Journal of Fractional Calculus and Applications,
Vol. 7(1) Jan. 2016, pp. 1-11.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

SOLUTION OF A PARABOLIC WEAKLY-SINGULAR PARTIAL INTEGRO-DIFFERENTIAL EQUATION WITH MULTI-POINT NONLOCAL BOUNDARY CONDITIONS

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ABSTRACT. We present a finite difference solution for a parabolic weakly singular partial integro differential equation with multi-point nonlocal boundary conditions. The singularity in the considered equation is removed using Taylor's approximation. The stability analysis for the implicit and explicit finite difference schemes are studied. Then, the effect of the parameter of multi-point nonlocal boundary conditions on the eigenvalues of the transition matrix is studied via spectral analysis. We conclude this paper with the results of a numerical experiment to show the efficiency of the technique.

1. Introduction

A partial integro-differential equation (PIDE) is obtained when the unknown function appears with its derivatives and either the unknown function or its derivatives, or both, appear under the sign of integration. There are some different forms of PIDEs and we concentrate on the parabolic type. This class of equations is applied in compression of poro-viscoelastic media [1], reaction diffusion problems [2] and nuclear reactor dynamics [3, 4, 5]. The PIDEs are investigated by some numerical methods [6, 7, 8, 9, 10, 11, 12, 13]. In this chapter we consider a class of partial PIDEs with singular kernel having the form of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \int_0^t \frac{u(x,s)}{(t-s)^\alpha} ds + f(x,t), \qquad 0 < x < 1, \quad t > 0 \tag{1}$$

subject to a multipoint nonlocal boundary conditions of the form

$$u(0,t) = \sum_{i=1}^{m-1} \gamma_i u(x_i,t) + \mu_1(t), \qquad t > 0 \tag{2}$$

and a Dirichlet condition of the form

$$u(1,t) = \mu_2(t), \qquad \alpha \in (0,1), \qquad x_i = \frac{i}{m}, \quad t > 0 \tag{3}$$

2010 *Mathematics Subject Classification.* 65N25; 65N06; 35K20.

Key words and phrases. Partial weakly-singular integro-differential equation; Multi-point nonlocal boundary conditions; Eigenvalue and eigenvector problem; Finite difference method.
Submitted May 2, 2015.

and with initial condition

$$u(x,0) = u_0(x), \qquad 0 < x < 1. \tag{4}$$

It can be seen that in (1) because of the possible singularities of the kernel which induce sharp transitions in the solution, developing accurate numerical methods for integro-differential equations is still a challenge. This is particularly interesting in viscoelasticity, because it might smooth the solution when the boundary data is discontinuous [14]. Numerical investigations have been given by several authors [15, 16, 17, 18, 19], but most of them considered smooth integral kernels only.

Equations with nonlocal conditions gained a lot of interest since Cannon [20] and Batten[21] presented this concepts in 1963, independently. The interest in this type of problems increased due to the emergence of applications of differential equations with nonlocal conditions such as in biotechnology[22] and mathematical biology [23]. In particular, parabolic differential equations subject to nolocal conditions emerge in a wide variety of technical, physical, and biological problems. Many aspects, related to the applications of parabolic models with non-local boundary conditions, finite difference schemes and other algorithms for their numerical solution, have been presented in the review paper [24]. Models with nonlocal boundary conditions include elliptic equations [25, 26, 27] hyperbolic equations [28, 29], difference equations [30, 31]. In this article, we study the effect of multi-point nonlocal boundary conditions on the numerical solution of PIDE with linear weakly-singular kernel using finite difference method (FDM). The suggested numerical scheme starts by removing the singularity using Taylor's approximation. The second-order partial singular integro-differential equations is transformed into a partial differential equation with variable coefficients which is then discretized by FDM. Secondly, we deduce the condition that should be imposed on the FDM parameters to guarantee the stability of the method. Then, we adopt the proposed analysis in [32] to study the eigenvalue problem of the transition matrix. Finally, a numerical experiment is presented to illustrate how the stability of the solution is affected by the values of the parameters of the problem and the parameters chosen for the difference scheme.

2. Taylor approximation

We propose an approximate solution for solving weakly-singular parabolic partial integro-differential equations. The singularity of the kernel of weakly- singular parabolic partial integro-differential equation at $s = t$ by is removed by using Taylor's approximation. Firstly, we reformulation the equation (1) in the following

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \int_0^t \frac{u(x,s) - u(x,t) + u(x,t)}{(t-s)^\alpha} ds + f(x,t). \tag{5}$$

Then

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(x,t)\frac{t^{1-\alpha}}{1-\alpha} - \int_0^t (t-s)^{1-\alpha} \frac{u(x,s) - u(x,t)}{s-t} ds + f(x,t) \tag{6}$$

and by a first order Taylor 's expansion of $u(x,s)$ about $s = t$, we can write

$$u(x,s) = u(x,t) + (s-t)\frac{\partial u(x,t)}{\partial t} \tag{7}$$

after substituting (7) into (6), then problem (1) is approximated by

$$\left(1+\frac{t^{2-\alpha}}{2-\alpha}\right)\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{t^{1-\alpha}}{1-\alpha}u(x,t)+f(x,t). \quad (8)$$

3. The difference scheme

We begin this section by developing the transition matrix of the difference scheme for the proposed approximated model (8) for the case $m=4$. First, a uniform discrete grid is defined on the rectangular $\Omega=(0,1)\times[0,T)$ by the spatial discretization $x_i=ih, i=1,2,\cdots,N-1, h=1/N$ and the temporal discretization $t_j=j\tau, j=0,1,\cdots,M-1,\tau=T/M$. The spatial discretization step h should chosen such that the points of condition (2) all lie on the grid. The solution at the grid points is denoted by $u_i^j(x_i,t_j)$. Here, the multi-point nonlocal boundary conditions and initial condition (2)-(4) take the form

$$\left(1+\frac{t^{2-\alpha}}{2-\alpha}\right)\frac{u_i^{j+1}-u_i^j}{k}=\sigma\left(\Lambda u_i^{j+1}+f_i^j\right)+(1-\sigma)\left(\Lambda u_i^j+f_i^j\right)+\sigma\frac{t_j^{1-\alpha}}{1-\alpha}u_i^{j+1}$$
$$+(1-\sigma)\frac{t_j^{1-\alpha}}{1-\alpha}u_i^j, \quad (9)$$

$$\sigma u_0^{j+1}+(1-\sigma)u_0^j=\sigma\left(\gamma_1u_{N/4}^{j+1}+\gamma_2u_{N/2}^{j+1}+\gamma_3u_{3N/4}^{j+1}\right)+$$
$$(1-\sigma)\left(\gamma_1u_{N/4}^{j}+\gamma_2u_{N/2}^{j}+\gamma_3u_{3N/4}^{j}\right)+\sigma\mu_1^{j+1}+$$
$$(1-\sigma)\mu_1^j+\sigma u_N^{j+1}+(1-\sigma)u_N^j=\sigma\mu_2^{j+1}+(1-\sigma)\mu_2^j \quad (10)$$

$$\sigma u_N^{j+1}+(1-\sigma)u_N^j=\sigma\mu_2^{j+1}+(1-\sigma)\mu_2^j \quad (11)$$

$$u_i^0=u_0(ih), \quad (12)$$

where the discrete operator Λ is defined by $\Lambda u_i^j=\frac{u_{i-1}^j-2u_i^j+u_{i+1}^j}{h^2}$ and $0\le\sigma\le 1$. The cases $\sigma=1$ and $\sigma=0$ correspond to the implicit and explicit finite difference scheme, respectively. We consider the analysis of the difference scheme for the implicit case $\sigma=1$ and a similar analysis holds for the explicit case. The expressions (9)-(10) for u_0^{j+1} and u_N^{j+1} are substituted into system (9). Then, difference scheme for the case $\sigma=1$ takes the form

$$\left(1+\frac{t_{j+1}^{2-\alpha}}{2-\alpha}\right)u_i^{j+1}=\left(1+\frac{t_{j+1}^{2-\alpha}}{2-\alpha}\right)u_i^j+$$
$$\begin{cases}\frac{\tau}{h^2}\left(\gamma_1u_{N/4}^{j+1}+\gamma_2u_{N/2}^{j+1}+\gamma_3u_{3N/4}^{j+1}-\left(2-\frac{t_{j+1}^{1-\alpha}}{1-\alpha}h^2\right)u_1^{j+1}+u_2^{j+1}\right) & \\ \qquad +\tau\left(f_1^{j+1}+\frac{1}{h^2}\mu_1^{j+1}\right), & i=1;\\ \frac{\tau}{h^2}\left(u_{i-1}^{j+1}-\left(2-\frac{t_{j+1}^{1-\alpha}}{1-\alpha}h^2\right)u_i^{j+1}+u_{i+1}^{j+1}\right)+\tau f_i^{j+1}, & i=2,3,\cdots,N-2;\\ \frac{\tau}{h^2}\left(u_{i+1}^{N-2}-\left(2-\frac{t_{j+1}^{1-\alpha}}{1-\alpha}h^2\right)u_{N-1}^{j+1}\right)+\tau\left(f_{N-1}^{j+1}+\frac{1}{h^2}\mu_2^{j+1}\right), & i=N-1.\end{cases} \quad (13)$$

We define the square matrix A of order $N-1$ by

$$A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & -\gamma_1 & 0 & -\gamma_2 & 0 & \cdots & -\gamma_3 & 0 & 2 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \tag{14}$$

where γ_1 , γ_2 and γ_1 are positioned in the matrix entries that correspond to $u_{1/4}$, $u_{1/2}$, $u_{3/4}$, respectively. Then, the implicit finite difference system is written as

$$\left(\left(1+\frac{((j+1)\tau)^{2-\alpha}}{2-\alpha}-\frac{((j+1)\tau)^{1-\alpha}\,\tau}{1-\alpha}\right)E+\tau A\right)U^{j+1} = \left(1+\frac{((j+1)\tau)^{2-\alpha}}{2-\alpha}\right)U^{j}+\tau F^{j+1}, \tag{15}$$

where U^{j+1} and U^{j} are vectors of the solution at time t_j and t_{j+1} , respectively. The vector F^{j+1} contains the remaining terms of the system and E is the identity matrix. All the vectors and matrices are of order $N-1$. Similarly, for the case explicit where $\sigma = 0$, the following difference scheme is obtained

$$\left(1+\frac{t_j^{2-\alpha}}{2-\alpha}\right)u_i^j = \left(1+\frac{t_j^{2-\alpha}}{2-\alpha}\right)u_i^j + \begin{cases} \frac{\tau}{h^2}\left(\gamma_1 u_{N/4}^j+\gamma_2 u_{N/2}^j+\gamma_3 u_{3N/4}^j-\left(2-\frac{t_j^{1-\alpha}}{1-\alpha}h^2\right)u_1^j+u_2^j\right)+\tau\left(f_1^j+\frac{1}{h^2}\mu_1^j\right), & i=1; \\ \frac{\tau}{h^2}\left(u_{i-1}^j-\left(2-\frac{t_j^{1-\alpha}}{1-\alpha}h^2\right)u_i^j+u_{i+1}^j\right)+\tau f_i^j, & i=2,3,\cdots,N-2; \\ \frac{\tau}{h^2}\left(u_{i+1}^{N-2}-\left(2-\frac{t_j^{1-\alpha}}{1-\alpha}h^2\right)u_{N-1}^j\right)+\tau\left(f_{N-1}^j+\frac{1}{h^2}\mu_2^j\right), & i=N-1. \end{cases} \tag{16}$$

The matrix form of system (16) is

$$U^{j+1} = \frac{1}{\left(1+\frac{t_j^{2-\alpha}}{2-\alpha}\right)}\left(\left(1+\frac{t_j^{2-\alpha}}{2-\alpha}+\frac{t_j^{1-\alpha}}{1-\alpha}\tau\right)E-\tau A\right)U^j+\tau F^j. \tag{17}$$

It is known that for a difference scheme of the form

$$U^{j+1} = SU^j + \bar{F}^j$$

a sufficient stability condition is given by [32, 33, 34, 35, 36, 37, 37, 38]

$$\|S\| \le 1 + c_0\tau, \tag{18}$$

where c_0 is a constant independent of both τ and h . In the case of a symmetric matrix S , we can define

$$\|S\| = \rho(S) = \max_{1\le i\le N-1}|\lambda_i(S)|$$

where $\lambda_i(S)$ are the eigenvalues of S , and $\rho(S)$ is the spectral radius of S . Thus, the stability of the difference scheme is defined by the condition $\rho(S) \le 1$. In the case of nonlocal boundary conditions, S is a nonsymmetric matrix. The sufficient stability condition (18) is usually replaced by the necessary von Neumann condition given by

$$\lambda_i(S) \leq 1 + c_1\tau \tag{19}$$

where c_1 is a constant independent of both τ and h . In this case, the inequality $\rho(S) \leq 1$ is a necessary and sufficient condition to define a norm $\|S\|_*$ of the nonsymmetric matrix S such that $\|S\|_* \leq 1$ as in [38]. ?If the necessary von Neumann condition (19) is true, then it is always possible to define norms so that the difference scheme is stable. Whereas if condition (19) does not hold, then it is practically impossible to define the norms of vectors or matrices so that the difference scheme is stable.

Theorem 3.1. *If all eigenvalues of matrix are real and positive, then difference scheme (15) is stable if*

$$\tau_j < \frac{((1-\alpha)\lambda_i(A))^{\frac{1}{1-\alpha}}}{j+1} \tag{20}$$

and for the explicit difference scheme (17), the difference scheme is stable if

$$\tau_j < \min\left\{\frac{((1-\alpha)\lambda_i(A))^{\frac{1}{1-\alpha}}}{j}, \frac{2}{\lambda_i(A)}\right\}. \tag{21}$$

proof. For the implicit scheme, we have

$$|\lambda_i(S)| = \left|\left(1+\frac{((j+1)\tau)^{2-\alpha}}{2-\alpha}\right)\lambda_i\left(\left(\left(1+\frac{((j+1)\tau)^{2-\alpha}}{2-\alpha}-\frac{((j+1)\tau)^{1-\alpha}k}{1-\alpha}\right)E+\tau A\right)^{-1}\right|$$

$$= \frac{\left(1+\frac{((j+1)\tau)^{2-\alpha}}{2-\alpha}\right)}{\left(\left(1+\frac{((j+1)\tau)^{2-\alpha}}{2-\alpha}-\frac{((j+1)\tau)^{1-\alpha}\tau}{1-\alpha}\right)E+\tau\lambda_i(A)\right)}$$

which shows that $\rho(S) < 1$ for all τ is satisfied if

$$\frac{((j+1)\tau)^{1-\alpha}}{1-\alpha}\tau - \tau\lambda_i(A) < 2\left(1+\frac{((j+1)\tau)^{2-\alpha}}{2-\alpha}\right)$$

which yield condition (20). For the explicit scheme, we have

$$|\lambda_i(S)| = \left|\frac{\lambda_i}{\left(1+\frac{t_j^{2-\alpha}}{2-\alpha}\right)}\left(\left(1+\frac{t_j^{2-\alpha}}{2-\alpha}+\frac{t_j^{1-\alpha}}{1-\alpha}\tau\right)E-\tau A\right)\right|.$$

Thus, $\rho(S) < 1$ if

$$\left|\frac{1}{\left(1+\frac{t_j^{2-\alpha}}{2-\alpha}\right)}\left(\left(1+\frac{t_j^{2-\alpha}}{2-\alpha}+\frac{t_j^{1-\alpha}}{1-\alpha}\tau\right)-\tau\lambda_i(A)\right)\right| < 1.$$

So, $\tau_i < \frac{((1-\alpha)\lambda_i(A))^{\frac{1}{1-\alpha}}}{j}$ and $\tau_i < \frac{2}{\lambda_i(A)}$ Then, a sufficient condition to ensure both conditions are satisfied is given by (21). Evidently, as $\alpha \in (0,1)$ if $(1-\alpha)\lambda_i(A) > 1$

for all positive eigenvalues, then the term $\tau_i < \frac{((1-\alpha)\lambda_i(A))^{\frac{1}{1-\alpha}}}{j}$ is greater than one and any time step less than one satisfies this condition.

Lemma 3.1. The eigenvalue problem

$$AU = \lambda U, \tag{22}$$

For the matrix A is equivalent to the difference eigenvalue problem

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \lambda u_i = 0, \qquad i = 1, 2, \cdots N-1, \tag{23}$$

$$u_0 = \gamma_1 u_{N/4} + \gamma_2 u_{N/2} + \gamma_3 u_{3N/4}, \tag{24}$$

$$u_N = 0, \tag{25}$$

Lemma 3.2. The necessary and sufficient condition for the difference problem (25)-(27) to have zero eigenvalue is

$$\frac{3}{4}\gamma_1 + \frac{1}{2}\gamma_2 + \frac{1}{4}\gamma_3 = 1. \tag{26}$$

proof. For $\lambda = 0$, the general solution of the difference equations (23) is

$$u = c_1 ih + c_2, \qquad i = 0, 1, \cdots, N, \tag{27}$$

where c_1 and c_2 are two arbitrary constants. Applying conditions (24) and (25), we get

$$\left(-\frac{\gamma_1}{4} - \frac{\gamma_2}{2} - \frac{3\gamma_1}{4}\right) c_1 + (1 - \gamma_1 - \gamma_2 - \gamma_3)\, c_2 = 0,$$
$$c_1 + c_2 = 0. \tag{28}$$

System (28) has a nontrivial solution (c_1, c_2) if the determinant of the system equals zero

$$\begin{vmatrix} -\frac{\gamma_1}{4} - \frac{\gamma_2}{2} - \frac{3\gamma_1}{4} & 1 - \gamma_1 - \gamma_2 - \gamma_3 \\ 1 & 1 \end{vmatrix}$$

The lemma is proved.

Lemma 3.3. The difference eigenvalue problem (23)-(25) has a negative eigenvalue, provided that it exists, given by $\lambda = -\frac{4}{h^2}\sinh^2(\frac{\omega h}{2})$, where ω is the positive parameter that satisfies the relation between $\gamma_1, \gamma_2, \gamma_3$ and ω in the following case

$$\tanh\omega = \left(\gamma_1 \sinh\frac{\omega}{4} + \gamma_2 \sinh\frac{\omega}{2} + \gamma_3 \sinh\frac{3\omega}{4}\right) - \tanh\omega\left(\gamma_1 \cosh\frac{\omega}{4} + \gamma_2 \cosh\frac{\omega}{2} + \gamma_3 \cosh\frac{3\omega}{4}\right)$$

The corresponding eigenvector is given by $u_i = c_1 \cosh(\omega ih) + c_2 \sinh(\omega ih)$, where c_1 and c_2 are arbitrary constants.

proof. If $\lambda < 0$,we have $1 - \frac{\lambda h^2}{2} > 1$. Denote $\cosh(\omega h) = 1 - \frac{\lambda h^2}{2}$ and we write difference equation (23) in the form

$$u_{i-1} - 2\cosh(\omega h) u_i + u_{i+1} = 0$$

The general solution of the latter equation is given by

$$u_i = c_1 \cosh(\omega ih) + c_2 \sinh(\omega ih).$$

By substituting this solution into nonlocal conditions (24) and (25), we obtain the following system of two linear algebraic equations with unknowns c_1 and c_2

$$\left(1-\gamma_1\cosh\frac{\omega}{4}-\gamma_2\cosh\frac{\omega}{2}-\gamma_3\cosh\frac{3\omega}{4}\right)c_1-\left(\gamma_1\sinh\frac{\omega}{4}+\gamma_2\sinh\frac{\omega}{2}+\gamma_3\sinh\frac{3\omega}{4}\right)c_2=0, \tag{29}$$

$$c_1=-(\tanh\omega)c_2. \tag{30}$$

By substitution (30) into (29), the lemma is proved.

Lemma 3.4. The difference eigenvalue problem (23)-(25) has a positive eigenvalue, provided that it exists, given by $\lambda=\frac{4}{h^2}\sin^2(\frac{\omega h}{2})$, where ω is the positive parameter that satisfies the relation between $\gamma_1,\gamma_2,\gamma_3$ and ω in the following case

$$\tan\omega=\tan\omega\left(\gamma_1\cos\frac{\omega}{4}+\gamma_2\cos\frac{\omega}{2}+\gamma_3\cos\frac{3\omega}{4}\right)-\left(\gamma_1\sin\frac{\omega}{4}+\gamma_2\sin\frac{\omega}{2}+\gamma_3\sin\frac{3\omega}{4}\right)$$

The corresponding eigenvector is given by $u_i=c_1\cos(\omega ih)+c_2\sin(\omega ih)$, where c_1 and c_2 are arbitrary constants.

proof. If $\lambda>0$,we have $1-\frac{\lambda h^2}{2}<1$. Denote $\cos(\omega h)=1-\frac{\lambda h^2}{2}$ and we write difference equation (23) in the form

$$u_{i-1}-2\cos(\omega h)u_i+u_{i+1}=0$$

The general solution of the latter equation is given by

$$u_i=c_1\cos(\omega ih)+c_2\sin(\omega ih)$$

. By substituting this solution into nonlocal conditions (24) and (25), we obtain the following system of two linear algebraic equations with unknowns c_1 and c_2

$$\left(1-\gamma_1\cos\frac{\omega}{4}-\gamma_2\cos\frac{\omega}{2}-\gamma_3\cos\frac{3\omega}{4}\right)c_1-\left(\gamma_1\sin\frac{\omega}{4}+\gamma_2\sin\frac{\omega}{2}+\gamma_3\sin\frac{3\omega}{4}\right)c_2=0, \tag{31}$$

$$c_1=-(\tan\omega)c_2. \tag{32}$$

By substitution (32) into (31), the lemma is proved.

4. Numerical experiment and discussion

In this section, we present the results of a numerical test example to illustrate the solution stability for different values of the boundary condition parameters $\gamma_1,\gamma_2,\gamma_3$ and the difference scheme parameters h and τ. In this example, the problem (1)-(4) is considered with the exact solution given by

$$u(x,t)=x^2t$$

with

$$f(x,t)=x^2-2t-\frac{t^{2-\alpha}x^2}{\alpha^2-3\alpha+2}$$

,

$$\mu_1(t)=-t\left(\left(\frac{1}{4}\right)^2\gamma_1+\left(\frac{1}{2}\right)^2\gamma_2+\left(\frac{3}{4}\right)^2\gamma_3\right),\qquad \mu_1(t)=t,\qquad \alpha\in(0,1)\text{and}u(x,0)=0.$$

To estimate the accuracy of the numerical solution, we calculated the maximum absolute relative error $\|\varepsilon\|=\frac{\max|u_{ex}-u_{app}|}{u_{ex}}$ over all the spatial nodes and for final time $T=1$ where u_{ex} and u_{app} denote the exact and approximate solution of the problem, respectively. In all figures, we adopt the logarithmic scale for the error $\|\varepsilon\|$.

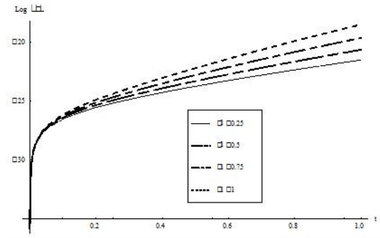

FIGURE 1. Absolute relative error at different values of γ_1 with $h = \frac{1}{12}, \alpha = 0.9, \tau = 0.0001$ and $\gamma_2 = \gamma_3 = 1$ for the implicit case.

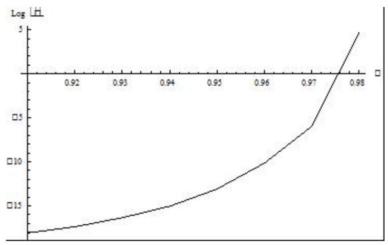

FIGURE 2. Effect of different values of α on the absolute relative error at $\tau = 0.0001, h = \frac{1}{12}$ and $\gamma_1 = \gamma_2 = \gamma_3 = 1$ for implicit scheme.

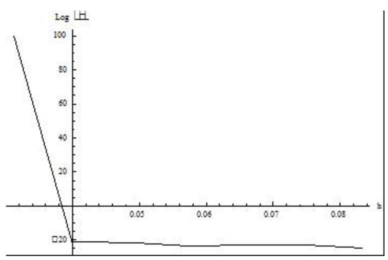

FIGURE 3. Effect of different values of h on the absolute relative error at $\tau = 0.0001, \alpha = 0.9$ and $\gamma_1 = \gamma_2 = \gamma_3 = 1$ for explicit case.

The following figures illustrate the effect of the model and the difference scheme parameters on the error. Also, to avoid repetition, when the behavior of the error is the same in both implicit and explicit case or for $\gamma_1, \gamma_2, \gamma_3$ and we present one figure for one case only. Although the parameters h, γ_1, γ_2 and γ_3 do not appear explicitly in the stability condition in Theorem 1, they affect the stability of the difference scheme as their values affect the values of the entries of matrix A and consequently its eigenvalues. Figure 1 shows that higher values for γ_1, γ_2 , or γ_3 yields a larger error. Figure (2) indicates that as α approaches one, the singularity in the integral term results in a blow up in the error. Figure (3) illustrates that though small spatial step h yields better results, a very small value of h yields eigenvalues that do not satisfy the stability condition of Theorem (3.1). Finally, Figure 4 shows that the time step in the explicit scheme should be chosen small enough to lie within the range of values that satisfy the conditions of Theorem (3.1)

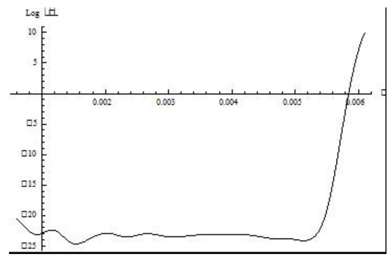

FIGURE 4. Effect of different values of τon the absolute relative error (explicit difference) at $h = \frac{1}{12}, \alpha = 0.9$ and $\gamma_1 = \gamma_2 = \gamma_3 = 1$ for explicit case.

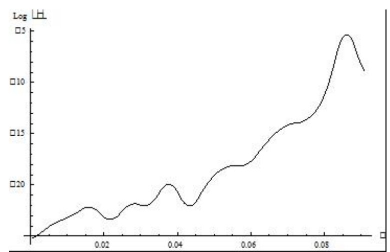

FIGURE 5. Effect of different values of τon the absolute relative error (implicit difference) at $h = \frac{1}{12}, \alpha = 0.9$ and $\gamma_1 = \gamma_2 = \gamma_3 = 1$ for explicit case.

Whereas Figure (5) asserts what we noticed that though a condition for stability is required for implicit scheme, it is satisfied for most cases.

5. Conclusion

In this paper, we proposed a parabolic weakly-singular partial integro-differential model with multi-point nonlocal integral boundary conditions and studied the stability of its finite difference solution. The weak singularity is removed by approximating the integrand by Taylor series. The resulting equation is a partial differential equation with variable coefficients. Thus, the performed analysis illustrates that stability conditions for choosing an appropriate time step are imposed in both implicit and explicit cases. Also, by relating the finite difference of the transition matrix of the considered model to the finite difference of differential equation, we proved some properties for the eigenvalues and eigenvectors problem of the proposed model.

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