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AN APPLICATION OF HOMOTOPY ANALYSIS TRANSFORM METHOD FOR RICCATI DIFFERENTIAL EQUATION OF FRACTIONAL ORDER

K. M. SAAD,A. A. AL-SHOMRANI

Abstract. We introduce an analytical method, namely the Homotopy Analysis Transform Method (HATM), which is a combination of the Homotopy Analysis Method (HAM) and the Laplace Decomposition Method (LDM). A new application of the HATM is presented for the solution of the fractional order Riccati differential equation. The accuracy and efficiency of the proposed method is verified through three examples and comparison with exact solutions.

1. Introduction

In recent years considerable interest in fractional differential equations has been stimulated by their numerous applications in many areas of physics and engineering [30]. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are well described by differential equations of fractional order [29, 23, 21, 24]. Historical summaries of the development of fractional calculus can be found in [21, 22, 23, 24]. Exact solutions of most fractional differential equations cannot be easily found. Thus analytical and numerical methods need to be used for their solution [1, 2, 3, 4, 5, 6].

In this paper, we introduce an approximate analytical method, namely the HATM, which is a combination of the HAM and the LDM. This scheme is simple to apply to linear and nonlinear fractional differential equations and requires less computational effort compared with other exiting methods. The most important advantage of this method is its ability to solve fractional nonlinear differential equation without using Adomian polynomials and He′s polynomials for the computation of the nonlinear terms. The proposed method has no linearization and restrictive assumptions for its stability. Recently, several mathematical methods for solving the fractional differential equations , including the homotopy analysis method HAM, have been proposed into [7, 8, 9, 10, 11].

The HAM was first proposed by Liao for solving linear and nonlinear differential and integral equations. In recent years, many authors have found solutions of linear

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and nonlinear partial differential equations using various methods in combination with the Laplace transform. Among these are the Laplace decomposition method [12, 13, 14, 15, 27] and the homotopy perturbation transform method [18, 19, 20, 28]. The paper is organized as follows. Section 2 contains the basic definition of the Caputo order fractional derivative. Section 3 outlines the basic idea of the Fractional Homotopy Analysis Transform Method (FHATM). Section 4 deals with an application of the FHATM to nonlinear Riccati equations and Section 5 contains the conclusions.

2. Fractional calculus

Well-known definitions of a fractional derivative of order $\alpha > 0$ have been given by Riemann, Liouville, Grunwald, Letnikow and Caputo [23, 29, 22, 21] and are based on generalized functions. The most commonly used definitions are those of Riemann and Liouville and Caputo. Here we give some basic definitions and properties of this fractional calculus theory.

Definition 2.1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in R$, if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$, and it is said to be in the space C_μ^m iff $f^m \in C_m$, $m \in N$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ for a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J_0^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau$$

$$J^0 f(t) = f(t).$$

It has the following properties. For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$, and $\gamma > 1$,

(1) $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$,
(2) $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$,
(3) $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

The Riemann-Liouville fractional derivative is mostly used in mathematics as this approach is not suitable for physical problems since it requires the use of fractional order initial conditions which have not been found to have physical meaning as yet. For this reason, Caputo introduced an alternative definition which has the advantage of defining integer order initial conditions for fractional order differential equations.

Definition 2.3. The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D_*^\nu f(t) = J_a^{m-\nu} D^m f(t) = \frac{1}{\Gamma(m-\nu)} \int_0^t (t-\tau)^{m-\nu-1} f^m(\tau)\, dt$$

for $m - 1 < \nu < m, m \in \mathbb{N}, t > 0, f \in C_\mu^m, \mu \geq -1$, then

$$D_*^\alpha J^\alpha f(t) = f(t)$$

$$J^\alpha D_*^\nu f(t) = f(t) - \sum_{k=0}^{m-1} f^k(0^+) \frac{t^k}{k!}, \quad x > 0$$

The Caputo fractional derivative will be used here as it allows traditional initial and boundary conditions to be used for differential equations.

3. Laplace Transform

Let $f(t)$ be defined for $0 \leq t < \infty$. Then, when the improper integral exists, the Laplace transform $F(s)$ of $f(t)$, written symbolically as $F(s) = \mathcal{L}\{f(t)\}$, is defined by

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

Lemma 3.1. If $m-1 < \alpha \leq m, m \in \mathbb{N}$, then the Laplace transform of the fractional derivative $D^\alpha f(t)$ is

$$\mathcal{L}(D^\alpha f(t)) = s^\alpha F(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-k-1}, t > 0, \tag{3.1}$$

where $F(s)$ is the Laplace transform of $f(t)$.

Proof. The convolution integral of two functions $f(t)$ and $g(t)$ is defined by

$$f(t) * g(t) = \int_0^t f(t-\tau) g(\tau) d\tau.$$

If $F(s)$ and $G(s)$ are the Laplace transforms of $f(t)$ and $g(t)$, respectively, then

$$\mathcal{L}\left(\int_0^t f(t-\tau) g(\tau) d\tau\right) = F(s)G(s).$$

The Laplace transform of the Riemann-Liouville fractional integral operator of order $\alpha > 0$ is

$$\mathcal{L}\left(J^\alpha f(t)\right) = \frac{1}{\Gamma(\alpha)} \mathcal{L}\left(\int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau\right) = \frac{1}{\Gamma(\alpha)} F(s)G(s),$$

where

$$G(s) = \mathcal{L}(t^{\alpha-1}) = \frac{\Gamma(\alpha)}{s^\alpha}.$$

If we take the Laplace transform of

$$J^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{t^k}{k!}, \; m-1 < \alpha \leq m,$$

we obtain

$$\mathcal{L}(J^\alpha D^\alpha f(t)) = L(f(t)) - \sum_{k=0}^{m-1} f^{(k)}(0^+) L(\frac{t^k}{k!}),$$

so that

$$\frac{\mathcal{L}\left(D^\alpha f(t)\right)}{s^\alpha} = F(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{-(k+1)}.$$

Hence

$$\mathcal{L}\left(D^\alpha f(t)\right) = s^\alpha F(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-k-1}, \; m-1 < \alpha \leq m.$$

We can also prove Lemma 3.1 using integral transforms as follows

$$
\begin{aligned}
\mathcal{L}(D^{\alpha} f(t)) =& \int_0^{\infty} e^{-st} D^{\alpha} f(t) dt \\
=& \int_0^{\infty} e^{-st} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(m)}(\tau) d\tau dt \\
=& \frac{1}{\Gamma(m-\alpha)} \int_0^{\infty} \int_t^{\infty} e^{-s} f^{(m)}(\tau)(t-\tau)^{n-\alpha-1} dt d\tau \\
=& \frac{1}{\Gamma(m-\alpha)} \int_0^{\infty} e^{-s\tau} f^{(m)}(\tau) \int_0^{\infty} e^{-s(y+\tau)} y^{n-\alpha-1} dy d\tau \\
=& \frac{1}{\Gamma(m-\alpha)} \int_0^{\infty} e^{-s\tau} f^{(m)}(\tau) \frac{\Gamma(m-\alpha)}{s^{m-\alpha}} d\tau \\
=& s^{\alpha-m} \int_0^{\infty} e^{-s\tau} f^{(m)}(\tau) d\tau = s^{\alpha-m} \mathcal{L}(f(t)) \\
=& s^{\alpha-m} \left\{ s^m \mathcal{L}(f(t)) - s^{m-1} f(0^+) - s^{m-2} f^{'}(0^+) - \cdots - f^{(m-1)}(0^+) \right\} \\
=& \left\{ s^{\alpha} \mathcal{L}(f(t)) - s^{\alpha-1} f(0^+) - s^{\alpha-2} f^{'}(0^+) - \cdots - s^{\alpha-m} f^{(m-1)}(0^+) \right\} \\
=& s^{\alpha} F(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-k-1}, \; m-1 < \alpha \leq m.
\end{aligned}
$$

□

4. Fractional Homotopy Analysis Transform Method(FHATM)

Consider the equation the $N(y(t))) = g(t)$, where N represents a general nonlinear ordinary differential equation both including both linear and nonlinear terms, y is the unknown function to be solved for and t is the independent variable. For simplicity, we ignore all boundary or initial conditions, as they can be treated in a similar way. The linear terms are decomposed into $L + R$, where L is the highest order linear operator, R is the other terms of the linear operator and $g(t)$ is a continuous function. Thus, the equation can be written as

$$Ly(t) + Ry(t) + Ny(t) = g(t), \tag{4.2}$$

where $Ny(t)$ indicate the nonlinear terms. Now if we let $L = D^{\alpha}(t)$ and apply the Laplace transform to both sides of Equation (4.2) we obtain

$$\mathcal{L}\left(D^{\alpha}(t)\right) + \mathcal{L}\left(Ry(t) + Ny(t)\right) = \mathcal{L}g(t). \tag{4.3}$$

Using (3.1) we then have

$$\mathcal{L}y(t) - \frac{1}{s^{\alpha}} \sum_{i=0}^{m-1} y^{(i)}(0) s^{\alpha-i-1} - \frac{1}{s^{\alpha}} \left(\mathcal{L}\left(Ry(t) + Ny(t)\right) - \mathcal{L}g(t)\right) = 0. \tag{4.4}$$

We define the nonlinear operator

$$N\left[\phi(t;q)\right] = \mathcal{L}\left[\phi(t;q)\right] - \frac{1}{s^{\alpha}} \sum_{i=0}^{m-1} \phi(t;q)^{(i)}(0) s^{\alpha-i-1} - \frac{1}{s^{\alpha}} \left(\mathcal{L}\left(R\phi(t;q) + N\phi(t;q)\right) - \mathcal{L}g(t)\right), \tag{4.5}$$

where $q \in [0,1]$ is an embedding parameter for $\phi(t;q)$, with ϕ a real function of t and q. The so-called zero-order deformation equations of the Laplace transform equation (4.5) have been shown by Liao to have the form

$$(1-q)\mathcal{L}[\phi(t;q) - y_0(t)] = qhH(t)N[\phi(t;q)] \tag{4.6}$$

when $q = 0$ and $q = 1$. Here we have $\phi(t;0) = u_0(t)$ and $\phi(t;1) = u(t)$, respectively. Thus, as q increases from 0 to 1, the solution $\phi(t;q)$ varies from the initial guess $y_0(t)$ to the solution $y(t)$. Expanding $\phi(t;q)$ in a Taylor series with respect to q we have

$$\phi(t;q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t)q^m, \tag{4.7}$$

where

$$y_m(t) = \frac{1}{m!}\frac{\partial^m \phi(t;q)}{\partial q^m}|_{q=0}. \tag{4.8}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h and the auxiliary function $H(t)$ are properly chosen, then the series (4.7) converges at $q = 1$ and we have [31, 8]

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t). \tag{4.9}$$

Let us now define the vector

$$\overrightarrow{y_n} = \{y_0(t), y_1(t), y_2(t), \ldots, y_n(t)\}. \tag{4.10}$$

Differentiating (4.6) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing by $m!$, we have the so called mth-order deformation equation

$$\mathcal{L}[y_m(t) - \chi_m y_{m-1}(t)] = hH(t)R_m(\overrightarrow{y}_{m-1}(t)), \quad m = 1, 2, 3, ..., n, \tag{4.11}$$

where

$$R_m(\overrightarrow{y}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(t;q)]}{\partial q^{m-1}}|_{q=0} \tag{4.12}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}. \tag{4.13}$$

On finding the inverse Laplace transform of (4.11) we then have a power series solution $y(t) = \sum_m^{\infty} y_m(t)$ of (4.2).

5. Applications

In this section, we present the solution of a nonlinear fractional Ricatti equation as an application of HATM.

Example 5.1. *Consider the following Riccati differential equation*

$$D^{\alpha} y(t) = 2y(t) - y^2(t) + 1, \quad 0 < \alpha \leq 1, \tag{5.14}$$

subject to the initial condition $y(0) = 0$.

The exact solution of (5.14) for $\alpha = 1$ is $y(t) = 1+\sqrt{2}\tanh\left(\sqrt{2}t + (1/2)\log\left((\sqrt{2}-1)/(\sqrt{2}+1)\right)\right)$. Applying the Laplace transform to both sides of equation (5.14) and using (3.1), we obtain

$$\mathcal{L}\{y(t)\} - \frac{1}{s}y(0) + \frac{1}{s^{\alpha}}\mathcal{L}\{2y(t)\} - \frac{1}{s^{\alpha}}\mathcal{L}\{y^2(t)\} + \frac{1}{s^{\alpha}}\mathcal{L}\{1\} = 0. \tag{5.15}$$

In view of the HAM technique and assuming $H(t) = 1$, we construct the zeroth-order deformation equation as follows

$$(1-q)\mathcal{L}[\phi(t;q) - y_0(t)] = qhH(t)N[\phi(t;q)], \tag{5.16}$$

where

$$N[\phi(t;q)] = \mathcal{L}\{\phi(t;q)\} - \frac{1}{s}y(0) + \frac{1}{s^{\alpha}}\mathcal{L}\{2\phi(t;q)\} - \frac{1}{s^{\alpha}}\mathcal{L}\{\phi(t;q)^2(t)\} + \frac{1}{s^{\alpha}}\mathcal{L}\{1\}. \tag{5.17}$$

The series solution of (5.14) is given by (4.9). Thus, we obtain the m-th order deformation equation

$$\mathcal{L}[y_m(t) - \chi_m y_{m-1}(t)] = hR_m(\overrightarrow{y}_{m-1}(t)), \quad m = 1,2,3,....,n, \tag{5.18}$$

with

$$R_m(\overrightarrow{y}_{m-1}(t)) = \mathcal{L}\{y_{m-1}(t)\} - \frac{1}{s}(1-\chi_m)y(0) + \frac{2}{s^{\alpha}}\mathcal{L}\{y_{m-1}(t)\} - \frac{1}{s^{\alpha}}\mathcal{L}\{\sum_{i=0}^{m-1} y_i(t)y_{m-1-i}(t)\} + \frac{1}{s^{\alpha}}\mathcal{L}\{1\}. \tag{5.19}$$

Finding inverse Laplace transform of (5.18), we obtain

$$y_m(t) = \chi_m y_{m-1}(t) + h\mathcal{L}^{-1}\{R_m(\overrightarrow{y}_{m-1}(t))\}, \quad m = 1,2,3,....,n. \tag{5.20}$$

Consequently, the first three terms of the HATM series approximate solution with $y_0(t) = 0$ are

$$y_1(t) = -\frac{ht^{\alpha}}{\Gamma(1+\alpha)}, \tag{5.21}$$

$$y_2(t) = (1+h)y_1 + \frac{2h^2t^{2\alpha}}{\Gamma(1+2\alpha)}, \tag{5.22}$$

$$y_3(t) = (1+h)y_2 + \left(1 + \frac{h\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)2ht^{\alpha}y_1 - h^2t^{\alpha}\left(\frac{ht^{2\alpha}\Gamma(1+\alpha)^2 - \Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}\right). \tag{5.23}$$

We therefore obtain the series solution from (4.9) as

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots \tag{5.24}$$

Example 5.2. *Consider the following Riccati differential equation*

$$D^{\alpha}y(t) = y^2(t) + 1, \quad 0 < \alpha \le 1, \tag{5.25}$$

subject to the initial condition $y(0) = 0$. *The exact solution of (5.25) for* $\alpha = 1$ *is* $y = \tan t$.

As for example 1 the HATM iteration is

$$y_m(t) = \chi_m y_{m-1}(t) + h\mathcal{L}^{-1}\{R_m(\overrightarrow{y}_{m-1}(t))\}, \quad m = 1,2,3,....,n, \tag{5.26}$$

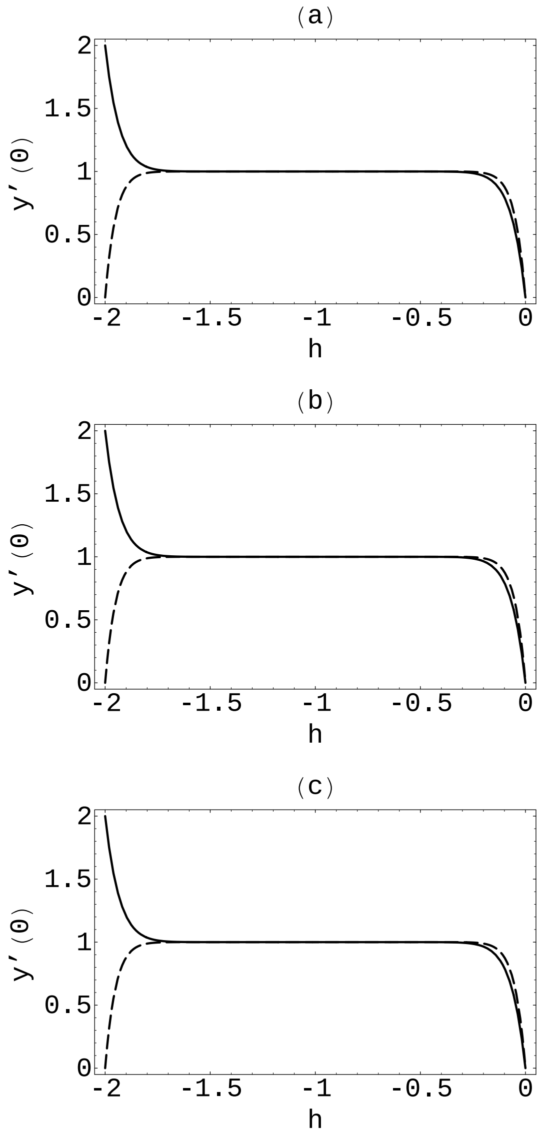

FIGURE 1. The h-curve of $y^{'}(0)$ at the 20th-term (*solid*) and 15-term ($--$) of the HATM series solutions (5.24), (5.31) and (5.38).

where

$$R_m(\overrightarrow{y}_{m-1}(t)) = \mathcal{L}\{y_{m-1}(t)\} - \frac{1}{s}(1-\chi_m)y(0) - \frac{1}{s^\alpha}\mathcal{L}\{\sum_{i=0}^{m-1} y_i(t)y_{m-1-i}(t)\} + \frac{1}{s^\alpha}\mathcal{L}\{1\}. \tag{5.27}$$

Consequently, the first three terms of the HATM series solution with $y_0(t) = 0$ are

$$y_1 = -\frac{ht^\alpha}{\Gamma(1+\alpha)}, \tag{5.28}$$

$$y_2 = (1+h)y_1, \tag{5.29}$$

$$y_3 = (1+h)y_2 - \frac{\Gamma(1+2\alpha)h^3}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}t^{3\alpha} \tag{5.30}$$

and the series solution from (4.9) is

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots \tag{5.31}$$

Example 5.3. *Consider the following Riccati differential equation*

$$D^\alpha y(t) = -y^2(t) + 1, \quad 0 < \alpha \leq 1, \tag{5.32}$$

subject to the initial condition $y(0) = 0$.

The exact solution of (5.32) for $\alpha = 1$ is $y = \frac{e^{2t}-1}{e^{2t}+1}$. As in example 1 the HATM iteration is

$$y_m(t) = \chi_m y_{m-1}(t) + h\mathcal{L}^{-1}\{R_m(\overrightarrow{y}_{m-1}(t))\}, \quad m = 1, 2, 3, ..., n, \tag{5.33}$$

where

$$R_m(\overrightarrow{y}_{m-1}(t)) = \mathcal{L}\{y_{m-1}(t)\} - \frac{1}{s}(1-\chi_m)y(0) - \frac{1}{s^\alpha}\mathcal{L}\{\sum_{i=0}^{m-1} y_i(t)y_{m-1-i}(t)\} + \frac{1}{s^\alpha}\mathcal{L}\{1\}. \tag{5.34}$$

Consequently, the first three terms of the HATM series solution with $y_0(t) = 0$ are

$$y_1 = -\frac{ht^\alpha}{\Gamma(1+\alpha)}, \tag{5.35}$$

$$y_2 = (1+h)y_1, \tag{5.36}$$

$$y_3 = (1+h)y_2 + \frac{\Gamma(1+2\alpha)h^3}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}t^{3\alpha} \tag{5.37}$$

and the series solution is

$$y(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots \tag{5.38}$$

As pointed by Liao [7] the expressions given by (4.9), (5.26) and (5.33) contain the auxiliary parameter h. This parameter determines the convergence region and rate of convergence of the approximation found using the HATM. This is shown in Figure 1. This Figure shows the 20-term and 15-term HATM approximate analytic solutions for $\alpha = 1$, as given by Example 1, Example 2 and Example 3, respectively. The figure shows $y'(0)$ was plotted against h. We chose the horizontal line parallel to the h-axis seen in this figure as the convergence region for the approximation, which provides us with a simple way to adjust and control the convergence region of the series solutions (5.24), (5.31) and (5.38). From this figure, the convergence of the method is guaranteed for $-1.8 \leq h \leq -0.2$. In Figure 2 we compare the HATM solutions (5.24), (5.31) and (5.38) for different values of h with the exact solutions of (5.14), (5.25) and (5.32), respectively. Figure 2 shows that the best result for the HATM solution occurs for $h = -0.5$ for Examples 1 and 3, while it is $h = -0.9$ for Example 2.

Next, we compute the HATM solution for different values of α with $h = -0.5$ for Examples 1 and 3 and at $h = -0.9$ for Example 2. Figure 3 shows the behavior of

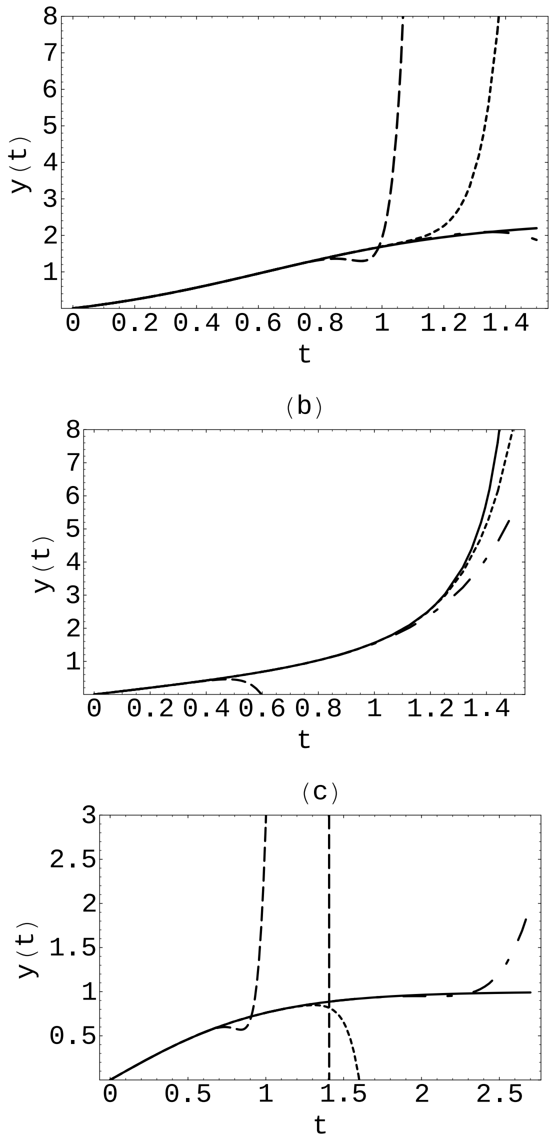

FIGURE 2. Comparison of the HATM series solutions (5.24), (5.31), and (5.38) with exact solutions of (5.14), (5.25), and (5.32) in (a), (b) and (c) respectively at $\alpha = 1$ for $h = -1.5\ (--), -1\ (\cdots)$, and $-0.5\ (-\cdot-)$.

the HATM for $\alpha = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and 1. The ADM results obtained for $\alpha = 1/2, 1/3, 1/4$ and $\alpha = 1$ are summarized in Figure 3. The comparison shows that as $\alpha \to 1$, the approximate solutions tend to the exact solution in the case $\alpha = 1$.

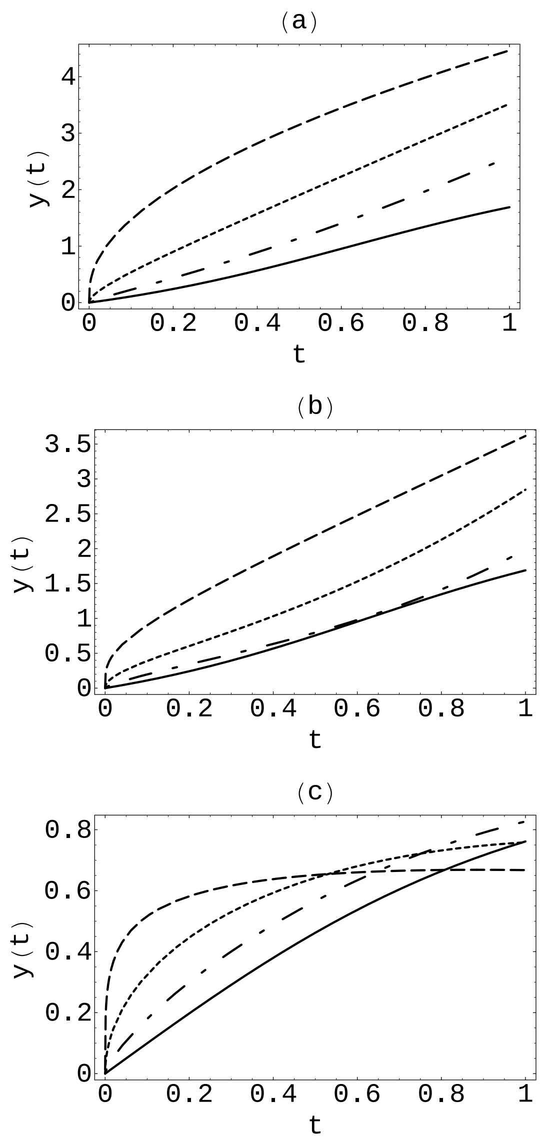

Figure 3. Plots of the HATM series solutions (5.24), (5.31), and (5.38) with exact solution $\alpha = 1(-)$ of (5.14), (5.25), and (5.32) in (a), (b) and (c) respectively at $h = -0.5$ for $\alpha = \frac{1}{4}(--)$, $\frac{1}{2}(\cdots)$, and $\frac{3}{4}(-\cdot-)$.

6. Conclusions

In this paper, the HATM was employed to analytically compute approximate solutions of a fractional-order Riccati differential equation. By comparing these approximate solutions with known exact solutions, it was shown that these solutions have high accuracy. The solution obtained by HATM is in good agreement for $\alpha = 1$. Mathematica was used for the computations in this paper.

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K. M. Saad
Faculty of Arts and Science, Najran University, Najran, Saudi Arabia. On leave from Taiz University , Yemen
E-mail address: `khaledma_sd@hotmail.com, kmalhomam@nu.edu.sa`

A. A. AL-Shomrani
Faculty of Arts and Science, Najran University, Najran, Saudi Arabia,
E-mail address: `aazzoz1234@hotmail.com`