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SERIES SOLUTION FOR FRACTIONAL RICCATI DIFFERENTIAL EQUATION AND ITS CONVERGENCE

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Abstract. In this paper, we present a study of the fractional order Riccati differential equation with variable coefficients. For the Caputo definition of the fractional derivative, the existence and uniqueness of the solution to this problem is proved. Then, the series solution is obtained and its convergence analysis is performed. Finally, for the Riemann-Liouville definition of fractional derivative, singularity analysis of the series solution of the considered equation is given.

1. Introduction

Fractional calculus provides a good description for modeling a physical phenomenon that depends on both the time instant and the prior time history. In recent years, fractional-order derivatives have been utilized in modeling phenomena in several fields of applied science such as engineering, physics, chemistry and hydrology [1–3]. Fractional differential equations (FDEs) are considered as generalizations of the classical differential equations of integer order.

For most nonlinear FDEs, exact solutions cannot be obtained and approximate techniques are employed to solve these equations. Several semi-analytical methods have been utilized for obtaining series solutions for FDEs. These methods include Adomian decomposition method [4], homotopy perturbation method [5], variational iterative method [6], homtopy analysis method [7] and fractional differential transform method [8].

One important FDE is the fractional Riccati differential equation (FRDE) which has different applications in engineering and applied science. The applications include random processes, optimal control, and diffusion problems [9], stochastic realization theory, optimal control, robust stabilization, network synthesis and financial mathematics [10, 11]. Approximate solutions for this equation have been obtained via different methods including the semi-analytic methods [12–16] and numerical methods [17].

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In this work, we consider the FRDE in the form

$$D^{\alpha}w(x) = a(x) + b(x)w(x) + c(x)w^2(x), \quad x > 0, \quad 0 < \alpha \leq 1, \tag{1}$$

with initial condition $w(0) = w_0$, and the variable coefficients $a(x)$, $b(x)$ and $c(x)$ are continuous functions with bound L.

The convergence analysis of the fractional power series solutions for FRDE (1) is discussed. Also, we illustrate how to numerically estimate the radius of convergence for the series solution of FRDE (1) based on the convergence of the series solution of the corresponding integer order problem. Finally, we study singularity behavior for the series solution of FRDE (1) when the fractional derivative is defined in Riemann-Liouville sense to obtain a series solution with determined pole.

2. Preliminaries

In this section, we introduce some definitions and properties that we use through the paper. There are several operators for defining fractional order derivative. The most common ones used in researches are the Riemann-Liouville and Caputo fractional differential operators.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f(x)$ is defined as

$$I^{\alpha}f(x) = \int_0^x \frac{(x-\zeta)^{\alpha-1}}{\Gamma(\alpha)} f(\zeta)d\zeta, \alpha > 0, \qquad I^0 f(x) = f(x). \tag{2}$$

The operator I^{α} satisfies the following properties, for $\alpha, \beta \geq 0$ and $m \geq -1$,

1. $I^{\alpha}I^{\beta}f(x) = I^{\alpha+\beta}f(x)$,
2. $I^{\alpha}I^{\beta}f(x) = I^{\beta}I^{\alpha}f(x)$,
3. $I^{\alpha}x^m = \frac{\Gamma(m+1)x^{m+\alpha}}{\Gamma(m+\alpha+1)}$.

Definition 2.2. The Riemann-Liouville fractional derivative operator of order α is defined as

$$D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{f(\zeta)}{(x-\zeta)^{\alpha-n+1}} d\zeta, \qquad \alpha > 0, x > 0, \tag{3}$$

where $n-1 < \alpha \leq n, n \in \mathbb{N}$.

Definition 2.3. The Caputo fractional derivative operator of order α is defined in the following form

$$D_c^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(\zeta)}{(x-\zeta)^{\alpha-n+1}} d\zeta \qquad \alpha > 0, x > 0, \tag{4}$$

where $n-1 < \alpha \leq n, n \in \mathbb{N}$.

Or by fractional integral operator as

$$D_c^{\alpha}f(x) = I^{n-\alpha}D^n f(x).$$

One of the main properties of Caputo fractional derivative operator is

$$D_c^{\alpha}x^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}, \tag{5}$$

where $m > 0, m \in \mathbb{R}$.

3. Existence and uniqueness

In this section, we prove the existence and uniqueness of the solution of FRDE (1) where the fractional derivative is defined in Caputo's sense.

Let $J = [0, \mathbb{B}]$, $\mathbb{B} < \infty$ and $C(J)$ be the class of all continuous functions defined on J equipped with the supermum norm $\| w \| = \sup_{x \in J} |w(x)|$.

From the properties of fractional calculus [18] FRDE (1) can be written in the following form

$$I^{1-\alpha} \frac{dw(x)}{dx} = a(x) + b(x)w(x) + c(x)w^2(x). \tag{6}$$

Operating with I^{α} we obtain the integral equation

$$w(x) = w_0 + I^{\alpha}(a(x) + b(x)w(x) + c(x)w^2(x)). \tag{7}$$

In [20], it is shown that

$$I^{\alpha}[a(x) + b(x)w(x) + c(x)w^2(x)] \mid_{x=0} = 0.$$

Then we have

$$\begin{aligned} w(x) = w_0 + (&\frac{x^{\alpha}}{\Gamma(\alpha+1)}(a_0 + b_0 w_0 + c_0 w_0^2) \\ &+ I^{\alpha+1}(a'(x) + b'(x)w(x) + b(x)w'(x) + c'(x)w^2(x) + 2c(x)w(x)w'(x))), \end{aligned}$$

from which we can infer that $w \in C(J)$.

From equation (7), we can write

$$\frac{dw}{dx} = \frac{d}{dx} I^{\alpha}(a(x) + b(x)w(x) + c(x)w^2(x)),$$

$$I^{1-\alpha}\frac{dw}{dx} = I^{1-\alpha}\frac{d}{dx} I^{\alpha}(a(x) + b(x)w(x) + c(x)w^2(x)),$$

$$D_c^{\alpha} w(x) = \frac{d}{dx} I(a(x) + b(x)w(x) + c(x)w^2(x)),$$

$$D_c^{\alpha} w(x) = a(x) + b(x)w(x) + c(x)w^2(x),$$

and

$$w(0) = w_0 + I^{\alpha}(a(x) + b(x)w(x) + c(x)w^2(x)) \mid_{x=0},$$

which yields

$$w(0) = w_0.$$

Then the integral equation (7) is equivalent to the initial value problem (1).

Let $f(x, w(x)) = a(x) + b(x)w(x) + c(x)w^2(x)$ where $f(x, w) : [0, \mathbb{B}]$ x $\mathbb{R} \to \mathbb{R}$, and let $Q_m = \{w \in C(J) : |w(x)| \leq w_m\}$. Then, for $w \in Q_m$ we have

$$|\frac{\partial f}{\partial w}| = |b(x) + 2c(x)w(x)| \leq L + 2Lw_m.$$

Then $f(x, w(x))$ satisfies Lipschitz condition with respect to the second argument with Lipschitz constant $\hat{L} = L + 2Lw_m$.

Theorem 3.1. *The initial value problem (1) has a unique solution $w \in C(J)$.*

Proof.
Let the operator F be defined as

$$Fw(x) = w_0 + I^{\alpha}(a(x) + b(x)w(x) + c(x)w^2(x)), \tag{8}$$

or equivalently,

$$Fw(x) = w_0 + \int_0^x (\frac{(x-u)^{\alpha-1}}{\Gamma(\alpha)} f(u, w(u)))du.$$

Let $w(x) \in Q_m, x_1, x_2 \in J$ such that $0 < x_1 < x_2$ then,

$$\begin{aligned}
&|Fw(x_2) - Fw(x_1)| \\
&= |\frac{1}{\Gamma(\alpha)} \int_0^{x_2} (x_2-u)^{\alpha-1} f(u, w(u))du - \frac{1}{\Gamma(\alpha)} \int_0^{x_1} (x_1-u)^{\alpha-1} f(u, w(u))du| \\
&= |\frac{1}{\Gamma(\alpha)} \int_0^{x_1} (x_2-u)^{\alpha-1} f(u, w(u))du - \frac{1}{\Gamma(\alpha)} \int_0^{x_1} (x_1-u)^{\alpha-1} f(u, w(u))du \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2-u)^{\alpha-1} f(u, w(u))du| \\
&\le \frac{L_1}{\Gamma(\alpha)} \int_0^{x_1} (x_1-u)^{\alpha-1} - (x_2-u)^{\alpha-1} du + \frac{L_1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2-u)^{\alpha-1} du \\
&= \frac{L_1}{\Gamma(\alpha+1)} (2(x_2-x_1)^{\alpha} + x_1^{\alpha} - x_2^{\alpha}) \\
&\le \frac{2L_1(x_2-x_1)^{\alpha}}{\Gamma(\alpha+1)},
\end{aligned}$$

where $L_1 = \sup_{x\in J} \mid f(x, w(x)) \mid$. This implies that $F : C(J) \to C(J)$.

Now, let w and $v \in Q_m$

$$\begin{aligned}
|Fw(x) - Fv(x)| &= |\int_0^x \frac{(x-u)^{\alpha-1}}{\Gamma(\alpha)} (f(u, w(u)) - f(u, v(u)))du| \\
&\le \hat{L} \int_0^x \frac{(x-u)^{\alpha-1}}{\Gamma(\alpha)} |w(u)) - v(u)| \\
\|Fw(x) - Fv(x)\| &\le \hat{L} \sup_{x\in J} |w(x)) - v(x)| \int_0^x \frac{(x-u)^{\alpha-1}}{\Gamma(\alpha)} du \\
&\le \hat{L} \|w(x) - v(x)\| \frac{\mathbb{B}^{\alpha}}{\Gamma(\alpha+1)},
\end{aligned}$$

if $\frac{\mathbb{B}^{\alpha}\hat{L}}{\Gamma(\alpha+1)} < 1$ then the operator F defined in equation (8) is contraction and the theorem is proved.

4. Change of Variable

In this section, we propose a change of variable by substitution to transform the general form of FRDE (1) to a simpler form with Caputo differential operator. Let

$$w(x) = c_1 y(x) + h(x), \tag{9}$$

where c_1 is a constant and consider $c(x) = c_2$ in equation (1) is a constant.

Substituting equation (9) in equation (1), we get

$$D_c^\alpha y(x) = A + By^2(x), \quad y(0) = c_0, \tag{10}$$

and

$$c_3 = a(x) - \frac{b^2(x)}{4c_2} + D^\alpha(\frac{b(x)}{2c_2}), \tag{11}$$

where c_3 is a constant, $h(x) = -\frac{b(x)}{2c_2}$, $A = \frac{c_3}{c_1}$ and $B = c_1 c_2$. Equation (11) is a constrain that relates the functions $a(x)$ and $b(x)$.

For the case of integer order derivative, i.e $\alpha = 1$, the exact solution of equation (10) is known [21] for the following cases

- Case 1: $A = B = 1$ and $y(0) = 0$, the exact solution is $y(x) = \tan(x)$.
- Case 2: $A = 1$, $B = -1$ and $y(0) = 0$, the exact solution is $y(x) = \tanh(x)$.
- Case 3: $A = 0$ and $y(0) = 1$, the exact solution is $y(x) = \frac{1}{1-Bx}$.

5. Power Series Solution

We use the fractional power series to solve equation (10) with Caputo differential operator. let

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k\alpha}. \tag{12}$$

Substituting (12) into (10), we have

$$\sum_{k=1}^{\infty} \frac{\Gamma(k\alpha+1)}{\Gamma((k-1)\alpha+1)} a_k x^{(k-1)\alpha} = A + B(\sum_{k=0}^{\infty} a_k x^{k\alpha})^2, \tag{13}$$

or equivalently,

$$\sum_{k=0}^{\infty} \frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)} a_{k+1} x^{k\alpha} = A + B \sum_{k=0}^{\infty} \sum_{j=0}^{k} a_j a_{k-j} x^{k\alpha}. \tag{14}$$

For k=0, we have

$$a_1 = \frac{A + Ba_0^2}{\Gamma(\alpha+1)}, \tag{15}$$

for $k \geq 1$ by comparing coefficients of identical power of x, we obtain

$$a_{k+1} = B \frac{\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha+1)} \sum_{j=0}^{k} a_j a_{k-j}. \tag{16}$$

Then the solution of equation (10) is given by

$$y(x) = a_0 + \frac{A + Ba_0^2}{\Gamma(\alpha+1)} x^\alpha + \sum_{k=1}^{\infty} B \frac{\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha+1)} \sum_{j=0}^{k} a_j a_{k-j} x^{(k+1)\alpha}. \tag{17}$$

Now, consider the three cases for integer order problem

- Case 1: equation (10) has solution [21]

$$y(x)=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}x^{3\alpha}+2\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}x^{5\alpha}+...,$$

- Case 2: equation (10) has solution

$$y(x)=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}x^{3\alpha}+2\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}x^{5\alpha}+...,$$

- Case 3: equation (10) has solution [23] as

$$y(x)=1+\frac{B}{\Gamma(\alpha+1)}x^{\alpha}+2\frac{B^2}{\Gamma(2\alpha+1)}x^{2\alpha}+\frac{(4(\Gamma(\alpha+1))^2)+\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2\Gamma(3\alpha+1)}B^3x^{3\alpha}+...$$

6. Convergence Analysis

In this section, we prove the convergence for series solution (17).

Theorem 6.1. *[22] The classical power series $\sum_{k=0}^{\infty} a_k x^k$,$-\infty < x < \infty$ has radius of convergence R if and only if the fractional power series $\sum_{k=0}^{\infty} a_k x^{k\alpha}$, $x \geq 0$ has radius of convergence $R^{\frac{1}{\alpha}}$.*

Proof. See [22].

Consider the series

$$\hat{y}(x)=\sum_{k=0}^{\infty} a_k x^k, \tag{18}$$

where a_k is the coefficient of fractional series solution (17) defined recursively by (15) and (16). From recurrence relation (16), we have

$$|a_{k+1}| \leq |B|\frac{|\Gamma(k\alpha+1)|}{|\Gamma((k+1)\alpha+1)|}\sum_{j=0}^{k}|a_j||a_{k-j}|,$$

$$|a_{k+1}| \leq M\sum_{j=0}^{k}|a_j||a_{k-j}|, \tag{19}$$

where

$$M=\max_k\{|B|\frac{|\Gamma(k\alpha+1)|}{|\Gamma((k+1)\alpha+1)|}\}.$$

Define the power series

$$\mu=P(x)=\sum_{k=0}^{\infty} p_k x^k,$$

by $p_0=|a_0|$, $p_1=|a_1|$ and p_{k+1}=M$\sum_{j=0}^{k} p_j p_{k-j}$, k=1,2,$\cdots$.

The series $P(x)$ is a majorant series of $\hat{y}(x)$ defined in (18). Note that by easy calculation, we have

$$\begin{aligned} P(x) &= p_0 + x\sum_{k=0}^{\infty} p_{k+1}x^k \\ &= p_0 + xM\sum_{k=0}^{\infty}(\sum_{j=0}^{k} p_j p_{k-j})x^k. \end{aligned}$$

Consider now the implicit functional equation

$$f(x,\mu)=\mu-p_0-xM\mu^2.$$

Since f is an analytic function in the (x,μ) -plane, and $f(0,p_0)=0$, $f_\mu(0,p_0)$ $=1\neq0$, by the implicit function theorem [24,25] $P(x)$ is analytic in a neighborhood of the point $(0,p_0)$ of the (x,μ) -plane and with a positive radius of convergence. This implies that power series (18) converges, hence by Theorem (6.1) series solution (17) converges.

In what follows, we propose a technique based on Theorem (6.1) to obtain an approximation for the radius of convergence for series solution (17) numerically. Consider the fractional series solution (17) in *Case 1.* Let the series

$$T(x)=\sum_{k=0}^{\infty}b_kx^k, \tag{20}$$

be Taylor series of function $f(x)=tan(\beta x)$. Then, series $T(x)$ has a radius of convergence $\varrho=\frac{\pi}{2\beta}$. By Theorem (6.1), the series $T_\alpha(x)$ defined by

$$T_\alpha(x)=\sum_{k=0}^{\infty}b_kx^{k\alpha}, \tag{21}$$

has a radius of convergence $\varrho=(\frac{\pi}{2\beta})^{\frac{1}{\alpha}}$. We seek a value for β such that

$$\mid b_k\mid>\mid a_k\mid,$$

where a_k is the coefficient of fractional series solution (17) in *Case 1.* By comparing coefficients a_k and b_k for $0.1<\alpha<0.9$, a numerical estimate for β that guarantees $\mid b_k\mid>\mid a_k\mid$ is given by

$$\beta=\frac{1}{(0.730235\alpha^2+0.355273)\Gamma(0.730235\alpha+1)}.$$

Then, series solution in *Case 1* has a radius of convergence at least $\varrho=(\frac{\pi}{2\beta})^{\frac{1}{\alpha}}$. Figure (1) shows the graph of fractional series solution and corresponding $tan(\beta x)$ at different values of α. It illustrates how the formula obtained for β guarantees that series $T(x)$ is a majorant for fractional series solution in the range specified for α.

7. Singularity Analysis

Classical integer-order Ricatti differential equation has also another two forms of solutions namely when $A=1,B=1$, $y(x)=-\cot(x)$ and $A=-1,B=1$, $y(x)=-\coth(x)$ [21]. These two solutions have movable singularity at $x=0$. In this section, we present singularity analysis for the series solution of FRDE (10) that corresponds to these two cases. To obtain these types of solutions, the fractional derivative in equation (10) is considered in Riemann-Liouville differential operator.

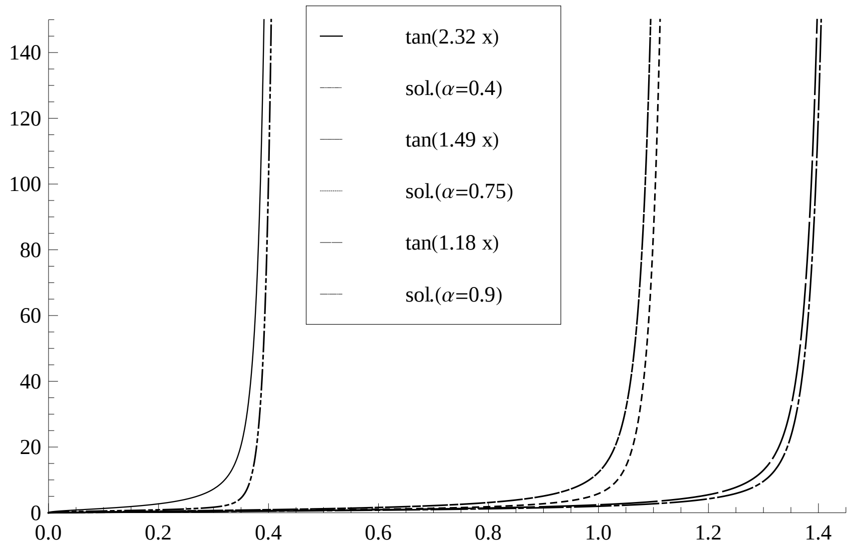

FIGURE 1. Series of solution for equation (10) *Case 1* denoted for parameter α_0 by sol.$(\alpha = \alpha_0)$ and the corresponding dominated series $tan(\beta x)$ at different vales of α.

7.1. **Singularity Behavior.** The negative power term of the series solution of equation (10) is important to know. The leading-order analysis [26] determines this power by setting

$$y(x) = g(x - x_0)^{\rho}, \tag{22}$$

where x_0 is the location of singularity, ρ is the power to be determined later and g is a constant. Substituting equation (22) into equation (10) using Riemann-Liouville differential operator (3), We get

$$g\frac{\Gamma(\rho+1)}{\Gamma(\rho-\alpha+1)}(x-x_0)^{\rho-\alpha} = A + Bg^2(x-x_0)^{2\rho}.$$

For the dominant terms,

$$g\frac{\Gamma(\rho+1)}{\Gamma(\rho-\alpha+1)}(x-x_0)^{\rho-\alpha} - Bg^2(x-x_0)^{2\rho} = 0, \tag{23}$$

from which we obtain

$$\rho = -\alpha, \tag{24}$$

and

$$g = \frac{\Gamma(\rho+1)}{B\Gamma(\rho-\alpha+1)}.$$

Therefore the singularity is a pole of order α.

Hence, for that case, the series solution takes the form

$$y(x) = a_0 x^{-\alpha} + \sum_{k=1}^{\infty} a_k x^{k\alpha}. \tag{25}$$

Substituting equation (25) into equation (10), we have

$$\begin{aligned} & a_0 \frac{\Gamma(-\alpha+1)}{\Gamma(-2\alpha+1)} x^{-2\alpha} + a_1\Gamma(\alpha+1) + a_2 \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} x^{\alpha} + \sum_{k=2}^{\infty} a_{k+1} \frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)} x^{k\alpha} \\ & = A + B(a_0^2 x^{-2\alpha} + 2a_0a_1 + 2a_0a_2x^{\alpha} + 2a_0 \sum_{k=2}^{\infty} a_{k+1}x^{k\alpha} + \sum_{k=2}^{\infty}\sum_{j=1}^{k-1} a_j a_{k-j} x^{k\alpha}). \end{aligned} \tag{26}$$

Comparing coefficients of equal exponents, we find

$$a_1 = \frac{A}{\Gamma(\alpha+1) - 2a_0B},$$

and

$$a_2 (\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} - 2a_0B) = 0,$$

then $a_2 = 0$. Finally, we obtain the recurrence relation

$$a_{k+1} = \frac{B\Gamma(k\alpha+1)}{\Gamma((k+1)\alpha+1) - 2a_0B\Gamma(k\alpha+1)} \sum_{j=1}^{k-1} a_j a_{k-j}, k = 2, 3, \cdots$$

and the series solution is given by

$$\begin{aligned} y(x) = & \frac{a_0}{x^{\alpha}} + \frac{A}{\Gamma(\alpha+1) - 2a_0B} x^{\alpha} + \frac{a_1^2 B\Gamma(2\alpha+1)}{\Gamma(3\alpha+1) - 2a_0B\Gamma(2\alpha+1)} x^{3\alpha} \\ & + \frac{2a_1a_3B\Gamma(4\alpha+1)}{\Gamma(5\alpha+1) - 2a_0B\Gamma(4\alpha+1)} x^{5\alpha} + \cdots, \end{aligned} \tag{27}$$

where $x \neq 0$.

Solution (27) is referred to as psi-series solution [26] of the FRDE (10).

Conclusion

In this paper, fractional order Riccati differential equation with variable coefficients is considered. For the Caputo definition of the fractional derivative, the conditions that guarantee the existence and uniqueness of the solution to this problem are deduced. Using a simple transformation, the considered equation can be transformed into a constant coefficient Riccati equation that is considered as the generalization to a well known integer order one. The convergence of the fractional power series solution is proved using the implicit function theorem. Also, a technique is proposed to numerically evaluate an approximation to the radius of convergence for the fractional power series from the radius of convergence of the corresponding integer-order series. Finally, for the Riemann-Liouville definition of fractional derivative, singularity analysis is performed and the psi-series solution of the fractional order Riccati differential equation is obtained.

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