Journal of Fractional Calculus and Applications,
Vol. 7(1) Jan. 2016, pp. 24-35.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

EXISTENCE AND APPROXIMATE SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

B.C. DHAGE, S.B. DHAGE, S.K. NTOUYAS

Abstract. In this paper the authors prove existence, uniqueness and approximation of the solutions for initial value problems of nonlinear fractional differential equations with nonlocal conditions, using the operator theoretic technique in a partially ordered metric space. The main results rely on the Dhage iteration principle embodied in the recent hybrid fixed point theorem of Dhage (2014) in a partially ordered normed linear space. The approximation of the solutions of the considered nonlinear fractional differential equations are obtained under weaker mixed partial continuity and partial Lipschitz conditions. Our hypotheses and result are also illustrated by a numerical example.

1. Introduction

The Dhage iteration principle or method (in short DIP or DIM) is relatively new to the literature on nonlinear analysis, particularly in the theory of nonlinear differential and integral equations, but it has been becoming more popular among the mathematicians all over the world because of the utility of its applications to nonlinear equations for different qualitative aspects of the solutions. Very recently, the above method has been applied in Dhage [10, 11, 12, 13], Dhage and Dhage [14, 15, 16, 17] and Dhage *et al.* [18] to nonlinear ordinary differential equations for proving the existence and algorithms of the solutions. Similarly, DIM has also some interesting applications in the theory of nonlinear fractional differential and integral equations and some basic results concerning the existence, uniqueness and algorithms for initial value problems of fractional differential equations with local conditions have been proved in Dhage [13]. In the present paper we prove the existence as well as algorithms for the solutions of the initial value problems of fractional differential equations with nonlocal conditions.

Before stating the main problem of the paper, we recall the following basic definitions of fractional calculus [21, 24] which are useful in what follows.

2010 *Mathematics Subject Classification.* 34A08, 47H07, 47H10.
Key words and phrases. Fractional differential equation; fixed point theorem; Dhage iteration principle; existence and uniqueness theorems.
Submitted April 5, 2015.

Definition 1.1. *Let $I = [t_0, t_0 + a]$ be a closed and bounded interval of the real line $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $t_0 \geq 0$ and $a > 0$. If $x \in AC^n(J, \mathbb{R})$, then the Caputo derivative of fractional order q is defined as*

$$^cD^q x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} x^{(n)}(s) ds, \ n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q, and Γ is the Euler's gamma function. Here $AC^n(I, \mathbb{R})$ denote the space of real valued functions $x(t)$ which have continuous derivatives up to order $n-1$ on I such that $x^{n-1}(t) \in AC(I, \mathbb{R})$.

Definition 1.2. *If $I_\infty = [t_0, \infty)$ be an interval of the real line $\mathbb{R}$ for some $t_0 \in \mathbb{R}$ with $t_0 \geq 0$, then for any $x \in C(J, \mathbb{R})$, the Riemann-Liouville fractional integral of order $q > 0$ is defined as*

$$I^q x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} \frac{x(s)}{(t-s)^{1-q}} ds, \ t \in I_\infty,$$

provided the right hand side is pointwise defined on (t_0, ∞).

Consider the following initial value problem of fractional differential equations with nonlocal condition,

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \ t \in J := [0,1], \\ \qquad\qquad x(0) - g(x) = x_0, \end{cases} \tag{1}$$

where $^cD^q$ denotes the Caputo fractional derivative of order q, $0 < q < 1$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and $g \in C(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function.

Fractional differential equations have aroused great interest, which is caused by both the intensive development of the theory of fractional calculus and the applications of physics, mechanics and chemistry engineering.

The importance of non-local problems appears to have been first noted in the literature by Bitsadze-Samarski [3]. As remarked by Byszewski [5, 7], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For example, $g(x)$ may be given by $g(x) = \sum_{i=1}^{p} c_i x(t_i)$, where $c_i, i = 1, \ldots, p$ are given constants and $0 < t_1 < \ldots < t_p \leq 1$. For more details on nonlocal problems we refer to [1, 2, 4, 19, 23, 25] and the references cited therein.

In the present paper we prove the existence, uniqueness and approximations of the solutions of problem (1) under weaker partially compactness and partially Lipschitz type conditions via Dhage's iteration method.

The rest of the paper will be organized as follows. In Section 2 we give some preliminaries and key fixed point theorems that will be used in subsequent part of the paper. In Section 3 we discuss the main existence, uniqueness and approximation results for initial value problems of fractional differential equations (1). An illustrative example is also discussed.

2. Auxiliary Results

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm

$\|\cdot\|$ in which the addition and the scalar multiplication by positive real numbers are preserved by $\preceq$. A few details of such partially ordered spaces appear in Dhage [8] and the references therein.

Two elements x and y in E are said to be **comparable** if either the relation $x \preceq$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all elements of C are comparable. We say that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [20] and the references therein.

We need the following definitions in the sequel.

Definition 2.1. *A mapping $\mathcal{T} : E \to E$ is said to be* **isotone** *or* **monotone nondecreasing** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is* **monotone nonincreasing** *if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is said to be* **monotonic** *or simply* **monotone** *if it is either monotone nondecreasing or monotone nonincreasing on E.*

Definition 2.2 (Dhage [8])**.** *A mapping $\mathcal{B} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{B}x - \mathcal{B}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{B}$ called a partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{B}$ is a partially continuous on E, then it is continuous on every chain C contained in E.*

Definition 2.3. *A non-empty subset S of the partially ordered Banach space E is called partially bounded if every chain C in S is bounded. A mapping $\mathcal{B} : E \to E$ is called* **partially bounded** *if $\mathcal{B}(C)$ is bounded for every chain C in E. $\mathcal{B}$ is called* **uniformly partially bounded** *if all chains $\mathcal{B}(C)$ in E are bounded by a unique constant. $\mathcal{B}$ is called* **bounded** *if $\mathcal{B}(E)$ is a bounded subset of E.*

Definition 2.4. *A non-empty subset S of the partially ordered Banach space E is called partially compact if every chain C in S is compact. A mapping $\mathcal{B} : E \to E$ is called* **partially compact** *if $\mathcal{B}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{B}$ is called* **uniformly partially compact** *if $\mathcal{B}(C)$ is a uniformly partially bounded and partially compact on E. $\mathcal{B}$ is called* **partially totally bounded** *if for any totally ordered and bounded subset C of E, $\mathcal{B}(C)$ is a relatively compact subset of E. If $\mathcal{B}$ is partially continuous and partially totally bounded, then it is called* **partially completely continuous** *on E.*

Definition 2.5 (Dhage [8])**.** *The order relation $\preceq$ and the metric d on a non-empty set E are said to be* ***compatible*** *if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* implies that the original sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.*

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property.

Definition 2.6 (Dhage [8]). *A upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(0) = 0$. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{T} : E \to E$ is called* ***partially nonlinear $\mathcal{D}$-Lipschitz*** *if there exists a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|) \tag{2}$$

for all comparable elements $x, y \in E$. If $\psi(r) = k\,r$, $k > 0$, then $\mathcal{T}$ is called a partially Lipschitz with a Lipschitz constant k. Furthermore, if $0 < \psi(r) < r$, $r > 0$, $\mathcal{T}$ is called a partially nonlinear $\mathcal{D}$-contraction on E.

The Dhage iteration principle are embodied in the following hybrid fixed point theorems proved of Dhage [9] which form the useful tools in what follows. A few other such hybrid fixed point theorems containing Dhage iteration principle along with applications appear in Dhage [8, 9].

Theorem 2.7 (Dhage [9]). *Let $(E, \preceq, \|\cdot\|)$ be a partially ordered Banach space and let $\mathcal{T} : E \to E$ be a nondecreasing and partially nonlinear $\mathcal{D}$-contraction. Suppose that there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$. If $\mathcal{T}$ is continuous or E is regular, then $\mathcal{T}$ has a fixed point x^* and the sequence $\{\mathcal{T}^n x_0\}$ of successive iterations converges monotonically to x^*. Moreover, the fixed point x^* is unique if every pair of elements in E has a lower and an upper bound.*

Theorem 2.8 (Dhage [9]). *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear algebra such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain C of E. Let $\mathcal{A}, \mathcal{B} : E \to \mathcal{K}$ be two nondecreasing operators such that*

(a) *$\mathcal{A}$ is partially bounded and partially nonlinear $\mathcal{D}$-contraction,*
(b) *$\mathcal{B}$ is partially continuous and partially compact, and*
(c) *there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0\, \mathcal{B}x_0$.*

Then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a positive solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, $n = 0, 1, \ldots$ converges monotonically to x^*.

Remark 2.9. The compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E is held if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$. This simple fact is used to prove the desired characterization of the positive solution of the problem (1) on J.

3. Main Existence Results

The equivalent integral form of the problem (1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3}$$

and

$$x \leq y \iff x(t) \leq y(t) \tag{4}$$

for all $t \in J$. Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular and a lattice so that every pair of elements of E has a lower and an upper bound in it. It is known that the partially ordered Banach space $C(J,\mathbb{R})$ has some nice properties w.r.t. the above order relation in it. The following lemma follows by an application of Arzellá-Ascoli theorem.

Lemma 3.1. *Let $\big(C(J,\mathbb{R}),\leq,\|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3) and (4) respectively. Then $\|\cdot\|$ and $\leq$ are compatible in every partially compact subset of $C(J,\mathbb{R})$.*

Proof. The proof of the lemma is given in Dhage and Dhage [15]. Since the proof is not well-known, we give the details of proof. Let S be a partially compact subset of $C(J,\mathbb{R})$ and let $\{x_n\}_{n\in\mathbb{N}}$ be a monotone nondecreasing sequence of points in S. Then we have

$$x_1(t)\leq x_2(t)\leq\cdots\leq x_n(t)\leq\cdots, \tag{ND}$$

for each $t\in J$.

Suppose that a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}_{k\in\mathbb{N}}$ of the monotone real sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent. By monotone characterization, the whole sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is convergent and converges to a point $x(t)$ in $\mathbb{R}$ for each $t\in J$. This shows that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x point-wise in S. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence by Arzelá-Ascoli theorem. Hence $\{x_n\}_{n\in\mathbb{N}}$ is convergent and converges uniformly to x. As a result, $\|\cdot\|$ and $\leq$ are compatible in S. This completes the proof. □

We need the following definition in what follows.

Definition 3.2. *A function $u\in C(J,\mathbb{R})$ is said to be a lower solution of the problem (1) if it satisfies*

$$\left.\begin{aligned} {}^cD^q u(t)&\leq f(t,u(t)),\ t\in J,\\ u(0)-g(u)&\leq x_0.\end{aligned}\right\} \tag{$*$}$$

Similarly, an upper solution $v\in C(J,\mathbb{R})$ to the problem (1) is defined on J, by the above inequalities with reverse sign.

We consider the following set of assumptions in what follows:

- (A_1) There exists a constant $M_g>0$ such that $|g(x)|\leq M_g$ for all $x\in C(J,\mathbb{R})$.
- (A_2) There exists a constant $L_g>0$ such that $0\leq g(x)-g(y)\leq L_g\|x-y\|$ for all $x,y\in C(J,\mathbb{R})$, $x\geq y$.
- (A_3) There exists a constant $M_f>0$ such that $|f(t,x)|\leq M_f$ for all $t\in J$ and $x\in\mathbb{R}$.
- (A_4) The function $f(t,x)$ is monotone nondecreasing in x for each $t\in J$.
- (A_5) There exists a constant $L_f>0$ such that $0\leq f(t,x)-f(t,y)\leq L_f(x-y)$ for all $t\in J$ and $x,y\in\mathbb{R}$, $x\geq y$.
- (A_6) The FDE (1) has a lower solution $u\in C(J,\mathbb{R})$.

The following lemma is useful in what follows and may be found in Kilbas *et.al.* [21], Podlubny [24] and Lakshmikantham *et al.* [22, page 54].

Lemma 3.3. *For any $h \in C(J, \mathbb{R})$, if the function $x \in C^1(J, \mathbb{R})$ is a solution of the problem*

$$\begin{cases} {}^cD^q x(t) = h(t), \ 0 < q < 1, \ t \in J, \\ x(0) - y_0 = x_0, \end{cases} \tag{5}$$

then

$$x(t) = x_0 + y_0 + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s) ds, \ t \in J, \tag{6}$$

and vice-versa.

Theorem 3.4. *Assume that the hypotheses (A_1), (A_2), (A_3), (A_4) and (A_6) hold. If $L_g < 1$, then the problem (1) has a solution x^* defined on J and the sequence $\{x_n\}_{n=1}^{\infty}$ of successive approximations defined by*

$$x_{n+1}(t) = x_0 + g(x_n) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, x_n(s))\, ds, \tag{7}$$

for all $t \in \mathbb{R}$, where $x_1 = u$, converges monotonically to x^*.

Proof. By Lemma 3.3, the problem (1) is equivalent to the nonlinear Volterra integral equation

$$x(t) = x_0 + g(x) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, x(s))\, ds, \ t \in J. \tag{8}$$

Set $E = C(J, \mathbb{R})$. Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Define the operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = x_0 + g(x), \ t \in J, \tag{9}$$

and

$$\mathcal{B}x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds, \ t \in J. \tag{10}$$

From the continuity of the integrals, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the maps $\mathcal{A}, \mathcal{B} : E \to E$. Then, the problem (1) is equivalent to the operator equation

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \ t \in J. \tag{11}$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.8. This is achieved in the series of following steps.

Step I: *$\mathcal{A}$ and $\mathcal{B}$ are nondecreasing operators on E.*

Let $x, y \in E$ be such that $x \geq y$. Then by hypothesis (A_2), we obtain

$$\mathcal{A}x(t) = g(x) + x_0 \geq g(y) + x_0 = \mathcal{A}y(t),$$

for all $t \in J$. This shows that $\mathcal{A}$ is nondecreasing operator on E into E. Similarly, we have by (A_4),

$$\begin{aligned} \mathcal{B}x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \\ &\geq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s))\, ds \\ &= \mathcal{B}y(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{B}$ is nondecreasing operator on E into itself.

Step II: *$\mathcal{A}$ is partially bounded and partially $\mathcal{D}$-contraction on E.*

Let $x \in E$ be arbitrary. Then by (A_1),

$$|\mathcal{A}x(t)| \leq |g(x)| + |x_0| \leq M_g + |x_0|,$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{A}x\| \leq M_g + |x_0|$ and so, $\mathcal{A}$ is bounded. This further implies that $\mathcal{A}$ is partially bounded on E.

Next, let $x, y \in E$ be such that $x \geq y$. Then,

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = |g(x) - g(y)| \leq L_g \|x - y\|.$$

Then, $\|\mathcal{A}x - \mathcal{A}y\| \leq L_g \|x - y\|$ for all $x, y \in E$ with $x \geq y$ and hence $\mathcal{A}$ is a partially $\mathcal{D}$-contraction on E which further implies that $\mathcal{A}$ is a partially continuous on E.

Step III: *$\mathcal{B}$ is a partially continuous operator on E.*

Let $\{x_n\}$ be a sequence of points of a chain C in E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \left[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x_n(s))\, ds \right] \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[\lim_{n\to\infty} f(s, x_n(s))\, ds \right] ds \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \\ &= \mathcal{B}x(t), \end{aligned}$$

for all $t \in J$. This shows that $\{\mathcal{B}x_n\}$ converges to $\mathcal{B}x$ pointwise on J.

Next, we will show that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then

$$\begin{aligned} |\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| &\leq \frac{1}{\Gamma(q)} \left| \int_0^{t_2} |(t_2-s)^{q-1} - (t_1-s)^{q-1}|\, |f(s, x_n(s))|\, ds \right| \\ &\quad + \frac{1}{\Gamma(q)} \left| \int_{t_1}^{t_2} (t_1-s)^{q-1} |f(s, x_n(s))|\, ds \right| \\ &\leq \frac{M_f}{\Gamma(q+1)} (t_2^q - t_1^q). \end{aligned}$$

Consequently,

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniformly and hence $\mathcal{B}$ is a partially continuous on E.

Step IV: *$\mathcal{B}$ is a partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $x \in C$ be arbitrary. Then,

$$\begin{aligned} |\mathcal{B}x(t)| &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x(s))|\, ds \\ &\leq \frac{M_f}{\Gamma(q+1)} \\ &= r, \end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{B}x\| \leq r$ for all $x \in C$. Hence $\mathcal{B}(C)$ is a uniformly bounded subset of E. Next, we will show that $\mathcal{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ be arbitrary with $t_1 < t_2$. Then,

$$\begin{aligned} |\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| &\leq \frac{1}{\Gamma(q)} \left| \int_0^{t_2} |(t_2-s)^{q-1} - (t_1-s)^{q-1}|\, |f(s, x(s))|\, ds \right| \\ &\quad + \frac{1}{\Gamma(q)} \left| \int_{t_1}^{t_2} (t_1-s)^{q-1} |f(s, x(s))|\, ds \right| \\ &\leq \frac{M_f}{\Gamma(q+1)} (t_2^q - t_1^q). \end{aligned}$$

Thus we have that

$$|\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1$$

uniformly for all $x \in C$. This shows that $\mathcal{B}(C)$ is an equicontinuous set in E. Hence $\mathcal{B}(C)$ is compact subset of E and consequently $\mathcal{B}$ is a partially compact operator on E into itself.

Step V: *u satisfies the operator inequality $u \leq \mathcal{A}u + \mathcal{B}u$.*

Since the hypothesis (A_6) holds, u is a lower solution of (1) defined on J. Then,

$$^cD^q u(t) \leq f(t, u(t)), \tag{12}$$

satisfying,

$$u(0) \leq x_0 + g(u), \tag{13}$$

for all $t \in J$.

Integrating (12) from 0 to t, we obtain

$$u(t) \leq x_0 + g(u) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))\, ds, \tag{14}$$

for all $t \in J$. This show that u is a lower solution of the operator inequality $u \leq \mathcal{A}u + \mathcal{B}u$.

Thus, the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.8 in view of Remark 2.9 and we apply it to conclude that the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution defined on J. Consequently the integral equation and the problem (1) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by (7) converges monotonically to x^*. This completes the proof. $\square$

Next, we prove a uniqueness theorem for the problem (1) under weak partial Lipschitz condition on the nonlinearity f.

Theorem 3.5. *Assume that hypotheses* (A_2)*,* (A_5) *and* (A_6) *hold. Then the problem (1) has a unique solution* x^* *defined on* J, *provided*

$$L_g + \frac{L_f}{\Gamma(q+1)} < 1,$$

and the sequence $\{x_n\}$ *of successive approximations defined by* (7) *converges monotonically to* x^**.*

Proof. Set $E = C(J, \mathbb{R})$. Clearly, E is a lattice w.r.t. the order relation $\leq$ and so the lower and the upper bound for every pair of elements in E exist. Define the operator $\mathcal{T}$ by

$$\mathcal{T}x(t) = x_0 + g(x) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, x(s))\, ds,\ t \in J. \tag{15}$$

Then, the problem (1) is equivalent to the operator equation

$$\mathcal{T}x(t) = x(t),\ t \in J. \tag{16}$$

We shall show that $\mathcal{T}$ satisfies all the conditions of Theorem 2.7 in E. Clearly, $\mathcal{T}$ is a nondecreasing operator on E into itself. We shall simply show that the operator $\mathcal{T}$ is a partially $\mathcal{D}$-contraction on E. Let $x, y \in E$ be any two elements such that $x \geq y$. Then, by hypothesis (A_5),

$$\begin{aligned}
|\mathcal{T}x(t) - \mathcal{T}y(t)| &\leq L_g \|x-y\| \\
&\quad + \left| \frac{1}{\Gamma q} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s))\, ds \right| \\
&\leq L_g \|x-y\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x(s)) - f(s, x(s))|\, ds \\
&\leq L_g \|x-y\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} L_f |x(s) - y(s)|\, ds \\
&\leq L_g \|x-y\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} L_f \|x-y\|\, ds \\
&\leq \left(L_g + \frac{L_f}{\Gamma(q+1)} \right) \|x-y\|, \\
&= \psi(\|x-y\|),
\end{aligned}$$

for all $t \in J$, where $\psi(r) = \left(L_g + \dfrac{L_f}{\Gamma(q+1)} \right) r < r$, $r > 0$.

Taking the supremum over t, we obtain

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x-y\|)$$

for all $x, y \in E$, $x \geq y$. As a result $\mathcal{T}$ is a partially nonlinear $\mathcal{D}$-contraction on E. Furthermore, it can be shown as in the proof of Theorem 3.4 that the function u given in hypothesis (A_6) satisfies the operator inequality $u \leq \mathcal{T}u$ on J. Now a direct application of Theorem 2.7 yields that the problem (1) has a unique solution x^* and the sequence $\{x_n\}$ of successive approximations defined by (7) converges monotonically to x^*. □

Remark 3.6. The conclusion of Theorems 3.4 and 3.5 also remains true if we replace the hypothesis (A_6) with the following one:

(A_6') The problem (1) has an upper solution $v \in C(J, \mathbb{R})$.

Example 3.7. *Given a closed and bounded interval $J = [0, 1]$, consider the problem,*

$$\left.\begin{aligned} {}^cD^{1/2}x(t) &= \tanh x(t),\ t \in J, \\ x(0) &= \sum_{i=1}^{k} \lambda_i \arctan x(t_i), \end{aligned}\right\} \tag{17}$$

where $t_1, \ldots, t_k$ are given real numbers such that $0 < t_1 < t_2 < \ldots < t_k < 1$ and $\lambda_i \in \mathbb{R}$, $\lambda_i \geq 0$, $i = 1, \ldots, k$.

Set $f(t, x) = \tanh x$ for all $t \in J$ and $x \in \mathbb{R}$, and $g(x) = \displaystyle\sum_{i=1}^{k} \lambda_i \arctan x(t_i)$ for all $x \in E$, where $0 < \sum\limits_{i=1}^{k} \lambda_i < 1$.

Clearly, the function g is bounded on E with bound $M_g = \dfrac{\pi}{2}$ and so hypothesis (A_1) is satisfied. Furthermore, the function g is nondecreasing and a partial $\mathcal{D}$-contraction on E. To see this, let $x, y \in E$ be such that $x \geq y$. Then,

$$\begin{aligned} 0 &\leq g(x) - g(y) \\ &= \sum_{i=1}^{k} \lambda_i (\arctan x(t_i) - \arctan y(t_i)) \\ &\leq \left(\sum_{i=1}^{k} \frac{\lambda_i}{1 + \xi_i^2} \right) (x(t_i) - y(t_i)) \quad (\text{because } x(t_i) > \xi_i > y(t_i)) \\ &\leq \left(\sum_{i=1}^{k} \lambda_i \right) (x(t_i) - y(t_i)) \\ &\leq L_g \left\| x - y \right\| \end{aligned} \tag{18}$$

where, $L_g = \sum\limits_{i=1}^{k} \lambda_i < 1$. Therefore, g is satisfies the hypothesis (A_2) on E.

Clearly, the functions f is continuous, nondecreasing, and bounded on $J \times \mathbb{R}$ with bound $M_f = 1$ and so hypotheses (A_3) and (A_4) are satisfied. Finally, a solution u of the problem

$$\left.\begin{aligned} {}^cD^{1/2}x(t) &= -1,\ t \in J, \\ x(0) &= -\frac{\pi}{2}, \end{aligned}\right\} \tag{19}$$

is a lower solution to the problem (17) on J. Solving problem (19) for u, we obtain

$$u(t) = -\frac{\pi}{2} + \frac{t^{1/2}}{\sqrt{\pi}}$$

for $t \in J$. Thus, hypothesis (A_6) of Theorem 3.4 is satisfied. Now we apply Theorem 3.4 and conclude that the problem (19) has a solution and the sequence $\{x_n\}$ of successive approximations defined by

$$x_{n+1}(t) = \sum_{i=1}^{k} \lambda_i \arctan x_n(t_i) + \frac{1}{\Gamma(1/2)} \int_0^t (t-s)^{-1/2} \tanh x_n(s)\, ds \tag{20}$$

for all $t \in J$, where $x_1 = u$, converges monotonically to x^*.

References

[1] B. Ahmad, S.K. Ntouyas, A four-point integral boundary value problem for fractional differential equations of arbitrary order, *Electronic J. Qualitative Theory of Diff. Equ. No. 22 (2011), pp. 1-15.*

[2] B. Ahmad, S.K. Ntouyas, A. Alsaedi, H. Al-Hutami, Nonlinear q-fractional differential equations with nonlocal and sub-strip type boundary conditions, *Electronic J. Qual. Theory Differ. Equ. 2014, No. 26, 1-12.*

[3] A. Bitsadze, A. Samarskii, On some simple generalizations of linear elliptic boundary problems, *Russian Acad. Sci. Dokl. Math.* **10** (1969), 398-400.

[4] O. Bolojan-Nica, G. Infante, R. Precup, Existence results for systems with coupled nonlocal initial conditions, *Nonlinear Anal.*, **94** (2014), 231-242.

[5] L. Byszewski, V.Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Appl. Anal.* **40** (1991) 1119.

[6] L. Byszewski, Theorems about existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.* **162** (1991) 494505.

[7] L. Byszewski, Existence and uniqueness of mild and classical solutions of semilinear functional-differential evolution nonlocal Cauchy problem, *Selected problems of mathematics, 50th Anniv. Cracow Univ. Technol. Anniv., Issue 6, Cracow Univ. Technol., Krakow, 1995, pp. 2533.*

[8] B.C. Dhage, Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations, *Differ. Equ. Appl.* **2** (2010), 465–486.

[9] B.C. Dhage, Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations, *Differ. Equ. Appl.* **5** (2013), 155-184.

[10] B.C. Dhage, Partially condensing mappings in partially ordered normed linear spaces and applications to functional integral equations, *Tamkang J. Math.* **45**(4) (2014), 397-426.

[11] B.C. Dhage, Nonlinear $\mathcal{D}$-set-contraction mappings in partially ordered normed linear spaces and applications to functional hybrid integral equations, *Malaya J. Mat.* **3**(1)(2015), 62-85.

[12] B.C. Dhage, Operator theoretic techniques in the theory of nonlinear hybrid differential equations, *Nonlinear Anal. Forum* **20** (2015), 15-31.

[13] B.C. Dhage, A new monotone iteration principle in the theory of nonlinear fractional differential equations, *Intern. J. Anal. Appl.* **9**(2) (2015), to appear.

[14] B.C. Dhage, S.B. Dhage, Approximating solutions of nonlinear first order ordinary differential equations, Global J. Math. Sci. **3** (2014), to appear.

[15] B. C. Dhage, S. B. Dhage, *Approximating solutions of nonlinear pbvps of hybrid differential equations via hybrid fixed point theory*, Indian J. Math. **57**(1) (2015), 103-119.

[16] B.C. Dhage, S.B. Dhage, Approximating positive solutions of nonlinear first order ordinary quadratic differential equations, *Cogent Mathematics* (2015), 2: 1023671.

[17] B.C. Dhage, S.B. Dhage, Approximating positive solutions of PBVPs of nonlinear first order ordinary quadratic differential equations, *Appl. Math. Lett.* **46** (2015), 133-142.

[18] B.C. Dhage, S.B. Dhage, S.K. Ntouyas, Approximating solutions of nonlinear hybrid differential equations, *Appl. Math. Lett.* **34** (2014), 76-80.

[19] C. Goodrich, Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions, *Comput. Math. Appl.* **61** (2011) 191-202.

[20] S. Heikkilä, V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker inc., New York 1994.

[21] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[22] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, *Theory of Fractional Differential Systems*, Cambridge Scientific Pblishers, Cambridge, U. K. 2009.

[23] S.K. Ntouyas, Nonlocal Initial and Boundary Value Problems: A survey, *Handbook on Differential Equations: Ordinary Differential Equations*, Edited by A. Canada, P. Drabek and A. Fonda, Elsevier Science B. V., 2005, 459-555.

[24] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[25] J.R.L. Webb, G. Infante, Non-local boundary value problems of arbitrary order, *J. London Math. Soc.* **79** (2009), 238-258.

Bapurao C. Dhage
Kasubai, Gurukul Colony, Ahmedpur-413 515, Dist: Latur, Maharashtra, India
E-mail address: `bcdhage@gmail.com`

Shyam B. Dhage
Kasubai, Gurukul Colony, Ahmedpur-413 515, Dist: Latur, Maharashtra, India
E-mail address: `sbdhage4791@gmail.com`

Sotiris K. Ntouyas
Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece; Nonlinear Analysis and Applied Mathematics-Research Group (NAAM), King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: `sntouyas@uoi.gr`