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STABILITY OF FRACTIONAL NEUTRAL AND INTEGRODIFFERENTIAL SYSTEMS

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Abstract. In this work, a brief overview on the recent stability results of fractional differential equations and the analytical methods used is given. The stability of the linear fractional system by analyzing the eigenvalues is discussed. Also the stability of the nonlinear fractional dynamical system is analyzed by giving conditions on the nonlinear term. Further the stability of fractional neutral and integrodifferential systems is studied. To show the applicability of fractional differential equations, some examples were presented.

1. Introduction

The Fractional Differential Equations (FDE) appear more and more frequently in different research areas and engineering applications. It has been found that the behavior of many physical systems can be properly described by using the fractional order theory. In fact, real world processes generally or most likely are fractional order systems. Fractional calculus can be an aid for explanation of discontinuity formation and singularity formation is an enriching thought experiment. Although fractional derivatives have a long mathematical history, for many years they were not used in physics. One possible explanation of such unpopularity could be that there are multiple nonequivalent definitions of fractional derivatives. Another difficulty is that fractional derivatives have no evident geometrical interpretation because of their nonlocal character.

However, during the last 10 years fractional calculus starts to attract much more attention of physicists and mathematicians. It was found that various, especially interdisciplinary applications can be elegantly modeled with the help of the fractional derivatives. The main reason for using the integer-order models was the absence of solution methods for fractional differential equations.

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications such as fluid mechanics, viscoelasticity, biology, physics and engineering and etc. (see [4, 16, 2]). The most important advantage of using fractional differential equation is their non-local property.

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The integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. These peculiar properties of FDE have attracted the researchers to do research in this area.

A list of mathematicians, who have provided important contributions up to the middle of our century, includes P.S. Laplace (1812), J.B.J. Fourier (1822), N.H. Abel (1823-1826), J. Liouville (1832-1873), B. Riemann (1847), A.K. Grounwall (1867-1872), A.V. Letnikov (1868-1872), J.Hadamard (1892), O. Heaviside (1892-1912), G.H. Hardy and J.E. Littlewood (1917-1928), H. Weyl (1917), P. Levy (1923), M. Riesz (1949) and etc.

The fundamental theorems of existence and uniqueness for ordinary fractional differential equations are presented in [2, 28]. The stability analysis is a central task in the study of FDE and the stability analysis has been performed by many authors see [1, 17, 23, 24]. Mittag-Leffler stability and the fractional Lyapunov direct method, which enriched the knowledge of both the system theory and the fractional calculus have been discussed in [18]. Mittag-Leffler stability is a generalization of exponential stability and power-law stability and also the convergence speed is more accurate than the exponential stability [18].

Next, the stability of a linear fractional differential equation by transforming the s-plane to the F-plane is reported in [5]. Finally, a procedure for studying the stability of a system having any number of fractional elements was explained in [27]. In [13] it is shown that the composite quadratic functions provide necessary and sufficient conditions for robust stability analysis and robust stabilization of linear differential inclusions discussed in [18, 21]. Recently, several approximated numerical methods for solving fractional differential equations have been given such as variational iteration method see [11], homotopy perturbation method see [12], Adomian's decomposition method [14], homotopy analysis method [9] and collocation method see [15].

This work is organized as follows. In section 2, some basic definitions and the stability of the linear and nonlinear fractional system are discussed by analyzing the eigenvalues and the application of the Lipschitz condition to the nonlinear term, which is based on the paper [1]. In section 3, the stability theorems for FDE, which cover the fractional integrodifferential systems and fractional neutral systems is analyzed. In section 4, the numerical method in which fractional Euler's method is used as a prediction, and the modified trapezoidal rule is used to make correction to obtain the finite value. For further details see [25, 26]. Some examples are solved and illustrated to explain the theorem.

2. Preliminaries

Basic definitions and results regarding the fractional calculus are given in this section.

Definition 1. [2]**(Riemann - Liouville Fractional Integral).** *The Riemann-Liouville fractional integral operator of order* $\alpha > 0$ *of a function* $f \in L^1(\mathcal{R}^+)$ *is defined by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s, \tag{1}$$

where $\Gamma(.)$ *is the Euler gamma function.*

Definition 2. [2]**(Riemann - Liouville Fractional Derivative).** *The Riemann-Liouville fractional derivative of order* $\alpha > 0, n-1 < \alpha < n, n \in N$, *is defined as*

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_{0}^{t} (t-s)^{n-\alpha-1} f(s) \mathrm{d}s, \tag{2}$$

where D^n *is the ordinary differential operator and the function* $f(t)$ *has absolutely continuous derivative upto order* $(n-1)$.

Definition 3. [2]**(Caputo Fractional Derivative).** *The Caputo fractional derivative of order* $\alpha > 0, n-1 < \alpha < n, n \in N$, *is defined as*

$$^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{n-\alpha-1} f^{n}(s) \mathrm{d}s, \tag{3}$$

where the function $f(t)$ *has absolutely continuous derivative upto order* $(n-1)$.

Mittag-Leffler Function and its Asymptotic Approximation. The exponential function plays a big role in the theory of ordinary differential equations. Another function, which plays a very important role in the fractional calculus,which is the generalization of the exponential function, it is proved that they are the solutions of superior differential equations called the fractional differential equations. Since a few decades the special transcendental function known as ML function has attracted an increasing attention of researchers because of its key role in treating problems related to integral and differential equations of fractional order. Due to its attractive feature it is referred as **the Queen Function of Fractional Calculus**.

Definition 4. [2]**Mittag-Leffler function.** *The one-parameter Mittag-Leffler function is defined as*

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad (\alpha > 0, z \in \mathbb{C}). \tag{4}$$

The two-parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad (\alpha, \beta > 0, z \in \mathbb{C}). \tag{5}$$

The Mittag-Leffler function of a matrix A is defined by

$$E_{\alpha,\beta}(At) = \sum_{k=0}^{\infty} \frac{(At)^k}{\Gamma(\alpha k + \beta)}, \qquad (\alpha, \beta > 0, A \in \mathbb{R}^{n\times n}). \tag{6}$$

The Laplace transform of Mittag-Leffler functions are given by

$$\mathcal{L}\{t^{\beta-1} E_{\alpha,\beta}(-\lambda t^{\alpha})\} \quad = \quad \frac{s^{\alpha-\beta}}{s^{\alpha} + \lambda}, \ (\mathcal{R}(s) > |\lambda|^{\frac{1}{\alpha}}), \tag{7}$$

where $t \geq 0, \lambda \in \mathbb{R}$.

Lemma 1. [2] *Let* $0 < \alpha < 2, \beta$ *be a an arbitrary complex number and* μ *be an arbitrary real number such that* $\frac{\pi\alpha}{2} < \mu < min\{\pi, \pi\alpha\}$. *Then, for an arbitrary integer* $p \geq 1$, *we have the following expansions:*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} exp(z^{1/\alpha}) - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O(|z|^{-1-p}), \tag{8}$$

when $|\arg(z)| \leq \mu$ *and* $|z| \to \infty$;

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O(|z|^{-1-p}), \tag{9}$$

when $\mu \leq |\arg(z)| \leq \pi$ *and* $|z| \to \infty$.

In particular If $\alpha = \beta$, then we have

$$E_{\alpha,\alpha}(z) = \frac{1}{\alpha} z^{(1-\alpha)/\alpha} exp(z^{1/\alpha}) - \sum_{k=2}^{p} \frac{z^{-k}}{\Gamma(\alpha - \alpha k)} + O(|z|^{-1-p}),$$

when $|\arg(z)| \leq \mu$ and $|z| \to \infty$;

$$E_{\alpha,\alpha}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\alpha - \alpha k)} + O(|z|^{-1-p}),$$

when $\mu \leq |\arg(z)| \leq \pi$ and $|z| \to \infty$.

Lemma 2. [5] *The following properties hold.*

(1) *There exists finite real constants* $M_1, M_2 \geq 1$ *such that for any* $0 < \alpha < 1$,

$$||E_\alpha(At^\alpha)|| \leq M_1 ||e^{At}||, \tag{10}$$

$$||E_{\alpha,\alpha}(At^\alpha)|| \leq M_2 ||e^{At}||, \tag{11}$$

where A *denotes matrix,* $||\cdot||$ *denotes any matrix norm.*

(2) *If* $\alpha \geq 1$, *then for* $\beta = 1, 2, \alpha$

$$||E_{\alpha,\beta}(At^\alpha)|| \leq ||e^{At^\alpha}||. \tag{12}$$

In addition,if A is a diagonal stability matrix then there exists a constant $N > 0$ such that for $t \geq 0$

$$\begin{aligned} ||E_{\alpha,\beta}(At^\alpha)|| &\leq Ne^{-\lambda t}, 0 < \alpha < 1; \\ ||E_{\alpha,\beta}(At^\alpha)|| &\leq e^{-\lambda t}, 1 \leq \alpha < 2, \end{aligned} \tag{13}$$

where λ is the largest eigenvalue of the diagonal matrix. In Particular $E_\alpha(z)$ is bounded in the region $\frac{\alpha\pi}{2} < \arg(z) < 2\pi - \frac{\alpha\pi}{2}$. In particular when $\alpha = 1$, that is, e^z is bounded when $|\arg(z)| > \frac{\pi}{2}$. The asymptotic behavior of Mittag-Leffler function is not of exponential form but it is the form of $t^{-\alpha}, (\alpha \in \mathbb{R})$. In particular consider the Mittag-Leffler function $E_\alpha(-t^\alpha)$. This function interpolates the negative power law due to its very slow decay for long times. Thus,

$$E_\alpha(-t^\alpha) \approx \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad t \to \infty. \tag{14}$$

Also, it may be observed that the behavior of Mittag-Leffler function is relaxation for $\alpha < 1$, is exponential for $\alpha = 1$, becomes a damped oscillation for $1 < \alpha < 2$ and oscillates for $\alpha = 2$. The decay is very fast as $t \to 0^+$ and very slow as $t \to \infty$.

3. **Stability of a Fractional Nonlinear Systems**

This section describes the basic definition of stability and the prevailing research work on the stability of a nonlinear dynamical system which is related to the work .

Definition 5. [1]**(Stability of a Linear System).** *Consider the following fractional differentialsystem involving Caputoderivative*

$$^{C}D^{\alpha}x(t) = Ax(t), \tag{15}$$

with initial value $x(0) = x_0 = (x_{10}, x_{20}, \dots, x_{n0})^T$*, where* $x = (x_1, x_2, \dots, x_n)^T, \alpha \in (0, 1)$ *and* $A \in \mathbb{R}^{n\times n}$*. The autonomous system* (15) *is said to be*

(i) *stable iff for any* x_0*, there exists* $\epsilon > 0$ *such that* $||x(t)|| \le \epsilon$ *for* $t \ge 0$*,*
(ii) *asymptotically stable iff* $\lim_{t\to\infty} ||x(t)|| = 0$.

Theorem 1. [1] *The autonomous system* (15) *is asymptotically stable iff*

$$|\arg(spec(A))| > \frac{\alpha\pi}{2} \tag{16}$$

In this case, the components of the state decay towards 0 *like* $t^{-\alpha}$.

The proof of the above theorem was simply sketched in [1, 23] and Zeng.et al [10] have proved in details by using the Mittag-Leffler function.
Consider the nonlinear system of the form

$$^{C}D^{\alpha}x(t) = Ax(t) + f(t, x(t)), \qquad \alpha \in (0, 1) \tag{17}$$

where $f(t, x) \in \mathrm{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n), f(t, 0) = 0$, and the initial condition is given by $x(0) = x_0$, where $x = (x_1, \dots, x_n)^T$, and $A \in \mathbb{R}^{n\times n}$.

Theorem 2. [1] *Suppose* $||f(t, x(t))|| \le M||x||$ *and all the eigenvalues of* A *satisfy (16). Then, the zero solution of* (17) *is asymptotically stable.*

The above theorem was analyzed in [1, 23].
This section consists of a Neutral and integrodifferential system of fractional order and its stability is analyzed with the help of eigenvalues and the conditions on the nonlinear system. Examples are given to illustrate the theorem.

4. **Fractional Neutral Differential Equations**

Consider the nonlinear system of the form

$$^{C}D^{\alpha}[x(t) - g(t, x(t))] = Ax(t) + f(t, x(t)), \tag{18}$$

where the initial condition is given by $x(0) = x_0$ and $f(t, x)$, $g(t, x) \in \mathbb{C}^1(J \times \mathbb{R}^n, \mathbb{R}^n)$, $f(t, 0) = 0$, $g(0, x_0) \ne x_0$, where $0 < \alpha < 1$, $x = (x_1, \dots, x_n)^T$, and $A \in \mathbb{R}^{n\times n}$.
In order to obtain main results, we need the following lemmas and make the following assumptions.

A1. The function $g(t, x, y)$ is Lipchitz continuous, that is, there exists posivive constants C_1 and C_2 such that

$$||g(t, x, y)|| \le C_1||x|| + C_2||y||, x, y \in \mathbb{R}^n$$

Lemma 3. [6]**(Gronwall Inequality)**
Suppose that $g(t)$ and $\phi(t)$ are continuous in $[t_0, t_1], g(t) \geq 0, \lambda \geq 0$ and $r \geq 0$ are two constants. If

$$\phi(t) \leq \lambda + \int_{t_0}^{t} [g(\tau)\phi(\tau) + r]\mathrm{d}\tau, \tag{19}$$

then

$$\phi(t) \leq (\lambda + r(t_1 - t_0)) \exp\left(\int_{t_0}^{t} g(\tau)\mathrm{d}\tau\right), t_0 \leq t \leq t_1. \tag{20}$$

Lemma 4. [1] *If all the eigenvalues of A satisfy*

$$|\arg(spec(A))| > \frac{\alpha\pi}{2}, \tag{21}$$

then there exists a constant $K > 0$ such that,

$$\int_0^t ||\theta^{\alpha-1} E_{\alpha,\alpha}(A\theta^\alpha)||\mathrm{d}\theta \leq K. \tag{22}$$

Theorem 3. *Suppose $f(t, x(t))$ and $g(t, x(t))$ satisfies the condition*

$$\begin{aligned} ||f(t, x(t))|| &\leq M_1||x||, \\ ||g(t, x(t))|| &\leq M_2||x||, \end{aligned} \tag{23}$$

with $M_2 \neq 1$, Assumption **A1** *holds and all the eigenvalues of A satisfy (30). Then, the zero solution of* (18) *is asymptotically stable.*

Proof. The given equation (18) can be written as

$${}^CD^\alpha x(t) = Ax(t) + f(t, x(t)) +^C D^\alpha g(t, x(t)).$$

The solution of the system can be written as,

$$\begin{aligned} x(t) =& \; E_\alpha(A(t)^\alpha)x_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)\{f(s, x(s)) +^C D^\alpha g(s, x(s))\}\mathrm{d}s, \\ =& \; E_\alpha(At^\alpha)x_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) f(s, x(s))\mathrm{d}s \\ &+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)^C D^\alpha g(s, x(s))\mathrm{d}s, \\ =& I_1 + I_2. \end{aligned}$$

Let evaluate I_2,

$$\begin{aligned} I_2 &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)^C D^\alpha g(s, x(s))\mathrm{d}s, \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^s (t-s)^{\alpha-1}(s-\tau)^{-\alpha} E_{\alpha,\alpha}(A(t-s)^\alpha) g'(\tau, x(\tau))\mathrm{d}\tau\mathrm{d}s, \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t g'(\tau, x(\tau)) \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \alpha)} \int_\tau^t (t-s)^{\alpha k+\alpha-1}(s-\tau)^{-\alpha}\mathrm{d}s\mathrm{d}\tau, \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t E_\alpha(A(t-\tau)^\alpha) g'(\tau, x(\tau))\mathrm{d}\tau. \end{aligned}$$

By using integration by parts,

$$I_2 = g(t, x(t)) - E_\alpha(At^\alpha)g(0, x_0) + A\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^\alpha)g(\tau, x(\tau))\mathrm{d}\tau.$$

Hence,

$$\begin{aligned} x(t) &= E_\alpha(At^\alpha)\{x_0 - g(0, x_0)\} + g(t, x(t)) \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)\{f(s, x(s)) + Ag(s, x(s))\}\mathrm{d}\tau. \end{aligned}$$

From this,

$$\begin{aligned} ||x(t)|| &\le ||E_\alpha(At^\alpha)(x_0 - g(0, x_0))|| + ||g(t, x(t))|| \\ &\quad + \int_0^t ||(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)||\ ||f(s, x(s)) + Ag(s, x(s))||\mathrm{d}\tau, \\ &\le ||E_\alpha(At^\alpha)(x_0 - g(0, x_0))|| + M_2||x|| + \{M_1 + ||A||M_2\} \\ &\quad \int_0^t ||(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)||\ ||x||\mathrm{d}\tau, \\ &\le ||E_\alpha(At^\alpha)\ \frac{(x_0 - g(0, x_0))}{1 - M_2}|| + \left\{\frac{M_1 + ||A||M_2}{1 - M_2}\right\} \\ &\quad \int_0^t ||(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)||\ ||x||\mathrm{d}\tau. \end{aligned}$$

By using Lemma 1,

$$||x(t)|| \le ||E_\alpha(At^\alpha)C_1||\exp\{C_2\int_0^t ||(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)||\mathrm{d}\tau\},$$

where $C_1 = \dfrac{(x_0 - g(0, x_0))}{1 - M_2},\ C_2 = \dfrac{M_1 + ||A||M_2}{1 - M_2}$.

Further, $||E_\alpha(At^\alpha)x_0|| \to 0$ as $t \to \infty$. [5]
Hence we have $\lim_{t\to\infty} x(t) = 0$.
Therefore, the zero solution of the given system is asymptotically stable. □

5. Nonlinear Fractional Integrodifferential Equations

Consider the fractional integro-differential equation

$$^{C}D^\alpha x(t) = Ax(t) + I^\alpha g(t, x(t)),\ 0 < \alpha \le 1, \tag{24}$$

where $g(t, x(t)) \le 0, g(t, 0) = 0$, with the initial condition $x(0) = x_0$, where $g \in C[J \times \mathbb{R}^n, \mathbb{R}^n], J = [0, a]$

Lemma 5. [1] *If all the eigenvalues of A satisfy*

$$|\arg(spec(A))| > \frac{\alpha\pi}{2} \tag{25}$$

Then there exists a constant $K > 0$ such that,

$$\int_0^t ||\theta^{2\alpha-1}E_{\alpha,2\alpha}(A\theta^\alpha)||\mathrm{d}\theta \le K. \tag{26}$$

Theorem 4. *Let $g(t, x(t))$ satisfy*

$$||g(t, x(t))|| \leq M_1 ||x||, \tag{27}$$

and all the eigenvalues of A satisfies (30). Then the zero solution of (24) is asymptotically stable.

Proof. The solution can be represented by

$$\begin{aligned} x(t) =& E_\alpha(At^\alpha)x_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) I^\alpha g(s, x(s)) \mathrm{d}s \\ =& E_\alpha(At^\alpha)x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) \int_0^s (s-\tau)^{\alpha-1} g(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s \\ =& I_1 + I_2. \end{aligned}$$

Let evaluate I_2

$$\begin{aligned} I_2 =& \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^s (t-s)^{\alpha-1} (s-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha) g(\tau, x(\tau)) \mathrm{d}\tau \mathrm{d}s \\ =& \frac{1}{\Gamma(\alpha)} \int_0^t g(\tau, x(\tau)) \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \alpha)} \int_\tau^t (t-s)^{\alpha k + \alpha - 1} (s-\tau)^{\alpha-1} \mathrm{d}s \mathrm{d}\tau \\ =& \int_0^t (t-\tau)^{2\alpha-1} \sum_{k=0}^{\infty} \frac{(A(t-\tau)^\alpha)^k}{\Gamma(\alpha k + 2\alpha)} g(\tau, x(\tau)) \mathrm{d}\tau \\ =& \int_0^t (t-\tau)^{2\alpha-1} E_{\alpha,2\alpha}(A(t-\tau)^\alpha) g(\tau, x(\tau)) \mathrm{d}\tau. \end{aligned}$$

Hence

$$\begin{aligned} x(t) = & E_\alpha(At^\alpha)x_0 \\ & + \int_0^t (t-\tau)^{2\alpha-1} E_{\alpha,2\alpha}(A(t-\tau)^\alpha) g(\tau, x(\tau)) \mathrm{d}\tau. \end{aligned}$$

Then,

$$||x(t)|| \leq ||E_\alpha(At^\alpha)x_0|| + M \int_0^t ||(t-\tau)^{2\alpha-1} E_{\alpha,2\alpha}(A(t-\tau)^\alpha)|| \; ||x|| \mathrm{d}\tau.$$

By using Lemma 3,

$$||x(t)|| \leq ||E_\alpha(At^\alpha)x_0|| \exp\{M \int_0^t ||(t-\tau)^{2\alpha-1} E_{\alpha,2\alpha}(A(t-\tau)^\alpha)|| \mathrm{d}\tau\}.$$

By using the Lemma 1,

$$||x(t)|| \leq C ||E_\alpha(At^\alpha)x_0||.$$

Further the proof is similar to the theorem 3. □

Consider the fractional integrodifferential equation

$${}^C D^\alpha x(t) = Ax(t) + g(t, x(t), \int_0^t K(t, s, x(s)) \mathrm{d}s), \tag{28}$$

with the initial condition $x(0) = x_0$, where $0 < \alpha \leq 1$,$J = [0, a]$, $g \in C[J \times \mathbb{R}^n, \times \mathbb{R}^n, \mathbb{R}^n]$, and $K \in C[J \times J \times \mathbb{R}^n, \mathbb{R}^n]$ with $g(t, 0) = 0, K(t, s, 0) = 0$ for all $t \in J$.

Theorem 5. *Let $K(t,s,x(s))$ satisfy*

$$||K(t,s,x(s))|| \leq M_1||x||, \qquad s \in [0,t] \tag{29}$$

and all the eigenvalues of A satisfies

$$|\arg spec(A)| > \frac{\alpha\pi}{2} \tag{30}$$

Then the zero solution of (28) is asymptotically stable.

Proof. Comparing the equation (28) with the equation (17) , the nonlinear term is given by

$$f(t,x(t)) = g(t,x(t), \int_0^t K(t,s,x(s))\mathrm{d}s.$$

Then the condition for the stability is given by

$$\begin{aligned} ||f(t,x(t))|| &= ||g(t,x(t), \int_0^t K(t,s,x(s))\mathrm{d}s)|| \\ &\leq C_1||x|| + C_2||\int_0^t K(t,s,x(s))\mathrm{d}s||. \end{aligned}$$

By using the condition (29), we have

$$\begin{aligned} &||f(t,x(t), \int_0^t K(t,s,x(s))\mathrm{d}s)|| \\ &\leq C_1||x|| + C_2 a M_1||x||, \\ &\leq M||x||. \end{aligned}$$

Here the nonlinear term satisfies the required condition of the Theorem 3. Further the proof is similar to the Theorem 3. □

Corollary 1. *Consider the nonlinear system of the form*

$${}^{C}D^{\alpha}x(t) = Ax(t) + \int_0^t g(\tau, x(\tau))\mathrm{d}\tau, \qquad \alpha \in (0,1) \tag{31}$$

where $g(t,x) \in \mathbb{C}(J \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t,x) \leq 0$, $g(t,0) = 0$ and the initial condition is given by $x(0) = x_0$, where $x = (x_1, \ldots, x_n)^T$, and $A \in \mathbb{R}^{n\times n}$. Suppose

$$||g(t,x(t))|| \leq M||x|| \tag{32}$$

and all the eigenvalues of A satisfy (30). Then, the zero solution of (31) is asymptotically stable.

Consider the nonlinear system of the form

$${}^{C}D^{\alpha}x(t) - AI^{\alpha}x(t) = f(t,x(t)), \qquad 0 < \alpha < 1, \tag{33}$$

where the initial condition is given by $x(0) = x_0$ and $f(t,0) = 0$, where $x = (x_1, \ldots, x_n)^T$, and $A \in \mathbb{R}^{n\times n}$.

Theorem 6. *Suppose $f(t,x(t))$ satisfies the condition*

$$||f(t,x(t))|| \leq M_1||x||, \tag{34}$$

all the eigenvalues of A satisfy (30). Then, the zero solution of (18) *is asymptotically stable.*

Proof. The given equation (33) can be written as

$$^{C}D^{\alpha}x(t) = AI^{\alpha}x(t) + f(t, x(t))$$

Taking Laplace transform on both sides, we get

$$X(s) = \frac{s^{2\alpha-1}}{s^{2\alpha} - A}x_0 + L\{t^{2\alpha-1}E_{2\alpha,\alpha}(At^{2\alpha}) * f(t, x(t))\}$$

By taking inverse Laplace transform, the solution representation of the system can be written as,

$$x(t) = E_{2\alpha}(At^{2\alpha})x_0 + \int_0^t (t-s)^{\alpha-1}E_{2\alpha,\alpha}(A(t-s)^{2\alpha})f(s, x(s))\mathrm{d}s$$

From this we have,

$$||x(t)|| = ||E_{2\alpha}(At)^{2\alpha})x_0|| + \int_0^t ||(t-s)^{\alpha-1}E_{2\alpha,\alpha}(A(t-s)^{2\alpha})||\ ||f(s, x(s))||\mathrm{d}s$$

By using the condition (34)

$$||x(t)|| \leq ||E_{2\alpha}(At^{2\alpha})x_0|| + M\int_0^t (t-s)^{\alpha-1}E_{2\alpha,\alpha}(A(t-s)^{2\alpha})||\ ||f(s, x(s))||\mathrm{d}s$$

By using Gronwalls inequality,

$$||x(t)|| = ||E_{2\alpha}(At^{2\alpha})x_0|| \exp\left\{M\int_0^t s^{\alpha-1}E_{2\alpha,\alpha}(As^{2\alpha})\right\}\mathrm{d}s$$

By using Lemma 3,

$$\exp\left\{M\int_0^t s^{\alpha-1}E_{2\alpha,\alpha}(As^{2\alpha})\right\}\mathrm{d}s \quad \text{is bounded}$$

Also $||E_{2\alpha(At^{2\alpha})}x_0|| \to 0$ as $t \to \infty$ [5].
Hence we have $\lim_{t\to\infty} x(t) = 0$. □

6. Examples

This section includes an example of a linear system and a nonlinear system which is stable in fractional order and not stable in integer order. From this it is known that the fractional order system has a peculiar properties which is occured in nature. Many physical systems can be properly described by using the fractional systems. Due to the absence of method of solving fractional systems, it is converted into an integer order system and solved. Also this section provides some examples to illustrate the above theories. These problems are solved by using the numerical method in which fractional Euler's method is used as a prediction, and the modified trapezoidal rule is used to make correction to obtain the finite value. For further details see [25, 26]. **A system asymptotically stable in fractional order not stable in integer order**

Example 6.1. *Now we shall discuss the stability of the linear system*
$^{C}D^{\alpha}x(t) = Ax(t), \quad 0 < \alpha \leq 1,$
with initial condition $x(0) = (1, 1)^T$*, where*

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}.$$

The eigenvalues of A are given by $\lambda = 1 \pm i\sqrt{3}$. Corresponding eigenvectors are given by
$\mathbf{u} = (-0.5i, -0.866)^T, \mathbf{v} = (0.5i, -0.866)^T$. It can be written in the standard form $\mathbf{u} = \mathbf{a} + i\mathbf{b}, \mathbf{v} = \mathbf{a} - i\mathbf{b}$, where $\mathbf{a} = (0, -0.866)^T, \mathbf{b} = (-0.5, 0)^T$.

Case 1:
In the integer order case, that is when $\alpha = 1$, eigenvalues does not have the negative real part. That is $|\arg(spec(A))| > \dfrac{\pi}{4}$ is not satisfied. The given system does not satisfies the necessary conditions of the Theorem 2.1. So the given system is not stable.

Case 2:
In the fractional order case, let $\alpha = \frac{1}{2}$, the eigenvalues satisfy $|\arg(\lambda_1)| = \dfrac{\pi}{3}$ and $|\arg(\lambda_2)| = \dfrac{\pi}{3}$. Since all the eigenvalues satisfy $|\arg(spec(A))| > \dfrac{\pi}{4}$. The given system does not satisfies the necessary conditions of the Theorem 2.1. Hence the given system is asymptotically stable.

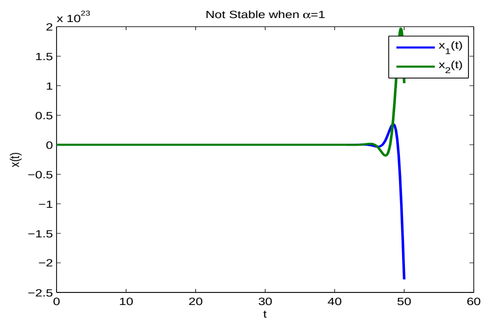

Fig.1

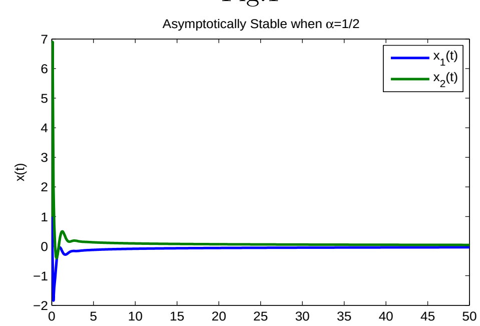

Fig.2

Nonlinear system stable in fractional order not stable in integer order

Example 6.2. *Now we shall discuss the stability of the nonlinear system*

$$^{C}D^{\alpha}x(t) + Ax(t) = f(t,x), \qquad 0 < \alpha \leq 1, \tag{35}$$

where $A = \begin{bmatrix} -3 & 5 \\ -2 & 3 \end{bmatrix}$ *and* $f(t,x) = (0, -\sin(x_1(t)))^T$.

The eigenvalues of the matrix A is given by $\pm i$. Let us consider the two cases.

Case 1:
Integer order case when $\alpha = 1$. Then it will become an ordinary differential equation. First consider the homogeneous system, since the eigenvalues of A does not contain negative real part, it is not asymptotically stable.
Case 2:
Fractional order case when $\alpha = 1/2$. First consider the homogeneous system, here the eigenvalues of A satisfies $|\arg(spec(A))| = \frac{\pi}{2} > \frac{\pi}{4}$. Also the nonlinear term $f(t,x) = (0, -sin(x_1))^T$ satisfies $||f(t,x)|| \leq M||x||$. Hence the given nonlinear system is asymptotically stable.

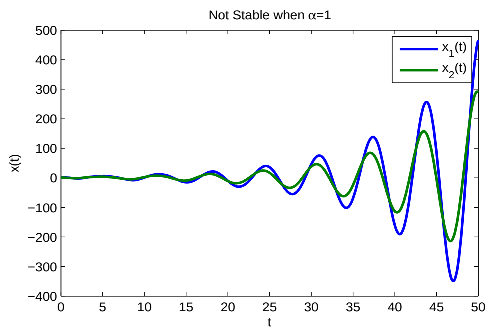

Fig.3

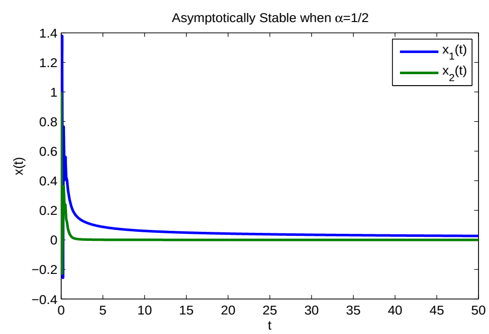

Fig.4

Stability of Nonlinear Fractional Dynamical Systems

Example 6.3. (Duffing Equation) *We shall discuss the stability of the system,*

$$^{C}D^{\alpha}x(t) = -x(t) - x(t)^3, \quad 0 < \alpha < 2.$$

This type of equations are discussed in [19].

When $\alpha = 1/2$ the nonlinear first order Duffing equation is given by

$$\begin{aligned} ^{C}D^{\frac{1}{2}}x(t) &= -x(t) - x(t)^3, \\ x(0) &= 1. \end{aligned} \tag{36}$$

The homogeneous system $^{C}D^{\frac{1}{2}}x(t) = -x(t)$, which satisfies $|\arg(-1)| > \frac{\pi}{4}$. Here $f(t,x) = -x^3$, which satisfies $f(t,0) = 0$. Since all the conditions of the Theorem 2.2 are satisfied. Hence the given system (36) is asymptotically stable.

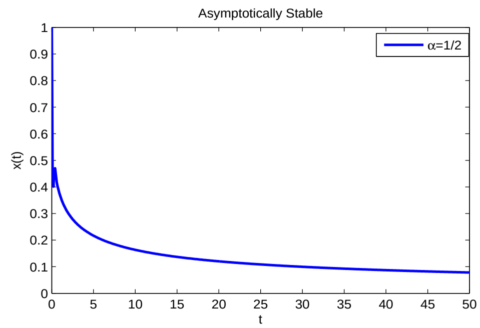

Fig.5

When $\alpha = 3/2$, the nonlinear second order Duffing equation is given by

$$\begin{aligned} {}^{C}D^{\frac{3}{2}}x(t) &= -x(t) - x(t)^3, \\ x(0) &= 0,\ x'(0) = 1. \end{aligned} \tag{37}$$

The given equation can be converted into a system of fractional differential equation, by using the substitution

$$\begin{aligned} {}^{C}D^{\frac{1}{2}}x_1(t) &= x_2(t), \\ {}^{C}D^{\frac{1}{2}}x_2(t) &= x_3(t), \\ {}^{C}D^{\frac{1}{2}}x_3(t) &= -x_1(t) - x_1^3(t), \end{aligned} \tag{38}$$

where $x_1(t) = x(t)$, with initial condition $x_1(0) = 0, x_2(0) = 1, x_3(0) = 0$. This can be written in the standard form, ${}^{C}D^{1/2}x(t) = Ax(t) + f(t,x)$, and $f(t,x) = (0, 0, x_1^3(t))^T$ where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of A are $-1, 0.5 \pm 0.8660i$, which satisfies $|\arg(spec(A))| > \frac{\pi}{4}$. Here the nonlinear term $f(t,x)$ satisfies $f(t,0) = 0$. Since all the conditions of the Theorem 2.2 are satisfied. Hence the given system (37) is asymptotically stable.

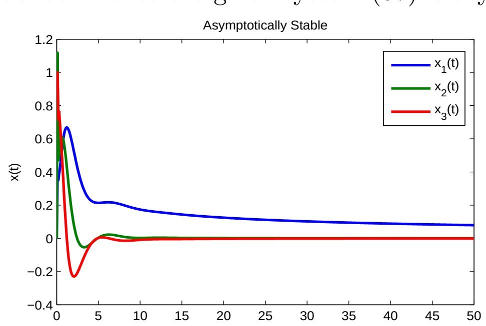

Fig.6

Example 6.4. *We shall discuss the stability of the nonlinear system*

$$\begin{aligned} {}^{C}D^{\alpha}x_1(t) &= x_2(t), \\ {}^{C}D^{\alpha}x_2(t) &= -x_1(t) - (1 + x_2(t))^2 x_2(t), \end{aligned} \tag{39}$$

with the initial condition $x_1(0) = 0.14,\ x_2(0) = 0.125,$ *when* $\alpha = 0.9$.

This can be written in the standard form, ${}^{C}D^{\alpha}x(t) = Ax(t)+f(t,x)$, where $f(t,x) = (0, -(1+x_2(t))^2x_2(t))^T$, where,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of A are $\pm i$, which satisfies $|\arg(spec(A))| > \frac{0.9\pi}{2}$. Since the system (39) satisfies all the conditions of the Theorem 2.2. Hence it is asymptotically stable.

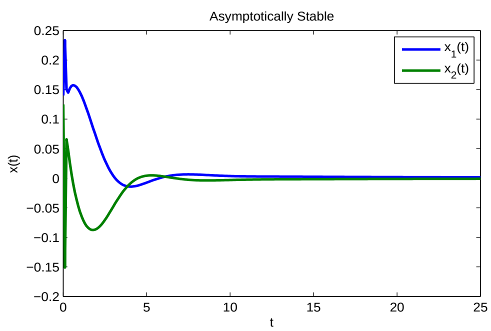

Fig.7

Example 6.5. *Consider the linear integrodifferential system of the form*

$$ {}^{C}D^{\alpha}x(t) + 3x(t) = -I^{\alpha}x(t), \tag{40}$$

where $\alpha = 1/2$, *and with the initial condition* $x(0) = 2$.

This can be written in the standard form ${}^{C}D^{\alpha}x(t) = Ax(t)+I^{\alpha}g(t,x(t))$, where $A = -3$ and $g(t,x) = -x(t)$. The given system (40) satisfies the necessary conditions of the Theorem 3.2. Hence it is asymptotically stable.

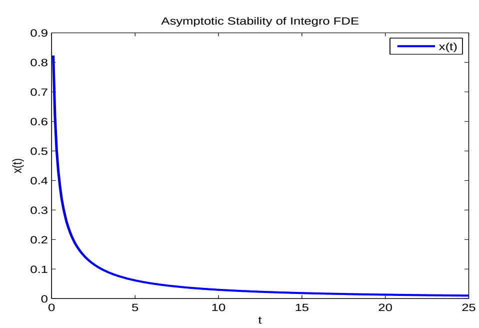

Fig.8

Example 6.6. *We shall consider linear integrodifferential system of the form*

$$ {}^{C}D^{\alpha}x(t) + 2x(t) = -5\int_0^t x(s)\mathrm{d}s + h(t), \tag{41}$$

where $\alpha = 1/2$, *with initial condition* $x(0) = 0$ *and* $h(t) = \begin{cases} 1, & 0 < t \le 1, \\ 0 & \textit{otherwise.} \end{cases}$

This can be written in the standard form ${}^{C}D^{\alpha}x(t) = Ax(t) + f(t, x(t))$, where $A = -2$ and $f(t, x) = \int_0^t g(t, x(s))\mathrm{d}s + h(t)$, where $g(t, x(t)) = -5x(t)$ and $h(t) = \begin{cases} 1, & 0 < t \le 1, \\ 0 & \text{otherwise.} \end{cases}$

The given system (41) satisfies the necessary conditions of the Theorem 3.1. Hence it is asymptotically stable.

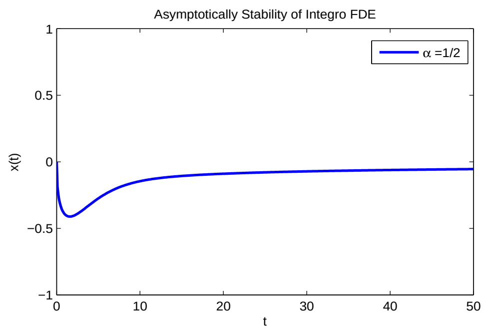

Fig.9

7. Conclusion

In nature every system occurs in the fractional order. From this it is conclude that the fractional order system has attractive feature comparing with integer order system. In this work, stability of nonlinear system having fractional order between 0 and 1 are studied. In this study, some simple sufficient conditions on the matrix and nonlinear term guarantees the stability of FDE's, were obtained. Also stability of the Fractional neutral and Fractional integrodifferential systems were discussed. To show the applicability of the obtained conditions, some examples were presented.

Future works include the stability of nonlinear fractional neutral and integrodifferential equations with delay by analyzing the eigenvalues and applying conditions on the nonlinear term.

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