac{1}{2}}, u^{n-\frac{1}{2}}) \\ &+2\rho_n(\delta_x^2 u^0, u^{n-\frac{1}{2}}) + 2(f^{n-\frac{1}{2}}, u^{n-\frac{1}{2}}), \qquad 1 \leq n \leq N. \end{aligned} \tag{3.8}$$

We easily obtain

$$2(\delta_t u^{n-\frac{1}{2}}, u^{n-\frac{1}{2}}) = \frac{1}{k}(\|u^n\|^2 - \|u^{n-1}\|^2). \tag{3.9}$$

When $N \geq 1$, we have

$$\begin{aligned} \sum_{n=1}^{N}(\|u^n\|^2 - \|u^{n-1}\|^2) &= 2k^{\alpha+1}\sum_{n=1}^{N}\sum_{p=0}^{n} \beta_p(\delta_x^2 u^{n-p-\frac{1}{2}}, u^{n-\frac{1}{2}}) \\ &+2\sum_{n=1}^{N} k\rho_n(\delta_x^2 u^0, u^{n-\frac{1}{2}}) + 2\sum_{n=1}^{N} k(f^{n-\frac{1}{2}}, u^{n-\frac{1}{2}}). \end{aligned} \tag{3.10}$$

Now each term will be estimated. First, the first term of the right equality is β_0-positive. This follows from Lemma 3.3, on permuting the summation indices and using, for each fixed j, Lemma 3.4, we obtain

$$\begin{aligned} \sum_{n=1}^{N}\sum_{p=0}^{n} \beta_p(\delta_x^2 u^{n-p-\frac{1}{2}}, u^{n-\frac{1}{2}}) &= -\sum_{n=1}^{N}\sum_{p=0}^{n}\beta_p\sum_{j=0}^{J-1} h\delta_x u_{j+\frac{1}{2}}^{n-p-\frac{1}{2}}\delta_x u_{j+\frac{1}{2}}^{n-\frac{1}{2}} \\ &= -h\sum_{j=0}^{J-1}\sum_{n=1}^{N}\sum_{p=0}^{n}\beta_p\delta_x u_{j+\frac{1}{2}}^{n-p-\frac{1}{2}}\delta_x u_{j+\frac{1}{2}}^{n-\frac{1}{2}} \\ &= -h\sum_{j=0}^{J-1}[\sum_{n=0}^{N}\sum_{p=0}^{n}\beta_p\delta_x u_{j+\frac{1}{2}}^{n-p-\frac{1}{2}}\delta_x u_{j+\frac{1}{2}}^{n-\frac{1}{2}} - \frac{1}{4}\beta_0(\delta_x u_{j+\frac{1}{2}}^0)^2] \\ &\leq \frac{1}{4}\beta_0\sum_{j=0}^{J-1} h(\delta_x u_{j+\frac{1}{2}}^0)^2 = \frac{1}{4}\beta_0\|\delta_x u^0\|^2 = -\frac{1}{4}\beta_0(\delta_x^2 u^0, u^0). \end{aligned} \tag{3.11}$$

Second, using Lemma 3.2 and Cauchy-Schwarz inequality, we have

$$\begin{aligned} &(\delta_x^2 u^0, u^{n-\frac{1}{2}}) \leq \frac{1}{h^2}\|u^0\|\|u^{n-\frac{1}{2}}\|, \\ &(f^{n-\frac{1}{2}}, u^{n-\frac{1}{2}}) \leq \|f^{n-\frac{1}{2}}\|\|u^{n-\frac{1}{2}}\|. \end{aligned} \tag{3.12}$$

Substituting (3.11)-(3.12) into (3.10), we have

$$\begin{aligned} \|u^N\|^2 - \|u^0\|^2 &\leq \frac{1}{2}\beta_0\frac{k^{\alpha+1}}{h^2}\|u^0\|\|u^0\| + 2\sum_{n=1}^{N}\frac{k}{h^2}\rho_n\|u^0\|\|u^{n-\frac{1}{2}}\| \\ &+2\sum_{n=1}^{N} k\|f^{n-\frac{1}{2}}\|\|u^{n-\frac{1}{2}}\|. \end{aligned} \tag{3.13}$$

With M chosen so that $\|u^M\| = \max_{0\le n\le N} \|u^n\|$, we have

$$\begin{aligned}\|u^M\|^2 \le \|u^0\|\|u^M\| + \tfrac{1}{2}\beta_0 \tfrac{k^{\alpha+1}}{h^2}\|u^0\|\|u^M\| + 2\sum_{n=1}^{N} \tfrac{k}{h^2}\rho_n\|u^0\|\|u^M\| \\ +2\sum_{n=1}^{N} k\|f^{n-\frac{1}{2}}\|\|u^M\|.\end{aligned} \tag{3.14}$$

Thus, we obtain

$$\begin{aligned}\|u^N\| \le \|u^M\| \le \|u^0\| + \tfrac{1}{2}\beta_0\tfrac{k^{\alpha+1}}{h^2}\|u^0\| \\ +2\sum_{n=1}^{N}\tfrac{k}{h^2}\rho_n\|u^0\| + 2\sum_{n=1}^{N} k\|f^{n-\frac{1}{2}}\|.\end{aligned} \tag{3.15}$$

Because of $|\rho_{n0}| \le Ckt_n^{\alpha-2}$(see [5], Theorem 3-(a)), we have

$$\sum_{n=1}^{N} |\rho_{n0}| \le C\sum_{n=1}^{N} kt_n^{\alpha-2} \le C(Nk)^{\alpha-1} \le C(T). \tag{3.16}$$

Using (3.15) and (3.16), we finish the proof.

3.3. Convergence

We can now establish the convergence of the scheme by means of the energy method.

Let

$$e_j^n = U_j^n - u_j^n, \qquad 0\le j\le J, \qquad 1\le n\le N.$$

Subtracting (2.16)-(2.18) from (2.11) and (2.14)-(2.15), we get the error equations

$$\delta_t e_j^{n-\frac{1}{2}} - (k^\alpha\sum_{p=0}^{n}\beta_p\delta_x^2 e_j^{n-p-\frac{1}{2}} + \rho_n\delta_x^2 e_j^0) = R_j^{n-\frac{1}{2}}, 1\le j\le J-1,\ 1\le n\le N. \tag{3.17}$$

$$e_0^n = e_J^n = 0, \qquad 1\le n\le N. \tag{3.18}$$

$$e_j^0 = 0, \qquad 0\le j\le J. \tag{3.19}$$

It follows from Theorem 3.6 that

$$\|e^n\| \le C(T)\|e^0\| + 2\sum_{l=1}^{n} k\|R^{l-\frac{1}{2}}\|.$$

Substituting (2.13) into the above inequality, we have

$$\|e^n\| \le C_R(h^2+k^2).$$

We have the convergence result.

Theorem 3.7. Assume that the problem (1.1)-(1.3) has smooth solution $u(x,t)$ in the domain $[0,L]\times(0,T]$ and $\{u_j^n|0\le j\le J,\ 1\le n\le N\}$ is the solution of the difference scheme (2.13)-(2.15). Then it holds that

$$\max_{0\le j\le J, 1\le n\le N} |u(x_j,t_n) - u_j^n| \le C(k^2+h^2).$$

4. Compact difference scheme

4.1. Numerical scheme and solvability

We can easily give a compact difference scheme utilizing compact difference operator when the spatial accuracy needs to be promoted. For any grid function $\mathbf{v} = \{v_j | 0 \leq j \leq J\}$ defined on Ω_h, denote

$$\mathcal{H}v_j = \begin{cases} (I + \frac{h^2}{12}\delta_x^2)v_j, & 1 \leq j \leq J-1, \\ v_j, & j = 0 \text{ or } J. \end{cases}$$

We construct the following compact difference scheme for the problem (1.1)-(1.3)

$$\begin{gathered} \mathcal{H}\delta_t u_j^{n-\frac{1}{2}} - (k^\alpha \sum_{p=0}^{n} \beta_p \delta_x^2 u_j^{n-p-\frac{1}{2}} + \rho_n \delta_x^2 u_j^0) = \mathcal{H} f_j^{n-\frac{1}{2}}, \\ 1 \leq j \leq J-1, \qquad 1 \leq n \leq N. \end{gathered} \tag{4.1}$$

$$u_0^n = u_J^n = 0, \qquad 1 \leq n \leq N. \tag{4.2}$$

$$u_j^0 = u_0(x_j), \qquad 0 \leq j \leq J. \tag{4.3}$$

Let $r = \frac{k^{\alpha+1}}{h^2}\beta_0$. The compact difference scheme (4.1)-(4.3) can be written in the following matrix form

$$C\mathbf{u}^n = D\mathbf{u}^{n-1} + \frac{k^{\alpha+1}}{h^2} \sum_{p=1}^{n} \beta_p B \mathbf{u}^{n-p-\frac{1}{2}} + \frac{k}{h^2}\rho_n B\mathbf{u}^0 + k\mathbf{f}^{n-\frac{1}{2}}, \qquad 1 \leq n \leq N, \tag{4.4}$$

where

$$C = \begin{bmatrix} \frac{5}{6}+r & \frac{1}{12}-\frac{r}{2} & & & \\ \frac{1}{12}-\frac{r}{2} & \frac{5}{6}+r & \frac{1}{12}-\frac{r}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{12}-\frac{r}{2} & \frac{5}{6}+r & \frac{1}{12}-\frac{r}{2} \\ & & & \frac{1}{12}-\frac{r}{2} & \frac{5}{6}+r \end{bmatrix}_{(J-1)\times(J-1)}, \tag{4.5}$$

$$D = \begin{bmatrix} \frac{5}{6}-r & \frac{1}{12}+\frac{r}{2} & & & \\ \frac{1}{12}+\frac{r}{2} & \frac{5}{6}-r & \frac{1}{12}+\frac{r}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{12}+\frac{r}{2} & \frac{5}{6}-r & \frac{1}{12}+\frac{r}{2} \\ & & & \frac{1}{12}+\frac{r}{2} & \frac{5}{6}-r \end{bmatrix}_{(J-1)\times(J-1)} \tag{4.6}$$

and

$$\mathbf{f}^{n-\frac{1}{2}} = \begin{bmatrix} \frac{1}{12}(f_0^{n-\frac{1}{2}} + 10f_1^{n-\frac{1}{2}} + f_2^{n-\frac{1}{2}}) + \frac{k^\alpha}{2h^2}\beta_n u_0^0 + \frac{1}{h^2}\rho_n u_0^0 \\ \frac{1}{12}(f_1^{n-\frac{1}{2}} + 10f_2^{n-\frac{1}{2}} + f_3^{n-\frac{1}{2}}) \\ \vdots \\ \frac{1}{12}(f_{J-3}^{n-\frac{1}{2}} + 10f_{J-2}^{n-\frac{1}{2}} + f_{J-1}^{n-\frac{1}{2}}) \\ \frac{1}{12}(f_{J-2}^{n-\frac{1}{2}} + 10f_{J-1}^{n-\frac{1}{2}} + f_J^{n-\frac{1}{2}}) + \frac{k^\alpha}{2h^2}\beta_n u_J^0 + \frac{1}{h^2}\rho_n u_J^0 \end{bmatrix}_{(J-1)\times 1}. \tag{4.7}$$

It is easy to see that the coefficient matrix of the linear system is tri-diagonal and strictly diagonally dominant at each time level, thus it is uniquely solvable and the Thomas algorithm can be used.

Theorem 4.1. The compact difference scheme (4.1)-(4.3) is uniquely solvable.

4.2. Stability

We now give the estimation of the truncation error and the following lemma may be used.

Lemma 4.2. [22]. Let function $y(x) \in C^6[0,L]$, and $\zeta(s) = 5(1-s)^3 - 3(1-s)^5$, then

$$\begin{aligned}&\tfrac{1}{12}[y''(x_{j+1}) + 10y''(x_j) + y''(x_{j-1})] - \tfrac{1}{h^2}[y(x_{j+1}) - 2y(x_j) + y(x_{j-1})]\\ &\quad= \tfrac{h^4}{360}\int_0^1 [y^{(6)}(x_j - sh) + y^{(6)}(x_j + sh)]\zeta(s)ds, \qquad 1 \le j \le J-1.\end{aligned}$$

Lemma 4.3. Suppose that $u(x,t) \in C^{6,3}_{x,t}([0,L]\times(0,T])$, then the truncation error of the scheme (4.1)-(4.3) satisfies

$$|\widetilde{R}_j^{n-\frac{1}{2}}| \le C_R(k^2 + h^4), \qquad 1 \le j \le J-1, \qquad 1 \le n \le N, \tag{4.8}$$

where C_R is a positive constant independent of k and h.

Lemma 4.4. For any grid function $\{u_j^n\}$ defined on $\Omega_h \times \Omega_k$ and $u_0 = u_J = 0$, it holds that

$$\sum_{n=1}^{N} k(\mathcal{H}\delta_t u^{n-\frac{1}{2}}, u^{n-\frac{1}{2}}) \ge \tfrac{2}{3}\|u^N\|^2 - \|u^0\|^2.$$

Proof. First, we have

$$\begin{aligned}k(\mathcal{H}\delta_t u^{n-\frac{1}{2}}, u^{n-\frac{1}{2}}) &= \tfrac{k}{12}h\sum_{j=1}^{J-1}(\delta_t u_{j-1}^n + 10\delta_t u_j^n + \delta_t u_{j+1}^n)u_j^{n-\frac{1}{2}}\\ &= kh\sum_{j=1}^{J-1}(\delta_t u_j^n + \tfrac{h^2}{12}\delta_x^2\delta_t u_j^n)u_j^{n-\frac{1}{2}} = k(\delta_t u^n, u^{n-\frac{1}{2}}) + \tfrac{kh^2}{12}(\delta_x^2\delta_t u^n, u^{n-\frac{1}{2}})\\ &= \tfrac{1}{2}(\|u^n\|^2 - \|u^{n-1}\|^2) - \tfrac{h^2}{12}(\|\delta_x u^n\|^2 - \|\delta_x u^{n-1}\|^2)\\ &= \tfrac{1}{2}[(\|u^n\|^2 - \tfrac{h^2}{12}\|\delta_x u^n\|^2) - (\|u^{n-1}\|^2 - \tfrac{h^2}{12}\|\delta_x u^{n-1}\|^2)],\end{aligned} \tag{4.9}$$

where

$$\|\delta_x u^n\|^2 = h\sum_{j=0}^{J-1}|\delta_x u_j^n|^2.$$

And then, the fact

$$\|\delta_x u^n\|^2 \le \tfrac{4}{h^2}\|u^n\|^2$$

follows

$$\tfrac{2}{3}\|u^n\|^2 \le \|u^n\|^2 - \tfrac{h^2}{12}\|\delta_x u^n\|^2 \le \|u^n\|^2.$$

Consequently,

$$\begin{aligned}&\sum_{n=1}^{N} k(\mathcal{H}\delta_t u^{n-\frac{1}{2}}, u^{n-\frac{1}{2}})\\ &\quad= \tfrac{1}{2}[(\|u^N\|^2 - \tfrac{h^2}{12}\|\delta_x u^N\|^2) - (\|u^0\|^2 - \tfrac{h^2}{12}\|\delta_x u^0\|^2)]\\ &\quad\ge \tfrac{2}{3}\|u^N\|^2 - \|u^0\|^2.\end{aligned} \tag{4.10}$$

Theorem 4.5. Suppose $\{u_j^n | 0 \le j \le J,\ 1 \le n \le N\}$ is the solution of the difference scheme (4.1)-(4.3), then it holds that

$$\|u^n\| \le C(T)\|u^0\| + 2\sum_{l=1}^{n} k\|\mathcal{H}f^{l-\frac{1}{2}}\|.$$

Proof. Taking the inner product of (4.1) with $u^{n-\frac{1}{2}}$, we have

$$\begin{aligned}(\mathcal{H}\delta_t u^{n-\frac{1}{2}}, u^{n-\frac{1}{2}}) &= k^\alpha\sum_{p=0}^{n}\beta_p(\delta_x^2 u^{n-p-\frac{1}{2}}, u^{n-\frac{1}{2}})\\ &\quad+\rho_n(\delta_x^2 u^0, u^{n-\frac{1}{2}}) + (\mathcal{H}f^{n-\frac{1}{2}}, u^{n-\frac{1}{2}}), \qquad 1 \le n \le N.\end{aligned} \tag{4.11}$$

When $N \geq 1$, we have

$$\begin{aligned}\sum_{n=1}^{N} k(\mathcal{H}\delta_t u^{n-\frac{1}{2}}, u^{n-\frac{1}{2}}) &= k^{\alpha+1} \sum_{n=1}^{N} \sum_{p=0}^{n} \beta_p(\delta_x^2 u^{n-p-\frac{1}{2}}, u^{n-\frac{1}{2}}) \\ &+ \sum_{n=1}^{N} k\rho_n(\delta_x^2 u^0, u^{n-\frac{1}{2}}) + \sum_{n=1}^{N} k(\mathcal{H} f^{n-\frac{1}{2}}, u^{n-\frac{1}{2}}).\end{aligned} \tag{4.12}$$

Using Lemma 4.4, (4.11) and (4.12), the following process is similar to the proof of Theorem 3.4 and we omit it.

4.3. Convergence

Applying Lemma 4.3 and Theorem 4.5, we have the following convergence result.

Theorem 4.6. Assume that the problem (1.1)-(1.3) has smooth solution $u(x,t)$ in the domain $[0,L] \times (0,T]$ and $\{u_j^n | 0 \leq j \leq J,\ 1 \leq n \leq N\}$ is the solution of the difference scheme (4.1)-(4.3). Then it holds that

$$\max_{0 \leq j \leq J, 1 \leq n \leq N} |u(x_j, t_n) - u_j^n| \leq C(k^2 + h^4).$$

5. Numerical experiment

In the calculation we set $L = 1$ and $T = 1$, and compute the problem (1.1)-(1.3) by using the fractional trapezoidal rule (FTR) type difference scheme (2.16)-(2.18) and the compact fractional trapezoidal rule (CFTR) type difference scheme (4.1)-(4.3). Let u_{FTR} and u_{CFTR} be the numerical solutions, respectively.

Denote

$$E_{FTR}(k,h) = \max_{1 \leq n \leq N} \|u^n - u_{FTR}^n\|_\infty,$$

$$rate_{FTR}^t = \log_2\left(\tfrac{E_{FTR}(2k,h)}{E_{FTR}(k,h)}\right), \qquad rate_{FTR}^x = \log_2\left(\tfrac{E_{FTR}(k,2h)}{E_{FTR}(k,h)}\right).$$

Notations $E_{CFTR}(k,h)$, $rate_{CFTR}^t$ and $rate_{CFTR}^x$ are defined similarly.

Example. In the example, the exact solution is given by

$$u(x,t) = \sin \pi x - \tfrac{2t^{\alpha+1}}{3\Gamma(\alpha+1)} \sin 2\pi x, \tag{5.1}$$

so the initial datum is $u_0(x) = \sin \pi x$ and the inhomogeneous term is

$$f(x,t) = \tfrac{t^\alpha}{\Gamma(\alpha+1)}(\pi^2 \sin \pi x - \tfrac{2(\alpha+1)}{3} \sin 2\pi x) - \tfrac{8\pi^2(\alpha+1)t^{2\alpha+1}}{3\Gamma(2\alpha+2)} \sin 2\pi x. \tag{5.2}$$

When the spatial step $J = 2000$ is fixed, Table 5.1 presents the maximum errors and the corresponding convergence order in time of the FTR type difference scheme (2.16)-(2.18) and the compact FTR type difference scheme (4.1)-(4.3) for $\alpha = 1.25, 1.5, 1.75$, respectively. The numerical results reflect the convergence rate ≈ 2 in time.

Table 5.1: The maximum errors and convergence orders when $J = 2000$

N	α	E_{FTR}	$rate^t_{FTR}$	E_{CFTR}	$rate^t_{CFTR}$
4		2.2316e-2	*	2.2361e-2	*
8	$\frac{5}{4}$	6.3134e-3	1.8214	6.3142e-3	1.8243
16		1.6522e-3	1.9342	1.6522e-3	1.9342
32		4.2253e-3	1.9673	4.2248e-4	1.9674
4		3.6688e-2	*	3.6688e-2	*
8	$\frac{3}{2}$	1.0382e-2	1.8212	1.0382e-2	1.8212
16		2.7017e-3	1.9421	2.7017e-3	1.9421
32		6.8469e-4	1.9803	6.8468e-4	1.9804
4		3.9030e-2	*	3.9030e-2	*
8	$\frac{7}{4}$	1.1195e-2	1.8017	1.1195e-2	1.8017
16		2.9149e-3	1.9413	2.9148e-3	1.9414
32		7.3755e-4	1.9440	7.3745e-4	1.9828

The numerical results in Table 5.2 show that the compact FTR type difference scheme (4.1)-(4.3) is more efficient than the FTR type difference scheme (2.16)-(2.18), and the numerical solutions are convergent with the fourth-order in spatial direction. They are in good agreement with the theoretical prediction of Theorem 4.6.

Table 5.2: The maximum errors and convergence orders when $N = 2000$

J	α	E_{FTR}	$rate^x_{FTR}$	E_{CFTR}	$rate^x_{CFTR}$
4		2.0833e-1	*	1.8702e-2	*
8	$\frac{5}{4}$	5.1009e-2	2.0300	1.1000e-3	4.0876
16		1.3026e-2	1.9694	6.7608e-5	4.0242
32		3.2516e-3	2.0022	4.1648e-6	4.0209
4		1.5916e-1	*	1.3286e-2	*
8	$\frac{3}{2}$	4.0477e-2	1.9753	7.8847e-4	4.0747
16		1.0557e-2	1.9389	4.8380e-5	4.0266
32		2.6401e-3	1.9995	2.8592e-6	4.0807
4		1.1007e-1	*	8.0276e-3	*
8	$\frac{7}{4}$	3.0211e-2	1.8653	4.7956e-4	4.0652
16		7.6084e-3	1.9894	2.9386e-5	4.0285
32		1.9182e-3	1.9878	1.6788e-6	4.1296

6. Conclusions

In this article, a fractional trapezoidal rule type difference scheme is formulated and analyzed for fractional order integro-differential equation. The L^2-stability and convergence are derived. Numerical experiment is reported, which is in accordance with the theoretical results. The numerical results show that the FTR difference scheme is convergent with the order $O(k^2 + h^2)$ and the compact FTR difference scheme is $O(k^2 + h^4)$.

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References

[1] C. Chen, V. Thomee and L. B. Wahlbin, Finite element approximation of a parabolic integro-differential equation with a weakly singular kernel, Math. Comp., Vol.58, (1992), pp.587-602.

[2] H. Chen, C. Chen and D. Xu, A second-order fully discrete difference scheme for a partial integro-differential equation (in Chinese), Math. Numer. Sinica, Vol.28, (2006), pp.141-154.

[3] H. Chen, D. Xu, A compact difference scheme for an evolution equation with a weakly singular kernel, Numer. Math. Theor. Meth. Appl.,Vol.5, (2012), pp.559-572.

[4] H. Chen, S. Gan, D. Xu and Q. Liu, A second order BDF compact difference scheme for fractional order Volterra equations, I. J. Computer Math.,(2015), DOI:10.1080/00207160.2015.1021695.

[5] E. Cuesta, C. Palencia, A fractional trapezoidal rule for integro-differential equations of fractional order in Banach spaces, Appl. Numer. Math., Vol.45, (2003), pp.139-159.

[6] E. Cuesta, C. Lubich and C. Palencia, Convolution quadrature time discretization of fractional diffusion-wave equations, Math. Comp., Vol.75, (2006), pp.673-696.

[7] Y. Fujita, Integro-differential equation which interpolates the heat equation and the wave equation, Osaka J. Math., Vol.27, (1990), pp.319-327.

[8] Y. Fujita, Integro-differential equation which interpolates the heat equation and the wave equation (II), Osaka J. Math., Vol.27, (1990), pp.797-804.

[9] R. Garrappa, Trapezoidal methods for fractional differential equations: theoretical and computational aspects, Math. Comp. Simulation, Vol.110, (2015), pp.96-112.

[10] Y. Lin, C. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation, J. Comp. Phys., Vol.225, (2007), pp.1533-1552.

[11] J. C. Lopez-Marcos, A difference scheme for a nonlinear partial integro-differential equation, SIAM. J. Numer. Anal., Vol.27, (1990), pp.20-31.

[12] C. Lubich, Discretized fractional calculus, SIAM. J. Math. Anal., Vol.17,(1986), pp.704-719.

[13] C. Lubich, Convolution quadrature and discretized operational calculus, I, Numer. Math., Vol.52,(1988), pp.187-199.

[14] C. Lubich, On convolution quadrature and Hille-Philips operational calculus, Appl. Numer. Math., Vol.9,(1992), pp.704-719.

[15] C. Lubich, I. H. Sloan and V.Thomee, Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term, Math. Comp., Vol.65, (1996), pp.1-17.

[16] W. Mclean, V. Thomee, Numerical solution of an evolution equation with a positive type memory term, J. Austral. Math. Soc. Ser. B, Vol.35, (1993), pp.23-70.

[17] W. Mclean, K. Mustapha, A second-order accurate numerical method for a fractional wave equation, Numer. Math., Vol.105, (2007), pp.481-510.

[18] I. Podlubny, Fractional differential equations, Academic Press, San Diego, 1999.

[19] J. M. Sanz-Serna, A Numerical method for a partial integro-differential equation, SIAM. J. Numer. Anal., Vol.25, (1988), pp.319-327.

[20] Z. Sun, X. Wu, A fully discrete difference scheme for a diffusion-wave system, Appl. Numer. Math., Vol.56, (2006), pp.193-209.

[21] Z. Sun, On the compact difference schemes for heat equation with Neumann boundary conditions, Numer. Methods Partial Differential Eqs., Vol.25, (2009), pp.1320-1341.

[22] Y. Zhang, Z. Sun and H. Wu, Error estimates of Crank-Nicolson type difference schemes for the sub-diffusion equation, SIAM. J. Numer. Anal., Vol.49, (2011), pp.2302-2322.

[23] Y. Zhang, Z. Sun and X. Zhao, Compact alternating direction implicit scheme for the two-dimensional fractional diffusion-wave equation, SIAM. J. Numer. Anal., Vol.50, (2012), pp.1535-1555.

[24] T. Tang, A finite difference scheme for partial integro-differential equations with a weakly singular kernel, Appl. Numer. Math., Vol.11, (1993), pp.309-319.

[25] D. Xu, The global behavior of time discretization for an abstract Volterra equation in Hilbert space, Calcolo, Vol.34, (1997), pp.71-104.

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