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WAVELET GALERKIN METHOD FOR SOLVING STOCHASTIC FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Stochastic fractional differential equations (SFDEs) have many physical applications in the fields of turbulance, heterogeneous, flows and matrials, viscoelasticity and electromagnetic theory. In this paper, a new wavelet Galerkin method is proposed for numerical solution of SFDEs. First, fractional and stochastic operational matrices for the Chebyshev wavelets are introduced. Then, these operational matrices are applied to approximate solution of SFDEs. The proposed method reduces the SFDEs to a linear system of algebraic equations that can be solved easily. A brief convergence and error analysis of the proposed method is given. Numerical examples are performed to test the applicability and efficiency of the method.

1. Introduction

Fractional integrals and derivatives have been applied for modeling many physical phenomena in fields of nonlinear oscillation of earthquake, fluid-dynamic traffic, continuum and statistical mechanics, signal processing, control theory, and dynamics of interfaces between nanoparticles and subtracts [1, 2, 3, 4, 5]. Accordingly, considerable attentions have been given for producing approximate solution of fractional differential and integral equations. Recently, several numerical methods such as Fourier transforms method [6], Laplace transforms method [7], fractional differential transform method [8], finite difference method [9], orthogonal functions [10, 11, 12, 13], wavelets method [14, 15], Adomian decomposition method [16], variational iteration method [17], and homotopy analysis method [18] have been used for producing approximate solution of fractional differential and integral equations.

Stochastic analysis has been an interesting research area in mathematics, fluid mechanics, geophysics, biology, chemistry, epidemiology, microelectronics, theoretical physics, economics, and finance. The behavior of dynamical systems in these fields are often dependent on a noise source and a Gaussian white noise, governed by certain probability laws. This noise might be either due to thermal fluctuations, noise in somecontrol parameter, coarse-graining of a high-dimensional deterministic system with random initial conditions or the stochastic parameterization of small

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scales. The dynamical systems subject to noise can be modeled accurately using stochastic differential equations, stochastic integral equations, stochastic integro-differential equations or in more complicated cases stochastic partial differential equation[19, 20, 21, 22, 23, 24]. In most cases analytical solution of stochasic differential and integral equations cannot be derived. Therefore, numerical methods have attracted many researcher and there are many research papers that deal with numerical solution of stochasic differential and integral equations. For example, Runge-Kutta method [26, 27, 25, 28], Galerkin fimite element method [29, 31], operational method and orthogonal functions [24, 30] and spectral methods [32] have been used for solving stochasic differential and integral equations.

In recent years, different orthogonal basis functions such as block pulse functions, Walsh functions, Fourier series, orthogonal polynomials and wavelets, were used to estimate solution of functional equations. Wavelet theory is a relatively new and an emerging area in scientific research. It has been applied in a wide range of fields including computational mathematics, signal processing, image processing and time-frequency analysis. [14, 15, 33, 34].

In this paper a wavelet Galerkin method will be used for approximate solution of the following SFDE

$$D^{\alpha}u(t) = f(t) + \int_0^t u(s)k_1(s,t)ds + \int_0^t u(s)k_2(s,t)dB(s),\ t \in [0,1]\,, \tag{1}$$

with these initial conditions

$$u^{(k)}(0) = u_k,\ k = 0, 1, ..., n-1,\ n-1 < \alpha \leq n, \tag{2}$$

where $u(t)$, $f(t)$ and $k_i(s,t), i = 1, 2$ are the stochastic processes defined on the same probability space (Ω, F, P), and $u(t)$ is unknown. Also $B(t)$ is a Brownian motion process and $\int_0^t k_2(s,t)u(s)dB(s)$ is the Itô integral. These SFDEs appear in modeling many phenomena in science that have some uncertainity [32, 36, 37, 35]. In order to compute the approximate solution of SFDEs (1) we first derive some operational matrices for the Chebyshev wavelets. Then, these operational matrices are applied to obtain approximate solution.

The reminder of the paper is organized as follows: In section 2 some preliminary definitions of stochastic calculus, fractional calculus and Block Pulse Functions (BPFs) are reviewed. Section 3 is devoted to the basic formulation of the Chebyshev wavelets and their properties. In section 4 general procedures for forming operational matrices of Chebyshev wavelets are explained. In section 5 a wavelet Galerkin method based on the Chebyshev wavelets and their operational matrices are proposed for solving SFDEs. Convergence and error analysis of the Chebyshev wavelets basis is considered in section 6. Numerical examples are included in section 7. Finally, a conclusion is given in section 8.

2. Preliminary definitions

In this section we review some necessary definitions and mathematical preliminaries about stochastic calculus, fractional calculus and BPFs which are required for establishing our results in the next sections.

2.1. **Fractional calculus.** Fractional order calculus is a branch of calculus which deal with integration and differentiation operators of non-integer order. Among the several formulations of the generalized derivative, the Riemann-Liouville and

Caputo definition are most commonly used, which can be described as follows [7]:
Definition 1 The Riemann-Liouville fractional integral operator of order $\alpha\,(\alpha \geq 0)$ is defined as

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \ t > 0, \tag{3}$$

with $J^0 f(t) = f(t)$.
Definition 2 The Caputo fractional derivative of order α is defined as

$$D^{\alpha} f(t) = J^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau) d\tau}{(t-\tau)^{\alpha-n+1}}, \ n-1 < \alpha \leq n, \ t > 0, \tag{4}$$

where D^n is the classical differential operator of order n.
Some of most important properties of the Riemann-Liouvill operator and Caputo operator operators D^{α} and J^{α} are given by the following expression

$$(a) \ J^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!}, \ n-1 < \alpha \leq n, \ t > 0. \tag{5}$$

$$(b) \ J^{\alpha} D^{\alpha} f(t) = f(t). \tag{6}$$

$$(c) \ D^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau) d\tau}{(t-\tau)^{1-\alpha}}, \ 0 < \alpha \leq 1, \ t > 0. \tag{7}$$

$$(d) \ D^{\alpha} t^{\beta} = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha} & \beta \geq \alpha, \\ 0 & \beta < \alpha. \end{cases} \tag{8}$$

For more details about fractional calculus please see [7].

2.2. **Stochastic calculus.** Let H be real separable Hilbert spaces equipped with norms $\|.\|_H$ and $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a right-continuous, increasing family of sub σ-algebras of $\mathcal{F}$. An random variable is an $\mathcal{F}$-measurable function $X : \Omega \to H$ and a collection of random variables $S = \{X(t,\omega) : \Omega \to H | 0 \leq t \leq T\}$ is called a stochastic process. Hereafter, for simplicity of notation we drop variable ω and write $X(t)$ instead of $X(t,\omega)$.
Definition 3 The collection of all strongly-measurable, square-integrable random variables, denoted by $L^2(\Omega, H)$, is a Banach space equipped with norm

$$\|X\| = \|X(t,\omega)\|_{L^2(\Omega,H)} = \left[E\left(\|X(t,\omega)\|_H^2\right)\right]^{\frac{1}{2}} = \left[\int_{\Omega} |X(t,\omega)|^2 dP(\omega)\right]^{\frac{1}{2}}. \tag{9}$$

Definition 4 (Brownian motion process) A real-valued stochastic process $B(t), t \in [0,T]$ is called Brownian motion, if it satisfies the following properties:
(i) The process has independent increments for $0 \leq t_0 \leq t_1 \leq ... \leq t_n \leq T$,
(ii) For all $t \geq 0$, $B(t+h) - B(t)$ is a normal distribution with mean 0 and variance h,
(iii) The function $t \to B(t)$ is a continuous function of t.
Definition 5 Let $\{\mathcal{N}_t\}_{t\geq 0}$ be an increasing family of σ-algebras of subsets of Ω. A process $g(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}^n$ is called $\mathcal{N}_t$-adapted if for each $t \geq 0$ the function $\omega \to g(t,\omega)$ is $\mathcal{N}_t$-measurable.
Definition 6 Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions $f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$ such that

(i) The function $(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel algebra on $[0,\infty)$ and $\mathcal{F}$ is the σ -algebra on Ω.
(ii) f is adapted to $\mathcal{F}_t$, where $\mathcal{F}_t$ is the σ -algebra generated by the random variables $B(s), s \le t$.
(iii)$E\left(\int_S^T f^2(t,\omega)dt\right) < \infty$.

Definition 7 (The Itô integral) Let $f \in \mathcal{V}(S,T)$, then the Itô integral of f is defined by

$$\int_S^T f(t,\omega)dB_t(\omega) = \lim_{n\to\infty} \int_S^T \varphi_n(t,\omega)dB_t(\omega),\ (\lim \text{in } L^2(\Omega, H)),$$

where, φ_n is a sequence of elementary functions such that

$$E\left(\int_s^T (f(t,\omega) - \varphi_n(t,\omega))^2\, dt\right) \to 0,\ as\ n \to \infty.$$

For more details about stochastic calculus and integration please see [20, 19, 38].

2.3. **Block pulse functions.** BPFs have been studied by many authors and applied for solving different problems. In this section we recall definition and some properties of the block pulse functions [24, 14].

The m-set of BPFs are defined as

$$b_i(t) = \begin{cases} 1 & (i-1)h \le t < ih \\ 0 & otherwise \end{cases} \tag{10}$$

in which $t \in [0,T)$, $i = 1,2,...,m$ and $h = \frac{T}{m}$. The set of BPFs are disjointed with each other in the interval $[0,T)$ and

$$b_i(t)b_j(t) = \delta_{ij}b_i(t), i,j = 1,2,...,m, \tag{11}$$

where δ_{ij} is the Kronecker delta. The set of BPFs defined in the interval $[0,T)$ are orthogonal with each other, that is

$$\int_0^T b_i(t)b_j(t) = h\delta_{ij},\ i,j = 1,2,...,m. \tag{12}$$

If $m \to \infty$ the set of BPFs is a complete basis for $L^2[0,T)$, so an arbitrary real bounded function $f(t)$, which is square integrable in the interval $[0,T)$, can be expanded into a block pulse series as

$$f(t) \simeq \sum_{i=1}^m f_i b_i(t), \tag{13}$$

where

$$f_i = \frac{1}{h}\int_0^T b_i(t)f(t),\ i = 1,2,...,m. \tag{14}$$

Rewritting Eq. (13) in the vector form we have

$$f(t) \simeq \sum_{i=1}^m f_i b_i(t) = F^T\Phi(t) = \Phi^T(t)F, \tag{15}$$

in which

$$\Phi(t) = [b_1(t), b_2(t), ..., b_m(t)]^T, \tag{16}$$

$$F = [f_1, f_2, ..., f_m]^T \,. \tag{17}$$

Morever, any two dimensional function $k(s,t) \in L^2([0,T_1]\times[0,T_2])$ can be expanded with respect to BPFs such as

$$k(s,t) = \Phi^T(t) K \Phi(t), \tag{18}$$

where $\Phi(t)$ is the m-dimensional BPFs vectors respectively, and K is the $m\times m$ BPFs coefficient matrix with (i,j)-th element

$$k_{ij} = \frac{1}{h_2 h_2}\int_0^{T_1}\int_0^{T_2} k(s,t) b_i(t) b_j(s)\,dtds,\ i,j = 1,2,...,m, \tag{19}$$

and $h_1 = \frac{T_1}{m}$ and $h_2 = \frac{T_2}{m}$. Let $\Phi(t)$ be the BPFs vector, then we have

$$\Phi^T(t)\Phi(t) = 1, \tag{20}$$

and

$$\Phi(t)\Phi^T(t) = \begin{pmatrix} b_1(t) & 0 & \dots & 0 \\ 0 & b_2(t) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b_m(t) \end{pmatrix}_{m\times m} . \tag{21}$$

For an m-vector F we have

$$\Phi(t)\Phi^T(t) F = \tilde{F}\Phi(t), \tag{22}$$

where $\tilde{F}$ is an $m\times m$ matrix, and $\tilde{F} = diag(F)$. Also, it is easy to show that for an $m\times m$ matrix A

$$\Phi^T(t) A \Phi(t) = \hat{A}^T\Phi(t), \tag{23}$$

where $\hat{A} = diag(A)$ is an m-vector

3. Chebyshev wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function ψ called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets

$$\psi_{a,b}(t) = a^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right),\ a,b\in\mathbb{R},\ a\neq 0. \tag{24}$$

The Chebyshev wavelets $\psi_{nm}(x) = \psi(k,n,m,x)$ are defined on the interval $[0,1)$ by

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k+1}{2}}\tilde{T}_m(2^k t - (2n+1)), & \frac{n}{2^k}\le x\le\frac{n+1}{2^k} \\ 0, & otherwise \end{cases}, \tag{25}$$

where

$$\tilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0 \\ \sqrt{\frac{2}{\pi}}T_m(t), & m > 0 \end{cases},$$

and $T_m(t)$ are the well-known Chebyshev polynomials of degree m. The Chebyshev wavelets $\{\psi_{nm}(x) | n = 0,1,\ldots,2^k-1, m = 0,1,2,...,M-1\}$ forms an orthonormal basis for $L^2_{w_n}[0,1]$ with respect to the weight function $w_n(t) = w(2^{k+1}t - (2n+1))$, in which $w(t) = \frac{1}{\sqrt{1-t^2}}$[15, 14].

By using the orthonormality of the Chebyshev wavelets, any function $f(t)$ over $[0,1)$; square-integrable with respect to the measure $\mathbf{w}(t)dt$; with $\mathbf{w}(t) = w_{nk}(t)$; for $\frac{n}{2^k} \leq t \leq \frac{n+1}{2^k}$; and $w_{nk}(t) = w(2^{k+1}t - 2n + 1)$; being $w(t) = \frac{1}{\sqrt{1-t^2}}$ can be expanded in terms of the Chebyshev wavelets as

$$f(t) \simeq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t) = C^T\Psi(t), \tag{26}$$

where $c_{mn} = (f(t), \psi_{mn}(t))_{w_{nk}}$ and $(.,.)_{w_{nk}}$ denotes the inner product on $L^2_{w_{nk}}[0,1]$. If the infinite series in (26) is truncated, then it can be written as

$$f(t) \simeq \sum_{n=0}^{2^k-1} \sum_{m=0}^{M-1} c_{mn}\psi_{mn}(x) = C^T\Psi(t), \tag{27}$$

where C and $\Psi(t)$ are $\hat{m} = 2^k M$ column vectors given by

$$C = \left[c_{00}, \ldots, c_{0(M-1)}|c_{10}, \ldots, c_{1(M-1)}|, \ldots, |c_{(2^k-1)0}, \ldots, c_{(2^k-1)(M-1)}\right]^T,$$
$$\Psi(x) = \left[\psi_{00}(t), \ldots, \psi_{0(M-1)}(t)|, \ldots, |\psi_{(2^k-1)0}(t), \ldots, \psi_{(2^k-1)(M-1)}(t)\right]^T.$$

By changing indices in the vectors $\Psi(t)$ and C the series (13) can be rewritten as

$$f(t) \simeq \sum_{i=1}^{\hat{m}} c_i\psi_i(t) = C^T\Psi(t), \tag{28}$$

where

$$C = [c_1, c_2, ..., c_{\hat{m}}], \ \Psi(x) = [\psi_1(x), \psi_2(x), ..., \psi_{\hat{m}}(x)],$$

and

$$c_i = c_{nm}, \ \psi_i(t) = \psi_{nm}(t), \ i = (n-1)M + m + 1.$$

Similarly, any two dimensional function $k(s,t) \in L^2_{\mathbf{w} \bigotimes \mathbf{w}}([0,1] \times [0,1])$ can be expanded into Chebyshev wavelets basis as

$$k(s,t) \approx \sum_{i=1}^{\hat{m}} \sum_{j=1}^{\hat{m}} k_{ij}\psi_i(s)\psi_j(t) = \Psi^T(s)K\Psi(t), \tag{29}$$

where $K = [k_{ij}]$ is an $\hat{m} \times \hat{m}$ matrix and $k_{ij} = \left(\psi_i(s), (k(s,t), \psi_j(t))_{w_{nk}}\right)_{w_{nk}}$.

3.1. **Chebyshev wavelets and BPFs.** In this section we will review the relation between the Chebyshev wavelets and BPFs. It is worth mention that here we set $T = 1$ in definition of BPFs.

Theorem 1. Let $\Psi(t)$ and $\Phi(t)$ be the $\hat{m}$-dimensional Chebyshev wavelets and BPFs vector respectively, the vector $\Psi(t)$ can be expanded by BPFs vector $\Phi(t)$ as

$$\Psi(t) \simeq Q\Phi(t), \tag{30}$$

where Q is an $\hat{m} \times \hat{m}$ block matrix and

$$Q_{ij} = \psi_i\left(\frac{2j-1}{2\hat{m}}\right), i, j = 1, 2, ..., \hat{m} \tag{31}$$

Proof. Let $\phi_i(t), i = 1, 2, ..., \hat{m}$ be the i-th element of Chebyshev wavelets vector. Expanding $\phi_i(t)$ into an $\hat{m}$-term vector of BPFs, we have

$$\psi_i(t) \simeq \sum_{i=1}^{\hat{m}} Q_{ij} b_j(t) = Q_i^T \Phi(t),\ i = 1, 2, ..., \hat{m}, \tag{32}$$

where Q_i is the i-th row and Q_{ij} is the (i, j)-th element of matrix Q. By using the orthogonality of BPFs we have

$$Q_{ij} = \frac{1}{h} \int_0^1 \psi_i(t) b_j(t) dt = \frac{1}{h} \int_{\frac{j-1}{\hat{m}}}^{\frac{j}{\hat{m}}} \psi_i(t) dt = \hat{m} \int_{\frac{j-1}{\hat{m}}}^{\frac{j}{\hat{m}}} \psi_i(t) dt, \tag{33}$$

by using mean value theorem for integrals in the last equation we can write

$$Q_{ij} = \hat{m} \left(\frac{j}{\hat{m}} - \frac{j-1}{\hat{m}} \right) \psi_i(\eta_i) = \psi_i(\eta_j),\ \eta_j \in \left(\frac{j-1}{\hat{m}}, \frac{j}{\hat{m}} \right), \tag{34}$$

now by choosing $\eta_j = \frac{2j-1}{2\hat{m}}$ we have

$$Q_{ij} = \psi_i \left(\frac{2j-1}{2\hat{m}} \right), i, j = 1, 2, ..., \hat{m}. \tag{35}$$

and this prove the desired result. □

The following Remark is the consequence of relations (22), (23) and Theorem 1.

Remark 1. For an $\hat{m}$-vector F we have

$$\Psi(t)\Psi^T(t)F = \tilde{F}\Psi(t), \tag{36}$$

in which $\tilde{F}$ is an $\hat{m} \times \hat{m}$ matrix as

$$\tilde{F} = Q\bar{F}Q^{-1}, \tag{37}$$

where $\bar{F} = diag\left(Q^T F\right)$. Moreover, it can be easy to show that for an $\hat{m} \times \hat{m}$ matrix A

$$\Psi^T(t)A\Psi(t) = \hat{A}^T\Psi(t), \tag{38}$$

where $\hat{A}^T = UQ^{-1}$ and $U = diag(Q^T AQ)$ is a $\hat{m}$-vector.

4. Operational matrices for Chebyshev wavelets

In this section some operational matrices for the Chebyshev wavelets vector $\Psi(t)$ are derived. Next theorems provide general procedures for forming these matrices. First, we remind some useful results for BPFs[24].

Lemma 1.[24] Let $\Phi(t)$ be the $\hat{m}$-dimensional BPFs vector defined in (16), then integration of this vector can be derived as

$$\int_0^t \Phi(s)ds \simeq P\Phi(t), \tag{39}$$

where P is called the operational matrix of integration for BPFs and is given by

$$P = \frac{h}{2} \begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{\hat{m}\times\hat{m}}. \tag{40}$$

Lemma 2. [24] Let $\Phi(t)$ be the $\hat{m}$-dimensional BPFs vector defined in (16), the Itô integral of this vector can be derived as

$$\int_0^t \Phi(s)dB(s) \simeq P_s\Phi(t), \tag{41}$$

where P_s is called the stochastic operational matrix of BPFs and is given by

$$P_s = \begin{bmatrix} B\left(\frac{h}{2}\right) & B(h) & B(h) & \dots & B(h) \\ 0 & B\left(\frac{3h}{2}\right) - B(h) & B(2h) - B(h) & \dots & B(2h) - B(h) \\ 0 & 0 & B\left(\frac{5h}{2}\right) - B(2h) & \dots & B(3h) - B(2h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B\left(\frac{(2\hat{m}-1)h}{2}\right) - B((\hat{m}-1)h) \end{bmatrix}.$$

Now we are ready to derive operational matrices of stochastic and fractional integration for the Chebyshev wavelets.

Theorem 2. Let $\Psi(t)$ be the $\hat{m}$-dimensional Chebyshev wavelets vector defined in (27), the operational matrix of the fractional order integration for $\Psi(t)$ can be derived as

$$J^\alpha\Psi(t) = QF^\alpha Q^{-1}\Psi(t) = P^\alpha\Psi(t), \tag{42}$$

where P^α is called the operational matrix of Chebyshev wavelets, Q is the matrix introduced in (30) and F^α is the operational matrix of integration for BPFs derived in [14].

Proof. By using Theorem 1 we have

$$J^\alpha\Psi(t) = J^\alpha Q\Phi(t) = QF^\alpha\Phi(t) = QF^\alpha Q^{-1}\Psi(t) = P^\alpha\Psi(t), \tag{43}$$

so, the Chebyshev wavelet operational matrix of the fractional order integration P^α is given by

$$P^\alpha = QF^\alpha Q^{-1}. \tag{44}$$

and this complete the proof. □

Theorem 3. Suppose $\Psi(t)$ be the $\hat{m}$-dimensional Chebyshev wavelets vector defined in (27), the integral of this vector can be derived as

$$\int_0^t \Psi(s)ds \simeq QPQ^{-1}\Psi(t) = \Lambda\Psi(t), \tag{45}$$

where Q is introduced in (30) and P is the operational matrix of integration for BPFs derived in (40).

Proof. Let $\Psi(t)$ be the Chebyshev wavelets vector, by using Theorem 1 and Lemma 1 we have

$$\int_0^t \Psi(s)ds \simeq \int_0^t Q\Phi(s)ds = Q\int_0^t \Phi(s)ds = QP\Phi(t), \tag{46}$$

now Theorem 1 give

$$\int_0^t \Psi(s)ds \simeq QP\Phi(t) = QPQ^{-1}\Psi(t) = \Lambda\Psi(t), \tag{47}$$

and using this identity we obtain the desired result. □

Theorem 4. Suppose $\Psi(t)$ be the $\hat{m}$-dimensional Chebyshev wavelets vector defined in (27), the Itô integral of this vector can be derived as

$$\int_0^t \Psi(s)dB(s) \simeq QP_sQ^{-1}\Psi(t) = \Lambda_s\Psi(t), \tag{48}$$

where Λ_s is called stochastic operational matrix for Chebyshev wavelets, Q is introduced in (30) and P_s is the stochastic operational matrix of integration for BPFs derived in (41).

Proof. Let $\Psi(t)$ be the Chebyshev wavelets vector, by using Theorem 1 and Lemma 2 we have

$$\int_0^t \Psi(s)dB(s) \simeq \int_0^t Q\Phi(s)dB(s) = Q\int_0^t \Phi(s)dB(s) = QP_s\Phi(t), \tag{49}$$

now Theorem 1 result

$$\int_0^t \Psi(s)dB(s) = QP_s\Phi(t) = QP_sQ^{-1}\Psi(t) = \Lambda_s\Psi(t), \tag{50}$$

and this complete the proof. □

5. Description of the numerical method

Here we present a wavelet Galerkin method based on the Chebyshev wavelets and their operational matrices for solving SFDEs (1). For this purpose, and by using the relation of the fractional derivative and integral in (5), the solution $u(t)$ can be derived as

$$u(t) = \sum_{k=0}^{n-1} u^{(k)}(0^+) + J^\alpha f(t) + J^\alpha\left(\int_0^t u(s)k_1(s,t)ds\right) + J^\alpha\left(\int_0^t u(s)k_2(s,t)dB(s)\right), \tag{51}$$

now functions $u(t)$, $f(t)$ and $k_i(s,t), i = 1, 2$, can be expanded in term of the Chebyshev wavelets as

$$f(t) \simeq F^T\Psi(t) = \Psi^T(t)F, \tag{52}$$

$$u(t) \simeq C^T\Psi(t) = \Psi^T(t)C, \tag{53}$$

$$k_i(s,t) \simeq \Psi^T(t)K_i\Psi(s) = \Psi^T(s)K_i^T\Psi(t),\ i = 1, 2, \tag{54}$$

where C and F are Chebyshev wavelets coefficients vector, and $K_i, i = 1, 2$, are Chebyshev wavelets coefficient matrices defined in Eqs. (27) and (29). Substituting above approximations in Eq. (51), we get

$$\begin{aligned} C^T\Psi(t) = F_0^T\Psi(t) + J^\alpha F^T\Psi(t) + J^\alpha\left(\Psi^T(t)K_1\int_0^t \Psi(s)\Psi^T(s)Cds\right) \\ + J^\alpha\left(\Psi^T(t)K_1\int_0^t \Psi(s)\Psi^T(s)CdB(s)\right), \end{aligned}$$

now Remark 1, Theorem 2 and 4 results

$$C^T\Psi(t) = F_0^T\Psi(t) + F^TP^\alpha\Psi(t)$$

$$+ J^\alpha\left(H^T(t)K_1\int_0^t \tilde{C}\Psi(s)ds\right) + J^\alpha\left(\Psi^T(t)K_2\int_0^t \tilde{C}\Psi(s)dB(s)\right)$$

$$= F_0^T\Psi(t) + F^TP^\alpha\Psi(t) + J^\alpha\left(\Psi^T(t)K_1\tilde{C}P\Psi(t)\right) + J^\alpha\left(\Psi^T(t)K_2\tilde{C}P_s\Psi(t)\right)$$

$$= F_0^T\Psi(t) + F^TP^\alpha\Psi(t) + J^\alpha\left(C_1^T\Psi(t)\right) + J^\alpha\left(C_2^T\Psi(t)\right)$$

$$= F_0^T\Psi(t) + F^TP^\alpha\Psi(t) + C_1^TP^\alpha\Psi(t) + C_2^TP^\alpha\Psi(t),$$

where $\tilde{C} = diag(C)$ is a $\hat{m}\times\hat{m}$ matrix, $C_1 = diag(K_1\tilde{C}P)$ and $C_2 = diag(K_2\tilde{C}P_s)$ are $\hat{m}$-vectors. As this equation is hold for all $t \in [0,1)$ we can write

$$C^T = F_0^T + F^TP^\alpha + C_1^TP^\alpha + C_2^TP^\alpha. \tag{55}$$

The vectors C_1 and C_2 are linear function of vector C, so Eq. (55) is a linear system of algebraic equations for unknown vector C. Solving this linear system we obtain vector C, which can be used to approximate solution of SFDE (1) by substituting in Eq. (53).

6. Convergence analysis

In this part, we consider the convergence and error analysis of the Chebyshev wavelets basis.

Theorem 5. Suppose $f(x) \in L^2_{w_n}[0,1]$ with bounded second derivative, say $|f''(x)| \le L$, and let $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{mn}\psi_{mn}(x)$ be its infinite Chebyshev wavelets expansion, then

$$|c_{mn}| \le \frac{\sqrt{2\pi}L}{(2n)^{\frac{5}{2}}(m^2-1)}, \tag{56}$$

this means the Chebyshev wavelets series converges uniformly to $f(x)$ and

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x), \tag{57}$$

Proof. Please see [39]. □

Theorem 6. Let $f(x)$ be a continuous function defined on $[0,1)$, with second derivatives $f''(x)$ bounded by L, then we have the following accuracy estimation

$$\sigma_{M,k} \le \left(\frac{\pi L^2}{2^4}\sum_{n=0}^{\infty}\sum_{m=M}^{\infty}\frac{1}{n^5(m^2-1)^2} + \frac{\pi L^2}{2^4}\sum_{n=2^k}^{\infty}\sum_{m=0}^{M-1}\frac{1}{n^5(m^2-1)^2}\right)^{\frac{1}{2}}, \tag{58}$$

where

$$\sigma_{M,k} = \left(\int_0^1\left(f(x) - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x)\right)^2 dx\right)^{\frac{1}{2}}.$$

Proof. We have

$$\sigma_{M,k}^2 = \int_0^1\left(f(x) - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x)\right)^2 dx$$

$$= \int_0^1\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x) - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x)\right)^2 dx$$

$$= \sum_{n=0}^{\infty} \sum_{m=M}^{\infty} c_{nm}^2 \int_0^1 \psi_{nm}^2(x)dx + \sum_{n=2^k}^{\infty} \sum_{m=0}^{M-1} c_{nm}^2 \int_0^1 \psi_{nm}^2(x)dx$$

$$= \sum_{n=0}^{\infty} \sum_{m=M}^{\infty} c_{nm}^2 + \sum_{n=2^k}^{\infty} \sum_{m=0}^{M-1} c_{nm}^2,$$

now by considering the relation (56) the desired result is achieved. □

7. Numerical results

In this section, some examples are given to demonstrate the applicability of the proposed method in section 5. In all examples the algorithms are performed by Maple 17 with 20 digits precision.

Example 1. Consider the following SFDE

$$D^{\alpha}u(t) = \frac{7}{12}t^4 - \frac{5}{6}t^3 + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + \int_0^t (s+t)u(s)ds + \int_0^t su(s)dB(s),$$
$$s, t \in [0,1],$$

subject to the initial condition $u(0) = 0$. The exact solution of this SFDE in unknown. The Chebyshev wavelet Galerkin method presented in section 5 are applied for deriving numerical solution of this SFDE. Fig.1 shows the approximate solution obtained by Galerkin wavelet method for different values of α and $\hat{m} = 128$. Table 1 shows the approximate solution for different values of t and α.

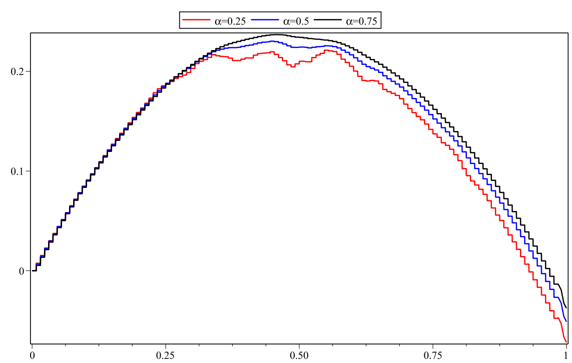

Figure 1. The approximate solution for $\alpha = 0.25, \alpha = 0.5$ and $\alpha = 0.75$.

Table 1. Numerical results for different values of t, α and $\hat{m} = 128$.

t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
0.1	0.0848476129	0.0845854008	0.0836981075
0.3	0.2030817351	0.2061982209	0.2070198626
0.5	0.4178064933	0.44860219373	0.4686001067
0.7	0.1671430738	0.1804842707	0.1890702089
0.9	0.0289527455	0.0480352233	0.0592956314

Example 2. Consider the following SFDE

$$D^{\alpha}u(t) = \frac{t^2}{2} + \frac{\Gamma(2)t^{1-\alpha}}{\Gamma(2-\alpha)} + \int_0^t u(s)ds + \int_0^t u(s)dB(s),\ \ s,t \in [0,1]\,,$$

subject to the initial condition $u(0) = 0$. The exact solution of this SFDE in unknown. Here we use the Chebyshev wavelet Galerkin method proposed in section 5 to solve it. The approximate solution derived by the wavelet Galerkin method for diffrent values of α and $\hat{m} = 128$ is plotted in Fig. 2. Moreover, Table 2 shows the approximate solutions obtained for different values of t and α.

Table 2. Numerical results for different values of t and $\hat{m} = 128$.

t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
0.1	0.1413354244	0.1147305355	0.1019923557
0.3	0.4848198910	0.4205839899	0.3744537669
0.5	1.5490799582	1.4084534350	1.3003772022
0.7	1.4520354033	1.2250944753	1.0684662718
0.9	2.3441217052	1.9033337794	1.6182147547

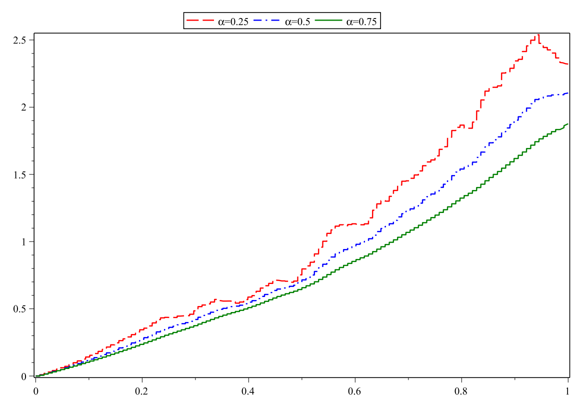

Figure 2. The approximate solution for $\alpha = 0.25, \alpha = 0.5$ and $\alpha = 0.75$.

8. Conclusion

A wavelet Galerkin method based on the Chebyshev wavelets and their operational matrices of fractional and stochastic integration is proposed for approximate solution of SFDEs. A general formulation of these operational matrices for the Chebyshev wavelets is derived. Then, the Chebyshev wavelets basis along with their operational matrices are used to approximate solution of SFDEs. Convergence and error analysis of the Chebyshev wavelets basis are also investigated. Numerical results confirm the efficiency of the proposed method.

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