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THE FEKETE-SZEGÖ PROBLEM FOR CERTAIN CLASS OF UNIFORMLY ANALYTIC FUNCTIONS

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Abstract. In this paper we used a family of linear operator defined on the space of univalent functions to introduce and investigate a new subclass related to uniformly starlike functions and we solve the Fekete-Szegö problem for this class.

1. Introduction

Let $\mathcal{A}$ denote the class of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc

$$U = \{z : |z| < 1\}$$

Let $f \in \mathcal{A}$ be given by (1) and g be given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{2}$$

The Hadamard product (or convolution) $(f * g)$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). \tag{3}$$

For two functions $f(z)$ and $F(z)$, analytic in U, we say that $f(z)$ is subordinate to $F(z)$, written symbolically as follows:

$$f \prec F \text{ in } U \text{ or } f(z) \prec F(z)(z \in U),$$

if there exists a Schwarz function $\omega(z) \in \Omega$, which (by definition) is analytic in U with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \quad (z \in U)$$

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such that

$$f(z) = F(\omega(z))(z \in U).$$

Indeed it is known that

$$f(z) \prec F(z)(z \in U) \implies f(0) = F(0) \text{ and } f(U) \subset F(U).$$

In particular, if the function $F(z)$ is univalent in U, we have the following equivalence

$$f(z) \prec F(z)(z \in U) \iff f(0) = F(0) \text{ and } f(U) \subset F(U) \text{ (see [23]).}$$

A function f in $\mathcal{A}$ is said to be uniformly convex in U if f is a univalent convex function along with the property that, for every circular arc γ contained in U, with center ξ also in U, the image curve $f(\gamma)$ is a convex arc. The class of uniformly convex functions is denoted by UCV (see [13]). It is well known [21] that $f \in UCV$ if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in U),$$

and the corresponding class UST is defined by the relation that $f \in UST$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in U).$$

Uniformly starlike and convex functions were first introduced by Goodman [13] and then studied by various other authors (e.g. [6, 16, 25]).

Also, a function $f \in \mathcal{A}$ is said to be in the class of uniformly convex functions of order γ and type β denoted by $UC(\beta, \gamma)$ (see [1] and [5]) if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \beta\left|\frac{zf''(z)}{f'(z)}\right| + \gamma \quad (\beta \geq 0, 0 \leq \gamma < 1; z \in U), \tag{4}$$

and is said to be in a corresponding class denoted by $SP(\beta, \gamma)$ if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq \beta\left|\frac{zf'(z)}{f(z)} - 1\right| + \gamma \quad (\beta \geq 0, 0 \leq \gamma < 1; z \in U). \tag{5}$$

We note that

$$f(z) \in UC(\beta, \gamma) \Leftrightarrow zf'(z) \in SP(\beta, \gamma) \tag{6}$$

Geometric interpretation. Let $f \in SP(\beta, \gamma)$ and $f \in UC(\beta, \gamma)$ if and only if $\frac{zf'(z)}{f(z)}$ and $1 + \frac{zf''(z)}{f'(z)}$, respectively, takes all the values in the conic domain $\mathcal{R}_{\beta,\gamma}$ which is included in the right half plane given by

$$\mathcal{R}_{\gamma,\beta} = \left\{w = u + iv \in \mathbb{C} : u > \beta\sqrt{(u-1)^2 + v^2} + \gamma, \beta \geq 0 \text{ and } \gamma \in [0, 1)\right\}, \tag{7}$$

with $p(z) = \frac{zf'(z)}{f(z)}$ or $p(z) = 1 + \frac{zf''(z)}{f'(z)}$ and considering the functions which map U onto conic domain $\mathcal{R}_{\beta,\gamma}$, such that $1 \in \mathcal{R}_{\beta,\gamma}$, we can write the conditions (4) or (5) in the form:

$$p(z) \prec P_{\beta,\gamma}(z) \tag{8}$$

Denote by $\mathcal{P}(P_{\beta,\gamma})$ $(\beta \geq 0, 0 \leq \gamma < 1)$, the family of functions p, such that $p \in \mathcal{P}$ and $p \prec P_{\beta,\gamma}$ in U, where $\mathcal{P}$ denotes the well-known class of Caratheodory functions and the function $P_{\beta,\gamma}$ maps the unit disk conformally onto the domain $\mathcal{R}_{\gamma,\beta}$ such that $1 \in \mathcal{R}_{\beta,\gamma}$ and $\partial\mathcal{R}_{\beta,\gamma}$ is a curve by the equality

$$\partial\mathcal{R}_{\gamma,\beta} = \left\{ w = u + iv \in \mathbb{C} : u^2 = \left(\beta\sqrt{(u-1)^2 + v^2} + \gamma \right)^2, \beta \geq 0 \text{ and } \gamma \in [0,1) \right\}.$$

From elementary computations we see that $\partial\mathcal{R}_{\beta,\gamma}$ represent the conic sections symmetric about the real axis. Thus $\mathcal{R}_{\beta,\gamma}$ is an elliptic domain for $\beta > 1$, a parabolic domain for $\beta = 1$, a hyperbolic domain for $0 < \beta < 1$ and a right half plain $u > \gamma$ for $\beta = 0$.
The functions that play the role of extremal functions of the class $\mathcal{P}(P_{\beta,\gamma})$, where obtained in [1] as follows:

$$P_{\beta,\gamma}(z) = \begin{cases} \frac{1+(1-2\gamma)z}{1-z}, & \beta = 0, \\ 1 + \frac{2(1-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & \beta = 1, \\ \frac{1-\gamma}{1-\beta^2}\cos\left\{\left(\frac{2}{\pi}\cos^{-1}\beta\right) i \log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\} - \frac{\beta^2-\gamma}{1-\beta^2}, & 0 < \beta < 1, \\ \frac{1-\gamma}{1-\beta^2}\sin\left(\frac{\pi}{2K(t)}\right)\int_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-t^2x^2}}dx + \frac{\beta^2-\gamma}{1-\beta^2}, & \beta > 1, \end{cases} \tag{9}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}, t \in (0,1), z \in U$ and t is chosen such that $\beta = \cosh\frac{\pi K'(t)}{4K(t)}, K(t)$ is Legendre's complete elliptic integral of the first kind and $K'(t)$ is the complementary integral of $K(t)$. The Jacobi elliptic integral (or normal elliptic integral) of first kind (see [37]) is defined by

$$\mathcal{F}(\omega, t) = \int_0^{\omega} \frac{dx}{(1-x^2)(1-t^2x^2)} \quad (0 < t < 1).$$

The function $\mathcal{F}(1,t) = K(t)$ is called the complete elliptic integral of the first kind. The following properties of $K(t)$ and $K'(t)$ are well known in [15].
$\lim_{t\to 0^+} K(t) = \frac{\pi}{2}, \qquad \lim_{t\to 1^-} K(t) = \infty.$
Moreover, the function

$$\nu(t) = \frac{\pi}{2}.\frac{K'(t)}{K(t)} \quad (t \in (0,1))$$

strictly decreases from ∞ to 0 as t moves from 0 to 1. Therefore every positive number β can be expressed as $\beta = \cosh(\nu(t))$ for some unique $t \in (0,1)$.
For $\beta = 0$ obviously $P_{0,\gamma}(z) = 1 + 2(1-\gamma)z + 2(1-\gamma)z^2 + ...$, for $\beta = 1$ (compare [21] and [31]) $P_{1,\gamma}(z) = 1 + \frac{8}{\pi^2}(1-\gamma)z + \frac{16}{3\pi^2}(1-\gamma)z^2 + ...$, by comparing Taylor series expansion in [16], we have for $0 < \beta < 1$

$$P_{\beta,\gamma}(z) = 1 + \frac{1-\gamma}{1-\beta^2}\sum_{n=1}^{\infty}\left[\sum_{\ell=1}^{2n} 2^{\ell}\binom{B}{\ell}\binom{2n-1}{2n-\ell}\right] z^n,$$

where $B = \cos^{-1}\beta$ and for $\beta > 1$,

$$P_{\beta,\gamma}(z) = 1 + \frac{\pi^2(1-\gamma)}{4\sqrt{t}(\beta^2-1)K^2(t)(1+t)} \times \left[z + \frac{4K^2(t)\left(t^2+6t+1\right) - \pi^2}{24\sqrt{t}K^2(t)(1+t)} z^2 + ... \right].$$

We consider the linear operator $\mathcal{J}_{\lambda,\ell}^{m}(a,c,A) : \mathcal{A} \longrightarrow \mathcal{A}$ which was introduced by Raina and Sharma [27]

$$\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z) = z + \frac{\Gamma(c+A)}{\Gamma(a+A)}\sum_{n=2}^{\infty}\left(1+\frac{\lambda(n-1)}{1+\ell}\right)^m \frac{\Gamma(a+nA)}{\Gamma(c+nA)} a_n z^n, \tag{10}$$

where

$$\varphi_{n,m}(\lambda,\ell,a,c,A) = \frac{\Gamma(c+A)}{\Gamma(a+A)}\left(1+\frac{\lambda(n-1)}{1+\ell}\right)^m \frac{\Gamma(a+nA)}{\Gamma(c+nA)}, \tag{11}$$

with, $m \in \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ and $A > 0, \lambda \geq 0, \ell > -1, a, c \in \mathbb{C}$ be such that $Re(c-a) > 0$ and $Re(a) > -A$.
We may point out here that some of the special cases of the operator defined by (10) can be found in [2, 4, 6, 7, 8, 9, 14, 10, 32, 33].

Now, we define new subclasses of univalent functions by the linear operator $\mathcal{J}^m_{\lambda,\ell}(a,c,A)(z)$ as follows:

Definition 1. *For $m \in \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ and $A > 0, \lambda \geq 0, \ell > -1, a, c \in \mathbb{C}$ be such that $Re(c-a) > 0$ and $Re(a) > -A$, a function $f(z) \in \mathcal{A}$ is said to be in the class $SP^m_{\lambda,\ell}(a,c,A,\beta,\gamma)$ if it is satisfies the following condition:*

$$Re\left\{\frac{z\left(\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z)\right)'}{\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z)}\right\} \geq \beta\left|\frac{z\left(\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z)\right)'}{\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z)} - 1\right| + \gamma$$
$$(\beta \geq 0, 0 \leq \gamma < 1; z \in U). \tag{12}$$

We note that by specializing the parameters $m, \lambda, \ell, a, c, A, \beta$ and γ the class $SP^m_{\lambda,\ell}(a,c,A,\beta,\gamma)$ reduces to several well-known subclasses of analytic functions. These subclasses are:
(i) $SP^m_{\lambda,0}(a,a,A,\beta,\gamma) = \beta\text{-}SP^{m,0}_{\lambda,0}(\gamma)$ (see [26]);
(ii) $SP^0_{0,0}(a,a,A,0,0) = S^*$ (see [12]);
(iii) $SP^1_{1,0}(a,a,A,0,0) = CV$ (see [12]);
(iv) $SP^m_{0,0}(a,a,A,0,\gamma) = ST^m(\gamma)$ (see [32]);
(v) $SP^m_{1,0}(a,a,A,\beta,0) = \beta\text{-}SP^m$ (see [16]);
(vi) $SP^1_{0,0}(a,a,A,1,0) = SP$ (see [31]);
(vii) $SP^1_{1,0}(a,a,A,1,0) = UCV$ (see [12, 21]);
(viii) $SP^1_{1,0}(a,a,A,\beta,0) = \beta\text{-}UCV$ (see [18]);
(ix) $SP^1_{0,0}(a,a,A,\beta,0) = \beta\text{-}SP$ (see [17]);
(x) $SP^1_{0,0}(a,a,A,\beta,\gamma) = \beta\text{-}SP(\gamma)$ (see [1]);
(xi) $SP^1_{\lambda,0}(a,a,A,\beta,\gamma) = \beta\text{-}UCV(\gamma)$ (see [1]);
(xii) $SP^0_{0,0}(a,a,A,0,\gamma) = S^*(\gamma)$ (see [30]);
(xiii) $SP^1_{0,0}(a,a,A,0,\gamma) = CV(\gamma)$ (see [30]).
Also, we note that:
(i) $f(z) \in SP^m_{\lambda,\ell}(a,c,A,\beta,\gamma)$ if and only if $\mathcal{J}^m_{\lambda,\ell}(a,c,A)f(z) \in SP(\beta,\gamma)$ (see [10]);
(ii) Using the Alexander type relation, we define the class $UC^m_{\lambda,\ell}(a,c,A,\beta,\gamma)$ as follows:
$f(z) \in UC^m_{\lambda,\ell}(a,c,A,\beta,\gamma)$ if and only if $zf'(z) \in SP^m_{\lambda,\ell}(a,c,A,\beta,\gamma)$;
(iii) $UC^m_{\lambda,\ell}(a,c,A,\beta,\gamma) \subset SP^m_{\lambda,\ell}(a,c,A,\beta,\gamma)$ (see also [10]).

In 1933, Fekete and Szego [11] proved that

$$\left|\mu a_2^2 - a_3\right| \leq \begin{cases} 4\mu - 3;\ \mu \geq 1, \\ 1 + \exp^{-\frac{2\mu}{1-\mu}}; 0 \leq \mu \leq 1, \\ 3 - 4\mu; \mu \leq 0 \end{cases} \tag{13}$$

holds for function $f \in S$ and the result is sharp. The problem of finding the sharp bounds for the non linear functional $\left|\mu a_2^2 - a_3\right|$ of any compact family of functions is popularly known as the Fekete-Szego problem. Several known authors at different time have applied the clssical Fekete- Szego to various classes to obtain different sharp bounds for example Keogh and Merkes in 1969 [19] obtained the upper bound of the Fekete-Szego funcnal $\left|\mu a_2^2 - a_3\right|$ for some classes of univalent functions S (see also [28, 29, 34, 35]).

To prove our results, we will need the following lemmas:

Lemma 1. [3]. *It can be verified that Rieman map $P_{\beta,\gamma}$ of U region $\mathcal{R}$, satisfying $P_{\beta,\gamma}(0) = 1$ and $P'_{\beta,\gamma}(0) > 0$, is given by*

$$P_{\beta,\gamma}(z) = 1 + P_1 z + P_2 z^2 + ... \quad (z \in U) \tag{14}$$

then

$$P_1 = \begin{cases} \frac{2(1-\gamma)\theta^2}{1-\beta^2}; & 0 \leq \beta < 1, \\ \frac{8(1-\gamma)}{\pi^2}; & \beta = 1, \\ \frac{\pi^2(1-\gamma)}{4\sqrt{t}(\beta^2-1)K^2(t)(1+t)} & \beta > 1, \end{cases} \tag{15}$$

and

$$P_2 = \begin{cases} \frac{\left(\theta^2+2\right)}{3} P_1; & 0 \leq \beta < 1, \\ \frac{2}{3} P_1; & \beta = 1, \\ \frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)(1+t)} P_1 & \beta > 1, \end{cases} \tag{16}$$

where

$$\theta = \frac{2}{\pi} \cos^{-1} \beta, \tag{17}$$

and $K(t)$ is Legendre's complete elliptic integral of the first kind

Lemma 2. [3, 25]. *Let $h(z) \in \mathcal{P}$ given by*

$$h(z) = 1 + c_1 z + c_2 z^2 + ... \qquad (z \in U).$$

Then $|c_n| \leq 2 \ \ (n \in \mathbb{N} = \{1, 2, 3, ...\})$,

$\left|c_2 - c_1^2\right| \leq 2$ and $\left|c_2 - \frac{1}{2} c_1^2\right| \leq 2 - \frac{1}{2} |c_1|^2$.

Furtheremore, we introduce the following functions which will be used in the discussion of sharpness of our results.

Now, we define the function $\phi_{n,m}(\lambda, \ell, a, c, A; z)$ by

$$\phi_{n,m}(\lambda, \ell, a, c, A; z) = z + \sum_{n=2}^{\infty} \varphi_{n,m}(\lambda, \ell, a, c, A) z^n,$$

where $\varphi_{n,m}(\lambda, \ell, a, c, A)$ is given by (11) and

$$\phi_{n,m}(\lambda, \ell, a, c, A; z) * \phi_{n,m}^{(\dagger)}(\lambda, \ell, a, c, A; z) = \frac{z}{1 - z}.$$

Also, define the function g in U by

$$g(z)=\frac{1}{z}\left[\phi_{n,m}^{(\mathfrak{t})}(\lambda,\ell,a,c,A;z)*\left\{z\exp\left(\int_0^z\frac{P_{\beta,\gamma}(\xi)-1}{\xi}d\xi\right)\right\}\right]\ \ (z\in U). \tag{18}$$

Finally, we consider the following extremal function $\mathcal{K}(z,\theta,\tau)$ in the class $SP_{\lambda,\ell}^m(a,c,A,\beta,\gamma)$ by

$$\begin{aligned}\mathcal{K}(z,\theta,\tau) &= \phi_{n,m}^{(\mathfrak{t})}(\lambda,\ell,a,c,A;z)*z\exp\left(\int_0^z\left[P_{\beta,\gamma}\left(\frac{e^{i\theta}\xi(\xi+\tau)}{1+\tau\xi}\right)-1\right]\frac{d\xi}{\xi}\right)\\ (0 &\le \theta\le 2\pi;0\le\tau\le 1).\end{aligned} \tag{19}$$

Note that $\mathcal{K}(z,0,1)=zg(z)$ defined by (18) and $\mathcal{K}(z,\theta,0)$ is an add function.
The object of this paper, we obtain the Fekete-Szegö inequalities for the class $SP_{\lambda,\ell}^m(a,c,A,\beta,\gamma)$.

2. Main results

Unless otherwise mentioned we shall assume throughout the paper that
$m\in\mathbb{Z}$, $\beta\ge 0, 0\le\gamma<1$, $A>0, \lambda\ge 0, \ell>-1, a,c\in\mathbb{C}$ be such that $Re(c-a)>0$ and $Re(a)>-A$.

Theorem 1. *Let the function $f(z)$ given by (1) be in the class $SP_{\lambda,\ell}^m(a,c,A,\beta,\gamma)$ $(0\le\gamma<1;0\le\beta<1)$.*

Then

$$\left|\mu a_2^2-a_3\right|\le\begin{cases}\frac{2(1-\gamma)\theta^2}{(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left(\frac{2(1-\gamma)\theta^2\varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu-\frac{\left(7-6\gamma-\beta^2\right)\theta^2}{6(1-\beta^2)}-\frac{1}{3}\right);\ \mu\ge\eta_1,\\ \frac{(1-\gamma)\theta^2}{(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)};\qquad \eta_2\le\mu<\eta_1,\\ \frac{2(1-\gamma)\theta^2}{(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left(\frac{\left(7-6\gamma-\beta^2\right)\theta^2}{6(1-\beta^2)}-\frac{2(1-\gamma)\theta^2\varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi_{3,m}^2(\lambda,\ell,a,c,A)}\mu+\frac{1}{3}\right);\mu\le\eta_2,\end{cases} \tag{20}$$

where $\varphi_{k,m}(\lambda,\ell,a,c,A)$ and θ are given by (11) and (17), respectively, and

$$\eta_1=\frac{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}{12(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left(\frac{5\left(1-\beta^2\right)}{\theta^2}+\left(7-6\gamma-\beta^2\right)\right), \tag{21}$$

$$\eta_2=\frac{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}{12(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left(\left(7-6\gamma-\beta^2\right)-\frac{\left(1-\beta^2\right)}{\theta^2}\right). \tag{22}$$

Proof. Let the function $f(z)$ is given by (1) be in class $SP_{\lambda,\ell}^m(a,c,A,\beta,\gamma)$, then there exists a function $\omega\in\mathcal{A}$ satisfies

$$\omega(0)=0 \text{ and } |\omega(z)|<1\ \ (z\in U),$$

such that

$$\frac{z\left(\mathcal{J}_{\lambda,\ell}^m(a,c,A)f(z)\right)'}{\mathcal{J}_{\lambda,\ell}^m(a,c,A)f(z)}=P_{\beta,\gamma}(\omega(z))\ (z\in U), \tag{23}$$

where $P_{\beta,\gamma}$ is defined by Lemma 1.
Define the function $h\in P$ as follows:

$$h(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+c_1z+c_2z^2+... \qquad (z\in U). \tag{24}$$

It folows that

$$\omega(z)=\frac{c_1}{2}z+\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)z^2+... \tag{25}$$

and

$$\begin{aligned}P_{\beta,\gamma}(\omega(z)) &= 1+P_1\left\{\frac{c_1}{2}z+\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)z^2+...\right\}\\ &\quad +P_2\left\{\frac{c_1}{2}z+\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)z^2+...\right\}^2+...\\ &= 1+\frac{P_1c_1}{2}z+\left\{\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)P_1+\frac{1}{4}c_1^2P_2\right\}z^2+.... \end{aligned} \tag{26}$$

Then, by using (23) and (26), we have

$$a_2=\frac{P_1}{2\varphi_{2,m}(\lambda,\ell,a,c,A)}c_1, \tag{27}$$

and

$$a_3=\frac{1}{2\varphi_{3,m}(\lambda,\ell,a,c,A)}\left\{\frac{P_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}c_1^2P_2+\frac{1}{4}c_1^2P_1^2\right\} \tag{28}$$

Putting the values of P_1 and P_2 for $0\le\beta<1$ from Lemma 1 in (27) and (28), we obtain

$$a_2=\frac{(1-\gamma)\,\theta^2}{\varphi_{2,m}(\lambda,\ell,a,c,A)\,(1-\beta^2)}c_1, \tag{29}$$

and

$$a_3=\frac{(1-\gamma)\theta^2}{2(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left\{c_2-\frac{1}{6}\left(1-\frac{\left(7-6\gamma-\beta^2\right)\theta^2}{(1-\beta^2)}\right)c_1^2\right\}. \tag{30}$$

From (29) and (30), we have

$$\mu a_2^2-a_3=\frac{(1-\gamma)\theta^2}{4(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left\{\left[\frac{4(1-\gamma)\theta^2\varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu+\frac{1}{3}\left(1-\frac{\left(7-6\gamma-\beta^2\right)\theta^2}{(1-\beta^2)}\right)\right]c_1^2-2c_2\right\}. \tag{31}$$

Therefore, using (31), we get

$$\left|\mu a_2^2-a_3\right|\le\frac{(1-\gamma)\theta^2}{4(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left\{\left[\frac{4(1-\gamma)\theta^2\varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu-\frac{5}{3}-\frac{\left(7-6\gamma-\beta^2\right)\theta^2}{3(1-\beta^2)}\right]|c_1|^2+2\left|c_1^2-c_2\right|\right\}. \tag{32}$$

If $\mu\ge\eta_1$, then by applying Lemma 2, we obtain

$$\begin{aligned}\left|\mu a_2^2-a_3\right| &\le \frac{(1-\gamma)\theta^2}{4(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left\{\left[\frac{4(1-\gamma)\theta^2\varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu-\frac{5}{3}-\frac{\left(7-6\gamma-\beta^2\right)\theta^2}{3(1-\beta^2)}\right]4+4\right\}\\ &= \frac{2(1-\gamma)\theta^2}{(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left\{\frac{2(1-\gamma)\theta^2\varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu-\frac{1}{3}-\frac{\left(7-6\gamma-\beta^2\right)\theta^2}{6(1-\beta^2)}\right\}, \end{aligned} \tag{33}$$

which is the frist part of assertion (20).

Next, if $\mu \leq \eta_2$, we can write (31) as

$$\begin{aligned} \left|\mu a_2^2 - a_3\right| &= \tfrac{(1-\gamma)\theta^2}{4(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)} \left| \left[\tfrac{\left(7-6\gamma-\beta^2\right)\theta^2}{3(1-\beta^2)} - \tfrac{1}{3} - \tfrac{4(1-\gamma)\theta^2 \varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu \right] c_1^2 + 2c_2 \right| \\ &\leq \tfrac{(1-\gamma)\theta^2}{4(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)} \left\{ \left(\tfrac{\left(7-6\gamma-\beta^2\right)\theta^2}{3(1-\beta^2)} - \tfrac{1}{3} - \tfrac{4(1-\gamma)\theta^2 \varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu \right) \left|c_1^2\right| + \left|2c_2\right| \right\} \end{aligned} \tag{34}$$

By applying Lemma 2, we obtain

$$\begin{aligned} \left|\mu a_2^2 - a_3\right| &\leq \tfrac{(1-\gamma)\theta^2}{4(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)} \left\{ \left(\tfrac{\left(7-6\gamma-\beta^2\right)\theta^2}{3(1-\beta^2)} - \tfrac{1}{3} - \tfrac{4(1-\gamma)\theta^2 \varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu \right) 4 + 4 \right\} \\ &= \tfrac{2(1-\gamma)\theta^2}{(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)} \left\{ \left(\tfrac{\left(7-6\gamma-\beta^2\right)\theta^2}{6(1-\beta^2)} + \tfrac{1}{3} - \tfrac{2(1-\gamma)\theta^2 \varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu \right), \right\} \end{aligned}$$

which is the third part of assertion (20).

$$\begin{aligned} \left|\mu a_2^2 - a_3\right| &= \tfrac{(1-\gamma)\theta^2}{4(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)} \left| \left\{ \tfrac{\left(7-6\gamma-\beta^2\right)\theta^2}{3(1-\beta^2)} + \tfrac{2}{3} - \tfrac{4(1-\gamma)\theta^2 \varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu \right\} c_1^2 + 2\left(c_2 - \frac{c_1^2}{2}\right) \right| \\ &\leq \tfrac{(1-\gamma)\theta^2}{4(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)} \left\{ \left| \tfrac{\left(7-6\gamma-\beta^2\right)\theta^2}{3(1-\beta^2)} + \tfrac{2}{3} - \tfrac{4(1-\gamma)\theta^2 \varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu \right| \left|c_1\right|^2 + 2\left|c_2 - \frac{c_1^2}{2}\right| \right\}. \end{aligned} \tag{35}$$

For $\eta_2 \leq \mu \leq \eta_1$, we have

$$\left| \tfrac{\left(7-6\gamma-\beta^2\right)\theta^2}{3(1-\beta^2)} + \tfrac{2}{3} - \tfrac{4(1-\gamma)\theta^2 \varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu \right| \leq 1.$$

By applying Lemma 2 to (35), we obtain

$$\left|\mu a_2^2 - a_3\right| \leq \tfrac{(1-\gamma)\theta^2}{(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}, \tag{36}$$

which the second part of (20). Each of the estimates in (20) is sharp for the function $\mathcal{K}(z,\theta,\tau)$ given by (19). □

Theorem 2. *Let the function $f(z)$ given by (1) be in the class $SP_{\lambda,\ell}^m(a,c,A,\beta,\gamma)$ $(0 \leq \gamma < 1; \beta = 1)$.*

Then

$$\left|\mu a_2^2 - a_3\right| \leq \begin{cases} \tfrac{8(1-\gamma)}{\pi^2\varphi_{3,m}(\lambda,\ell,a,c,A)} \left(\tfrac{8(1-\gamma)\theta^2\varphi_{3,m}(\lambda,\ell,a,c,A)}{\pi^2\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu - \tfrac{4(1-\gamma)}{\pi^2} - \tfrac{1}{3} \right); \ \mu \geq \delta_1, \\ \tfrac{4(1-\gamma)}{\pi^2\varphi_{3,m}(\lambda,\ell,a,c,A)}; \qquad \delta_2 \leq \mu < \delta_1, \\ \tfrac{8(1-\gamma)}{\pi^2\varphi_{3,m}(\lambda,\ell,a,c,A)} \left(\tfrac{4(1-\gamma)}{\pi^2} - \tfrac{8(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}{\pi^2\varphi_{3,m}^2(\lambda,\ell,a,c,A)}\mu + \tfrac{1}{3} \right); \mu \leq \delta_2, \end{cases} \tag{37}$$

where $\varphi_{k,m}(\lambda,\ell,a,c,A)$ is given by (11) and

$$\delta_1 = \frac{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}{2\varphi_{3,m}(\lambda,\ell,a,c,A)} \left(1 + \frac{5\pi^2}{24(1-\gamma)}\right), \tag{38}$$

$$\delta_2 = \frac{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}{2\varphi_{3,m}(\lambda,\ell,a,c,A)} \left(1 - \frac{\pi^2}{24(1-\gamma)}\right). \tag{39}$$

Proof. The proof of Theorem 2 is similar that to the proof of Theorem 1, with putting the values of P_1 and P_2 for $\beta = 1$ from Lemma 1 in (27) and (28), we obtain

$$a_2 = \frac{4(1-\gamma)}{\pi^2 \varphi_{2,m}(\lambda,\ell,a,c,A)} c_1,$$

and

$$a_3 = \frac{2(1-\gamma)}{\pi^2 \varphi_{3,m}(\lambda,\ell,a,c,A)} \left\{ c_2 - \frac{1}{6}\left(1 - \frac{24(1-\gamma)}{\pi^2}\right) c_1^2 \right\}$$

An easy computation shows that

$$\begin{aligned} \left|\mu a_2^2 - a_3\right| &= \frac{(1-\gamma)}{\pi^2 \varphi_{3,m}(\lambda,\ell,a,c,A)} \left| \left(\frac{16(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}{\pi^2 \varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu + \frac{1}{3} - \frac{8(1-\gamma)}{\pi^2} \right) c_1^2 - 2c_2 \right| \qquad (40) \\ &\leq \frac{(1-\gamma)}{\pi^2 \varphi_{3,m}(\lambda,\ell,a,c,A)} \left[\left| \frac{16(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}{\pi^2 \varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu - \frac{5}{3} - \frac{8(1-\gamma)}{\pi^2} \right| \left|c_1^2\right| + 2\left|c_1^2 - c_2\right| \right]. \end{aligned}$$

If $\mu \geq \delta_1$, then by applying Lemma 2, we obtain

$$\left|\mu a_2^2 - a_3\right| \leq \frac{8(1-\gamma)}{\pi^2 \varphi_{3,m}(\lambda,\ell,a,c,A)} \left(\frac{8(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}{\pi^2 \varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu - \frac{1}{3} - \frac{4(1-\gamma)}{\pi^2} \right), \qquad (41)$$

which is the frist part of assertion (37).

Next, if $\mu \leq \delta_2$, we can write (40) as

$$\begin{aligned} \left|\mu a_2^2 - a_3\right| &= \frac{(1-\gamma)}{\pi^2 \varphi_{3,m}(\lambda,\ell,a,c,A)} \left| \left(\frac{8(1-\gamma)}{\pi^2} - \frac{1}{3} - \frac{16(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}{\pi^2 \varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu \right) c_1^2 + 2c_2 \right| \\ &\leq \frac{(1-\gamma)}{\pi^2 \varphi_{3,m}(\lambda,\ell,a,c,A)} \left\{ \left| \frac{8(1-\gamma)}{\pi^2} - \frac{1}{3} - \frac{16(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}{\pi^2 \varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu \right| \left|c_1^2\right| + 2\left|c_2\right| \right\}. \end{aligned} \qquad (42)$$

By applying Lemma 2, we obtain

$$\left|\mu a_2^2 - a_3\right| \leq \frac{8(1-\gamma)}{\pi^2 \varphi_{3,m}(\lambda,\ell,a,c,A)} \left\{ \frac{4(1-\gamma)}{\pi^2} + \frac{1}{3} - \frac{8(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}{\pi^2 \varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu \right\}$$

which is the third part of assertion (37). Finally from (40), we have

$$\begin{aligned} \left|\mu a_2^2 - a_3\right| &= \frac{(1-\gamma)}{\pi^2 \varphi_{3,m}(\lambda,\ell,a,c,A)} \left| \left(\frac{8(1-\gamma)}{\pi^2} + \frac{2}{3} - \frac{16(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}{\pi^2 \varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu \right) c_1^2 + 2\left(c_2 - \frac{c_1^2}{2}\right) \right| \\ &\leq \frac{(1-\gamma)}{\pi^2 \varphi_{3,m}(\lambda,\ell,a,c,A)} \left\{ \left| \frac{8(1-\gamma)}{\pi^2} + \frac{2}{3} - \frac{16(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}{\pi^2 \varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu \right| \left|c_1\right|^2 + 2\left|c_2 - \frac{c_1^2}{2}\right| \right\}. \end{aligned} \qquad (43)$$

For $\eta_2 \leq \mu \leq \eta_1$, we have

$$\left| \frac{8(1-\gamma)}{\pi^2} + \frac{2}{3} - \frac{16(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}{\pi^2 \varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu \right| \leq 1.$$

By applying Lemma 2 to (43), we obtain

$$\left|\mu a_2^2 - a_3\right| \leq \frac{4(1-\gamma)}{\pi^2 \varphi_{3,m}(\lambda,\ell,a,c,A)}, \qquad (44)$$

which the second part of (37). Each of the estimates in (37) is sharp for the function $\mathcal{K}(z,\theta,\tau)$ given by (19). □

Theorem 3. *Let the function* $f(z)$ *given by (1) be in the class* $SP^m_{\lambda,\ell}(a,c,A,\beta,\gamma)$ $(0 \leq \gamma < 1; 1 < \beta < \infty)$ *and let* t *be the unique positive number in the open interval* $(0,1)$ *defined as in introduction. Then*

$$\left|\mu a_2^2 - a_3\right| \leq \begin{cases} \frac{P_1}{2\varphi_{3,m}(\lambda,\ell,a,c,A)}\left(\frac{2\varphi_{3,m}(\lambda,\ell,a,c,A)P_1}{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu - P_1 - \frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)(1+t)}\right); \ \mu \geq \nu_1, \\ \frac{P_1}{2\varphi_{3,m}(\lambda,\ell,a,c,A)}; \qquad \nu_2 \leq \mu < \nu_1, \\ \frac{P_1}{2\varphi_{3,m}(\lambda,\ell,a,c,A)}\left(\frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)(1+t)} + P_1 - \frac{2\varphi_{3,m}(\lambda,\ell,a,c,A)P_1}{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu\right); \mu \leq \nu_2, \end{cases} \tag{45}$$

where $K(t)$ is Legendre's complete elliptic integral of the first kind, $\varphi_{k,m}(\lambda,\ell,a,c,A)$ and P_1 are given by (11) and (15) respectively, and

$$\nu_1 = \frac{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}{2\varphi_{3,m}(\lambda,\ell,a,c,A)P_1}\left(1 + P_1 + \frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)\left(1+t\right)}\right), \tag{46}$$

$$\nu_2 = \frac{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}{2\varphi_{3,m}(\lambda,\ell,a,c,A)P_1}\left(P_1 - 1 + \frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)\left(1+t\right)}\right). \tag{47}$$

Proof. Putting the values of P_1 and P_2 for $1 < \beta < \infty$ from Lemma 1 in (27) and (28), we obtain

$$a_2 = \frac{P_1}{2\varphi_{2,m}(\lambda,\ell,a,c,A)}c_1,$$

and

$$a_3 = \frac{P_1}{4\varphi_{3,m}(\lambda,\ell,a,c,A)}\left\{c_2 - \frac{1}{2}\left(1 - P_1 - \frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)\left(1+t\right)}\right)c_1^2\right\}$$

An easy computation shows that

$$\begin{aligned} \left|\mu a_2^2 - a_3\right| &= \frac{P_1}{8\varphi_{3,m}(\lambda,\ell,a,c,A)}\left|\left(\frac{2\varphi_{3,m}(\lambda,\ell,a,c,A)P_1}{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu - P_1 - \frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)(1+t)} + 1\right)c_1^2 - 2c_2\right| \\ &\leq \frac{P_1}{8\varphi_{3,m}(\lambda,\ell,a,c,A)}\left\{\left|\frac{2\varphi_{3,m}(\lambda,\ell,a,c,A)P_1}{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu - P_1 - \frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)(1+t)} - 1\right|\left|c_1\right|^2 + 2\left|c_1^2 - c_2\right|.\right\} \end{aligned} \tag{48}$$

If $\mu \geq \nu_1$, then by applying Lemma 2, we obtain

$$\left|\mu a_2^2 - a_3\right| \leq \frac{P_1}{2\varphi_{3,m}(\lambda,\ell,a,c,A)}\left(\frac{2\varphi_{3,m}(\lambda,\ell,a,c,A)P_1}{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu - P_1 - \frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)(1+t)}\right), \tag{49}$$

which is the frist part of assertion (45).
Next, if $\mu \leq \nu_2$, then

$$\left(\frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)(1+t)} + P_1 - \frac{2\varphi_{3,m}(\lambda,\ell,a,c,A)P_1}{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu - 1\right) \geq 0. \tag{50}$$

By applying Lemma 2 in (48), we obtain

$$\begin{aligned}\left|\mu a_2^2 - a_3\right| &\leq \frac{P_1}{8\varphi_{3,m}(\lambda,\ell,a,c,A)}\left|\left|\left(\frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)(1+t)} + P_1 - \frac{2\varphi_{3,m}(\lambda,\ell,a,c,A)P_1}{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu - 1\right)\right|\left|c_1^2\right| + 2\left|c_2\right|\right| \\ &\leq \frac{P_1}{2\varphi_{3,m}(\lambda,\ell,a,c,A)}\left(\frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)(1+t)} + P_1 - \frac{2\varphi_{3,m}(\lambda,\ell,a,c,A)P_1}{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu\right)\end{aligned}$$

which is the third part of assertion (45). Finally, for $\nu_2 \leq \mu \leq \nu_1$, we have

$$\left|\frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)(1+t)} + P_1 - \frac{2\varphi_{3,m}(\lambda,\ell,a,c,A)P_1}{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu\right| \leq 1.$$

By applying Lemma 2 in (48), we have

$$\begin{aligned}\left|\mu a_2^2 - a_3\right| &= \frac{P_1}{8\varphi_{3,m}(\lambda,\ell,a,c,A)}\left|\left(\frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)(1+t)} + P_1 - \frac{2\varphi_{3,m}(\lambda,\ell,a,c,A)P_1}{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu\right)c_1^2 + 2\left(c_2 - \frac{c_1^2}{2}\right)\right| \\ &\leq \frac{P_1}{8\varphi_{3,m}(\lambda,\ell,a,c,A)}\left\{\left|c_1\right|^2 + 2\left|c_2 - \frac{c_1^2}{2}\right|\right\} \\ &\leq \frac{P_1}{2\varphi_{3,m}(\lambda,\ell,a,c,A)}\end{aligned}$$

which is the second part of (45). Each of the estimates in (45) is sharp for the function $\mathcal{K}(z,\theta,\tau)$ given by (19). □

Theorem 4. *Let the function $f(z)$ given by (1) be in the class $SP_{\lambda,\ell}^m(a,c,A,\beta,\gamma)$ $(0 \leq \gamma < 1; 0 \leq \beta < 1)$.*

Then

$$\begin{aligned}&\left|\mu a_2^2 - a_3\right| + \left\{\mu - \frac{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}{12(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left[\left(7-6\gamma-\beta^2\right) - \frac{1-\beta^2}{\theta^2}\right]\right\}\left|a_2\right|^2 \\ &\quad\leq \frac{(1-\gamma)\theta^2}{(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}; \qquad \eta_2 \leq \mu < \eta_3,\end{aligned} \tag{51}$$

and

$$\begin{aligned}&\left|\mu a_2^2 - a_3\right| + \left\{\frac{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}{12(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left[\frac{5\left(1-\beta^2\right)}{\theta^2} + \left(7-6\gamma-\beta^2\right)\right] - \mu\right\}\left|a_2\right|^2 \\ &\quad\leq \frac{(1-\gamma)\theta^2}{(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}; \qquad \eta_3 \leq \mu < \eta_1,\end{aligned} \tag{52}$$

where $\varphi_{k,m}(\lambda,\ell,a,c,A)$, θ, η_1 and η_2 are given by (11), (17),(21) and (22) respectively, and

$$\eta_3 = \frac{\left(1-\beta^2\right)\varphi_{2,m}^2(\lambda,\ell,a,c,A)}{12(1-\gamma)\theta^2\varphi_{3,m}(\lambda,\ell,a,c,A)}\left(2 + \frac{\left(7-6\gamma-\beta^2\right)}{(1-\beta^2)}\theta^2\right). \tag{53}$$

Proof. Suppose $0 \leq \beta < 1$ and $\eta_2 \leq \mu < \eta_3$. Using (35) for $\left|\mu a_2^2 - a_3\right|$ and (29) for $\left|a_2\right|$, we have $\left|\mu a_2^2 - a_3\right| + (\mu - \eta_2)\left|a_2\right|^2 = \left|\mu a_2^2 - a_3\right|$

$$\begin{aligned}&+ \left\{\mu - \frac{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}{12(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left(\left(7-6\gamma-\beta^2\right) - \frac{\left(1-\beta^2\right)}{\theta^2}\right)\right\}\left|a_2\right|^2 \\ &\leq \frac{(1-\gamma)\theta^2}{4(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left\{\left|\frac{\left(7-6\gamma-\beta^2\right)\theta^2}{3(1-\beta^2)} + \frac{2}{3} - \frac{4(1-\gamma)\theta^2\varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu\right|\left|c_1\right|^2 + 2\left|c_2 - \frac{c_1^2}{2}\right|\right\} \\ &+ \left\{\mu - \frac{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}{12(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left(\left(7-6\gamma-\beta^2\right) - \frac{\left(1-\beta^2\right)}{\theta^2}\right)\right\}\left(\frac{(1-\gamma)^2\theta^4}{\varphi_{2,m}^2(\lambda,\ell,a,c,A)(1-\beta^2)^2}\right)\left|c_1\right|^2 \\ &= \frac{(1-\gamma)\theta^2}{4(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left\{2\left|c_2 - \frac{c_1^2}{2}\right| + \left|\frac{\left(7-6\gamma-\beta^2\right)\theta^2}{3(1-\beta^2)} + \frac{2}{3} - \frac{4(1-\gamma)\theta^2\varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi_{2,m}^2(\lambda,\ell,a,c,A)}\mu\right|\left|c_1\right|^2\right.\end{aligned}$$

$$+\left(\frac{4(1-\gamma)\theta^2\varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi^2_{2,m}(\lambda,\ell,a,c,A)}\mu-\frac{\left(7-6\gamma-\beta^2\right)\theta^2}{3(1-\beta^2)}+\frac{1}{3}\right)|c_1|^2\right\}.$$

Observe that, since $\mu<\eta_3$

$$\frac{\left(7-6\gamma-\beta^2\right)\theta^2}{3(1-\beta^2)}+\frac{2}{3}-\frac{4(1-\gamma)\theta^2\varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi^2_{2,m}(\lambda,\ell,a,c,A)}\mu\geq 0$$

Therefore, by using Lemma 2, we obtain

$$\left|\mu a_2^2-a_3\right|+\left\{\mu-\frac{\varphi^2_{2,m}(\lambda,\ell,a,c,A)}{12(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left(\left(7-6\gamma-\beta^2\right)-\frac{\left(1-\beta^2\right)}{\theta^2}\right)\right\}|a_2|^2$$
$$\leq\frac{(1-\gamma)\theta^2}{4(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left\{|c_1|^2+2\left|c_2-\frac{c_1^2}{2}\right|\right\}$$
$$\leq\frac{(1-\gamma)\theta^2}{(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}\quad(\eta_2\leq\mu<\eta_3),$$

which proves (51). Similarly, for the value of μ given in (52), we get

$$\left|\mu a_2^2-a_3\right|+\left\{\frac{\varphi^2_{2,m}(\lambda,\ell,a,c,A)}{12(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left(\left(7-6\gamma-\beta^2\right)-\frac{5\left(1-\beta^2\right)}{\theta^2}\right)-\mu\right\}|a_2|^2$$
$$\leq\frac{(1-\gamma)\theta^2}{4(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left\{\left|\frac{\left(7-6\gamma-\beta^2\right)\theta^2}{3(1-\beta^2)}+\frac{2}{3}-\frac{4(1-\gamma)\theta^2\varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi^2_{2,m}(\lambda,\ell,a,c,A)}\mu\right||c_1|^2+2\left|c_2-\frac{c_1^2}{2}\right|\right\}$$
$$+\left\{\frac{\varphi^2_{2,m}(\lambda,\ell,a,c,A)}{12(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left(\left(7-6\gamma-\beta^2\right)+\frac{5\left(1-\beta^2\right)}{\theta^2}\right)-\mu\right\}\left(\frac{(1-\gamma)^2\theta^4}{\varphi^2_{2,m}(\lambda,\ell,a,c,A)(1-\beta^2)^2}\right)|c_1|^2$$
$$=\frac{(1-\gamma)\theta^2}{4(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left\{2\left|c_2-\frac{c_1^2}{2}\right|+\left|\frac{4(1-\gamma)\theta^2\varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi^2_{2,m}(\lambda,\ell,a,c,A)}\mu-\frac{\left(7-6\gamma-\beta^2\right)\theta^2}{3(1-\beta^2)}-\frac{2}{3}\right||c_1|^2\right.$$
$$\left.+\left(-\frac{4(1-\gamma)\theta^2\varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi^2_{2,m}(\lambda,\ell,a,c,A)}\mu+\frac{\left(7-6\gamma-\beta^2\right)\theta^2}{3(1-\beta^2)}+\frac{5}{3}\right)|c_1|^2\right\}.$$

Since $\mu\geq\eta_3$,

$$\frac{4(1-\gamma)\theta^2\varphi_{3,m}(\lambda,\ell,a,c,A)}{(1-\beta^2)\varphi^2_{2,m}(\lambda,\ell,a,c,A)}\mu-\frac{\left(7-6\gamma-\beta^2\right)\theta^2}{3(1-\beta^2)}-\frac{2}{3}\geq 0$$

and by using Lemma 2, we get

$$\left|\mu a_2^2-a_3\right|+\left\{\mu-\frac{\varphi^2_{2,m}(\lambda,\ell,a,c,A)}{12(1-\gamma)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left(\left(7-6\gamma-\beta^2\right)-\frac{\left(1-\beta^2\right)}{\theta^2}\right)\right\}|a_2|^2$$
$$\leq\frac{(1-\gamma)\theta^2}{4(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}\left\{|c_1|^2+2\left|c_2-\frac{c_1^2}{2}\right|\right\}$$
$$\leq\frac{(1-\gamma)\theta^2}{(1-\beta^2)\varphi_{3,m}(\lambda,\ell,a,c,A)}\quad(\mu\geq\eta_3),$$

which proves (52). The proof of Theorem 4 is completed. □

Theorem 5. *Let the function $f(z)$ given by (1) be in the class $SP^m_{\lambda,\ell}(a,c,A,\beta,\gamma)$ $(0\leq\gamma<1;\beta=1)$.*

Then

$$\begin{aligned}\left|\mu a_2^2-a_3\right|+\left\{\mu-\frac{\varphi^2_{2,m}(\lambda,\ell,a,c,A)}{2\varphi_{3,m}(\lambda,\ell,a,c,A)}\left[1-\frac{\pi^2}{24(1-\gamma)}\right]\right\}|a_2|^2\\ \leq\frac{24(1-\gamma)}{\pi^2\varphi_{3,m}(\lambda,\ell,a,c,A)};\qquad\delta_2\leq\mu<\delta_3,\end{aligned}\tag{54}$$

and

$$\begin{aligned}\left|\mu a_2^2-a_3\right|+\left\{\frac{\varphi^2_{2,m}(\lambda,\ell,a,c,A)}{2\varphi_{3,m}(\lambda,\ell,a,c,A)}\left[1+\frac{5\pi^2}{24(1-\gamma)}\right]-\mu\right\}|a_2|^2\\ \leq\frac{4(1-\gamma)}{\pi^2\varphi_{3,m}(\lambda,\ell,a,c,A)};\qquad\delta_3\leq\mu<\delta_1,\end{aligned}\tag{55}$$

where $\varphi_{k,m}(\lambda,\ell,a,c,A)$, δ_1 and δ_2 are given by (11), (38) and (39) respectively, and

$$\delta_3=\frac{\varphi^2_{2,m}(\lambda,\ell,a,c,A)}{2\varphi_{3,m}(\lambda,\ell,a,c,A)}\left(1+\frac{\pi^2}{12(1-\gamma)}\right).\tag{56}$$

Theorem 6. *Let the function $f(z)$ given by (1) be in the class $SP^m_{\lambda,\ell}(a,c,A,\beta,\gamma)$ $(0\leq\gamma<1;1<\beta<\infty)$ and let t be the unique positive number in the open interval*

$(0,1)$ *defined as in introduction. Then*

$$\begin{aligned}&\left|\mu a_2^2-a_3\right|+\left\{\mu-\frac{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}{2\varphi_{3,m}(\lambda,\ell,a,c,A)P_1}\left(P_1+\frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)\left(1+t\right)}-1\right)\right\}\left|a_2\right|^2\\ \leq\ &\frac{P_1}{2\varphi_{3,m}(\lambda,\ell,a,c,A)};\nu_2\leq\mu\leq\nu_3 \qquad (57)\end{aligned}$$

and

$$\begin{aligned}&\left|\mu a_2^2-a_3\right|+\left\{\frac{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}{2\varphi_{3,m}(\lambda,\ell,a,c,A)P_1}\left(1+P_1+\frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)\left(1+t\right)}-\mu\right)\right\}\left|a_2\right|^2\\ \leq\ &\frac{P_1}{2\varphi_{3,m}(\lambda,\ell,a,c,A)};\nu_3\leq\mu\leq\nu_1 \qquad (58)\end{aligned}$$

where $K(t)$ is Legendre's complete elliptic integral of the first kind, $\varphi_{k,m}(\lambda,\ell,a,c,A)$, P_1,ν_1 and ν_2 are given by (11), (15), (46) and (47), respectively, and

$$\nu_3=\frac{\varphi_{2,m}^2(\lambda,\ell,a,c,A)}{2\varphi_{3,m}(\lambda,\ell,a,c,A)P_1}\left(P_1+\frac{4K^2(t)\left(t^2+6t+1\right)-\pi^2}{24\sqrt{t}K^2(t)\left(1+t\right)}\right). \qquad (59)$$

The proof of Theorem 5 and Theorem 6 is similar to that of Theorem 4, except for obvious changes. Hence, we omit the details.

Remark 1. *(i) Taking $\ell=0$, $a=c$ and $m=1$ in our main results, we obtain the results obtained by Orhan et al.* [26]*, with $\alpha=\mu=0$.*

(ii) Taking $\ell=\gamma=\lambda=0$, $a=c$ and $m=1$ in Theorems 2 and 5, we obtain the results obtained by Srivastava and Mishra [36].
(iii) Taking $\ell=\gamma=0$, $a=c$ and $m=\lambda=1$ in Theorems 2, we obtain the results obtained by Ma and Minda [22].
(iv) Taking $\ell=\gamma=\beta=\lambda$, $a=c$ and $m=1$ in Theorems 1, we obtain the results obtained by Srivastava et al. [35], with $\alpha=0$.
(v) Taking $\ell=\lambda=\gamma=0$, $a=c$ and $m=1$ in all our main results, we obtain the results obtained by Mishra and Gochhayat [25], with $\alpha=0$.
(vi) Taking special cases of $\lambda,\ell,a,c,A,m,\gamma$ and β in all our main results, we obtain new results.

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