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EXISTENCE, UNIQUENESS AND WELL-POSED CONDITIONS ON A CLASS OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITION

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ABSTRACT. In this paper, we propose the conditions on which a class of boundary value problems, presented by fractional differential equations, is well-posed. First, under the suitable conditions, we will prove the existence and uniqueness of solution. Then, the stability of solution will be discussed under the perturbations of boundary condition, function exists in the problem and the fractional order derivative.

1. INTRODUCTION

In many applications Fractional differential equations (FDEs) present more accurate models of phenomena than the ordinary differential equations. Therefore they have obtained importance, due to their applications in the sciences and engineering such as, physics, chemistry, mechanics, fluid dynamic, etc [1, 2]. In last decade many papers have written in this field. Existence of solutions to FDEs have received considerable interest in recent years. There are several papers dealing with the existence and uniqueness of solution to initial and boundary value problem of fractional order in Caputo or Riemann-Liouville senses. For example, see [3, 4, 5] and references therein. Some authors have also investigated the existence and uniqueness solutions for a coupled system of multi-term FDEs [6, 7]. However, in general, the study of well posed conditions for FDEs is less considered in the literature.
In [8], Houas and Benbachir investigated the existence and uniqueness of solutions for

$$\begin{cases} D_*^{\alpha}x(t) + f(x(t), D_*^{\beta}x(t)) = 0, \quad 2 < \alpha \leq 3,\ 0 < \beta < 1,\ t \in [0,1], \\ x(0) = x_0,\ x'(0) = 0,\ x'(1) = \lambda J^{\sigma}x(\eta), \end{cases}$$

where, D_*^{α} is the Caputo's fractional derivative, $0 < \eta < 1$, f is continuous function on $\mathbb{R}^2$ and λ is a real constant. In [9], Authors studied existence and uniqueness of

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solution for fractional differential equation

$$\begin{cases} D_*^q u(t) = f(t, u(t), D_*^p u(t)), \quad 2 < q < 3,\ 0 < p < 1, \\ u(0) = 0,\ D_*^p u(1) = \sum_{i=1}^{m-2} \zeta_i D_*^p u(\eta_i),\ u''(1) = 0, \end{cases}$$

where, $0 < \zeta_i, \eta_i < 1$ and D_*^α is the Caputo's fractional derivative. In this article, we will prove the conditions on which the following class of FDEs is well-posed.

$$D_*^\alpha y(t) = f(y(t), D_*^\beta y(t)), \quad 2 < \alpha \leq 3,\ 0 < \beta < 1, \tag{1}$$

where $t \in [0, 1]$, and D_*^α is the standard Caputo derivative, subject to the boundary value condition

$$y(0) = y'(0) = 0, \quad y(1) = \lambda y(\xi), \tag{2}$$

where $\xi \in (0, 1)$ and $0 \leq \lambda < \frac{1}{\xi^2}$.
We recall that a problem is said to be well-posed if it has a uniqueness solution and this solution depends on a parameter in a continuous way. This parameter, in the classical order differential equations is dependent on the initial conditions and the function exists in the problem. Whereas, in the FDEs this dependency and the stability solution with respect to the perturbation of fractional order derivative, α, should be taken into the account too [10].
In this article, we have first proved the existence solution of (1) by means of Schauder fixed point theorem on the interval $[0, 1]$. Then, we have proved the uniqueness by using Banach contraction map theorem under a suitable condition. We have also investigated the stability of solutions under the perturbations on boundary condition, the function exists in the problem and the fractional order derivative α. Finally, we have brought some examples to illustrate our results.
Let us start with some basic preliminaries that we will use them shortly.

2. Preliminaries

There are various definitions for fractional integration and derivatives [1, 2]. Here, we have used the Caputo's definition which is more reliable in applications.

Definition 2.1. *A real function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in \mathbb{R}$, if there exists a real number $p(> \mu)$ such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0, \infty)$.*

Definition 2.2. [1, 2] *The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L^1(\mathbb{R}^+)$ is defined as*

$$I_{0^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.3. *Let $f \in C_\mu$ and $\mu \geq 1$, then Caputo's definition of the fractional-order derivative is defined as* [2]

$$D_*^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}} dt, \tag{3}$$

where $\alpha > 0$ is the order of the derivative and $n = \lceil \alpha \rceil$.

Lemma 2.4. *For the Caputo derivative we have*

$$D_*^\alpha c = 0 \qquad (c \text{ is a constant}), \tag{4}$$

and

$$D_*^\alpha x^\beta = \begin{cases} 0 & \text{for } \beta \in N_0 \text{ and } \beta < \lceil\alpha\rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha} & \text{for } \beta \in N_0 \text{ and } \beta \geq \lceil\alpha\rceil \text{ or } \beta \notin N_0 \text{ and } \beta > \lceil\alpha\rceil, \end{cases} \quad (5)$$

where $N_0 = \{0, 1, 2, \cdots\}$. *Note that for* $\alpha \in \mathbb{N}$ *the Caputo differential operator coincides with the usual differential operator of an integer order.*

Lemma 2.5. [1] *Let* $\alpha > 0$. *Then the following equality holds for* $u \in C(0,1) \cap L^1(0,1)$ *with a derivative of order* n *that belongs to* $C(0,1) \cap L^1(0,1)$.

$$I_{0^+}^\alpha D_*^\alpha u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}, i = 1, \cdots, n-1$, *where* $n - 1 < \alpha \leq n$.

Now, we consider the following important Lemma in our article.

Lemma 2.6. *Let* $h \in AC[0,1]$ *and* $2 < \alpha \leq 3$. *Then the fractional differential equation*

$$D_*^\alpha y(t) = h(t), \quad 2 < \alpha \leq 3, \quad (6)$$
$$y(0) = y'(0) = 0,$$
$$y(1) = \lambda y(\xi), \qquad \xi \in (0,1), \quad 0 \leq \lambda < \frac{1}{\xi^2}, \quad (7)$$

has a solution

$$y(t) = \int_0^1 G(t,s)h(s)ds + \frac{\lambda t^2}{(1-\lambda\xi^2)} \int_0^1 G(\xi,s)h(s)ds, \quad (8)$$

where

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1} - t^2(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s < t \leq 1, \\ \frac{-t^2(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t < s \leq 1. \end{cases} \quad (9)$$

Proof. By Lemma 2.5 the solution of (6) can be written as

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds - c_0 - c_1 t - c_2 t^2.$$

Since $y(0) = y'(0) = 0$, a simple calculation gives $c_0 = c_1 = 0$, and from boundary condition we obtain

$$\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds - c_2 = \frac{\lambda}{\Gamma(\alpha)} \int_0^\xi (\xi-s)^{\alpha-1} h(s)ds - c_2 \lambda \xi^2,$$

therefore,

$$c_2 = \frac{1}{(1-\lambda\xi^2)\Gamma(\alpha)} \Big(\int_0^1 (1-s)^{\alpha-1} h(s)ds - \lambda \int_0^\xi (\xi-s)^{\alpha-1} h(s)ds \Big).$$

Hence, the solution of boundary value problem (6) is

$$
\begin{aligned}
y(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \\
&\quad -\frac{t^2}{(1-\lambda\xi^2)\Gamma(\alpha)}\Big(\int_0^1 (1-s)^{\alpha-1}h(s)ds-\lambda\int_0^\xi (\xi-s)^{\alpha-1}h(s)ds\Big) \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds-\frac{t^2}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)ds \\
&\quad -\frac{\lambda\xi^2t^2}{(1-\lambda\xi^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}h(s)ds+\frac{\lambda t^2}{(1-\lambda\xi^2)\Gamma(\alpha)}\int_0^\xi (\xi-s)^{\alpha-1}h(s)ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t \big((t-s)^{\alpha-1}-t^2(1-s)^{\alpha-1}\big)h(s)ds-\frac{1}{\Gamma(\alpha)}\int_t^1 t^2(1-s)^{\alpha-1}h(s)ds \\
&\quad +\frac{\lambda t^2}{(1-\lambda\xi^2)\Gamma(\alpha)}\Big(\int_0^\xi \big((\xi-s)^{\alpha-1} \\
&\quad -\xi^2(1-s)^{\alpha-1}\big)h(s)ds-\int_\xi^1 \xi^2(1-s)^{\alpha-1}h(s)ds\Big) \\
&= \int_0^1 G(t,s)h(s)ds+\frac{\lambda t^2}{(1-\lambda\xi^2)}\int_0^1 G(\xi,s)h(s)ds,
\end{aligned}
$$

where

$$
G(t,s)=\begin{cases} \frac{(t-s)^{\alpha-1}-t^2(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s<t\le 1, \\ \frac{-t^2(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le t<s\le 1, \end{cases}
$$

which completes the proof. □

In order to check the existence of solutions, we prove some properties of the functions $G(t,s)$.

Lemma 2.7. *For any $t\in[0,1]$, the functions $G(t,.)$ and $\frac{\partial}{\partial t}G(t,.)$ are integrable and have following properties*

(i) $\int_0^1 |G(t,s)|ds \le \frac{2}{\Gamma(\alpha+1)}$.
(ii) $\int_0^1 |\frac{\partial}{\partial t}G(t,s)|ds \le \frac{3}{\Gamma(\alpha)}$.

Proof. Let $t\in[0,1]$. Then for (i) we have

$$
\begin{aligned}
\int_0^1 |G(t,s)|ds &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds+\frac{t^2}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}ds \\
&\le \frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{t^2}{\Gamma(\alpha+1)}\le\frac{2}{\Gamma(\alpha+1)}.
\end{aligned}
$$

For (ii),

$$
\begin{aligned}
\int_0^1 |\frac{\partial}{\partial t}G(t,s)|ds &\le \int_0^1 \frac{2t(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds \\
&\le \frac{2t}{\Gamma(\alpha+1)}+\frac{t^{\alpha-1}}{\Gamma(\alpha)}\le\frac{3}{\Gamma(\alpha)}.
\end{aligned}
$$

Hence, $G(t,.)$ and $\frac{\partial}{\partial t}G(t,.)$ are integrable. □

Let $I=[0,1]$ and $C^1(I)$ be the class of all continuous function with continuous first order derivative on, I. Since $D_*^\beta y(t)=I^{1-\beta}y'(t)$ for $\beta\in(0,1)$, the operator D_*^β is continuous for any $y\in C^1(I)$. Now, for $y\in C^1(I)$ define the maximum norm by

$$\|y\|=\max_{t\in I}|y(t)|+\max_{t\in I}|D_*^\beta y(t)|,$$

and the space $X=\{y(t)|y(t)\in C^1(I)\}$ endowed with the above norm.

Lemma 2.8. $(X,\|.\|)$ *is a Banach space.*

Proof. Let $\{y_n\}_{n=1}^\infty$ be a Cauchy sequence in the space $(X,\|.\|)$. Obviously, $\{y_n\}_{n=1}^\infty$ and $\{D_*^\beta y_n\}_{n=1}^\infty$ are Cauchy sequence in the space $C(I)$. Since $C(I)$ is compact, $\{y_n\}_{n=1}^\infty$ and $\{D_*^\beta y_n\}_{n=1}^\infty$ uniformly converge to some y,u in I. Furthermore $y,u\in C(I)$. In the following, we need to show that $u=D_*^\beta y$.
Now, by definition of fractional integral

$$\begin{aligned}|I^\beta D_*^\beta y_n(t)-I^\beta u(t)| &\le \frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}|D_*^\beta y_n-u|ds\\ &\le \frac{1}{\Gamma(\beta+1)}\max_{t\in I}|D_*^\beta y_n-u|.\end{aligned}$$

Therefore using the convergence of $\{D_*^\beta y_n\}_{n=1}^\infty$, implies that $\lim_{n\to\infty}I^\beta D_*^\beta y_n(t)=I^\beta u(t)$ uniformly on I. On the other hand, we know $I^\beta D_*^\beta y_n(t)=y_n$ for $t\in I, 0<\beta<1$. Hence $I^\beta u=y$ and this means $u=D_*^\beta y$. This completes the proof. □

Remark 2.9. *Lemma 2.5 implies that the solution of the problem (1) coincides with the fixed point of the the operator T defined as*

$$Ty(t)=\int_0^1 G(t,s)f(y(s),D_*^\beta y(s))ds+\frac{\lambda t^2}{(1-\lambda\xi^2)}\int_0^1 G(\xi,s)f(y(s),D_*^\beta y(s))ds. \quad (10)$$

3. Existence and Uniqueness

According to Schauder fixed point theorem, the existence result have been stated. For convenience throughout this paper take

$$M:=(1+\frac{\lambda}{1-\lambda\xi^2})\frac{2}{\Gamma(\alpha+1)}.$$

Theorem 3.1. *suppose $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ be a continuous function and one of the following conditions is satisfied*
(H1) *There exist nonnegative function $a(t)\in[0,1]$ such that*

$$|f(x,y)|\le a(t)+c_0|x|^{\gamma_0}+c_1|y|^{\gamma_1},$$

where $c_0,c_1\ge 0$, $0<\gamma_0,\gamma_1<1$.
(H2) *The function f satisfy*

$$|f(x,y)|\le c_0|x|^{\gamma_0}+c_1|y|^{\gamma_1},$$

where $c_0,c_1\ge 0$, $\gamma_0,\gamma_1>1$.
Then there exist a solution $y(t)$ for boundary value problem (1).

Proof. First, suppose that condition (**H1**) holds. define

$$\mathcal{A} = \{y(t)\big|\ \|y(t)\| \leq R,\ t \in I\},$$

where

$$\begin{aligned} R \quad \geq \quad & \max\Big\{(6Mc_0)^{\frac{1}{1-\gamma_0}}, (6Mc_1)^{\frac{1}{1-\gamma_1}}, 6Mk_1, \\ & (\frac{12Mc_0}{\Gamma(2-\beta)})^{\frac{1}{1-\gamma_0}}, (\frac{12Mc_1}{\Gamma(2-\beta)})^{\frac{1}{1-\gamma_1}}, \frac{16\lambda k_1}{\Gamma(2-\beta)(1-\lambda\xi^2)}, \frac{8k_2}{\Gamma(2-\beta)}\Big\}, \end{aligned}$$

and

$$\begin{aligned} k_1 &= \max_{t\in I}\{\frac{1}{\Gamma(\alpha)}\int_0^1 |G(t,s)a(s)|ds\}, \\ k_2 &= \max_{t\in I}\{\frac{1}{\Gamma(\alpha)}\int_0^1 |\frac{\partial}{\partial t}G(t,s)a(s)|ds\}. \end{aligned}$$

Clearly, $\mathcal{A}$ is a closed, bounded and convex subset of Banach space X. Here, we prove that $T : \mathcal{A} \longrightarrow \mathcal{A}$. For any $y \in \mathcal{A}$, we obtain

$$\begin{aligned} |Ty(t)| \quad & \leq \quad \int_0^1 |G(t,s)f(y(s), D_*^\beta y(s))|ds + \frac{\lambda t^2}{(1-\lambda\xi^2)}\int_0^1 |G(\xi,s)f(y(s), D_*^\beta y(s))|ds. \\ & \leq \quad \int_0^1 |G(t,s)a(s)|ds + (c_0R^{\gamma_0} + c_1R^{\gamma_1})\int_0^1 |G(t,s)|ds \\ & \quad + \frac{\lambda}{(1-\lambda\xi^2)}\Big(\int_0^1 |G(\xi,s)a(s)|ds + (c_0R^{\gamma_0} + c_1R^{\gamma_1})\int_0^1 |G(\xi,s)|ds\Big) \\ & \leq \quad \big(1 + \frac{\lambda}{1-\lambda\xi^2}\big)\big(k_1 + \frac{2}{\Gamma(\alpha+1)}(c_0R^{\gamma_0} + c_1R^{\gamma_1})\big) \\ & \leq \quad M(k_1 + c_0R^{\gamma_0} + c_1R^{\gamma_1}) \leq \frac{R}{2}. \end{aligned}$$

For $0 < \beta < 1$, we have

$$\begin{aligned} |D_*^\beta(Ty)(t)| \quad & = \quad \Big|\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(Ty)'(s)ds\Big| \\ & \leq \quad \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big(\int_0^1 |\frac{\partial}{\partial s}G(s,\tau)f(\tau, y(\tau), D_*^\beta y(\tau))|d\tau \\ & \quad + \frac{2\lambda s}{(1-\lambda\xi^2)}\int_0^1 |G(\xi,\tau)f(\tau, y(\tau), D_*^\beta y(\tau))|d\tau\Big)ds \end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big\{\int_0^1 |\frac{\partial}{\partial s}G(s,\tau)a(\tau)|d\tau \\
& +(c_0R^{\gamma_0}+c_1R^{\gamma_1})\int_0^1 |\frac{\partial}{\partial s}G(s,\tau)|d\tau \\
& +\frac{2\lambda s}{1-\lambda\xi^2}\big(\int_0^1 |G(s,\tau)a(\tau)|d\tau \\
& +(c_0R^{\gamma_0}+c_1R^{\gamma_1})\int_0^1 |G(\xi,\tau)|d\tau\big)\Big\}ds \\
\leq\ & \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\big(k_2+\frac{3}{\Gamma(\alpha)}(c_0R^{\gamma_0}+c_1R^{\gamma_1})\big)ds \\
& +\frac{2\lambda}{(1-\lambda\xi^2)\Gamma(1-\beta)}\int_0^t s(t-s)^{-\beta}\big(k_1+\frac{2}{\Gamma(\alpha+1)}(c_0R^{\gamma_0}+c_1R^{\gamma_1})\big)ds \\
\leq\ & \frac{1}{\Gamma(1-\beta)}\big(k_2+\frac{3}{\Gamma(\alpha)}(c_0R^{\gamma_0}+c_1R^{\gamma_1})\big)\frac{t^{1-\beta}}{1-\beta} \\
& +\frac{2\lambda}{(1-\lambda\xi^2)\Gamma(1-\beta)}\big(k_1+\frac{2}{\Gamma(\alpha+1)}(c_0R^{\gamma_0}+c_1R^{\gamma_1})\big)\frac{t^{2-\beta}}{(1-\beta)(2-\beta)} \\
\leq\ & \frac{1}{\Gamma(2-\beta)}\big(k_2+\frac{3}{\Gamma(\alpha)}(c_0R^{\gamma_0}+c_1R^{\gamma_1})\big) \\
& +\frac{2\lambda}{(1-\lambda\xi^2)\Gamma(3-\beta)}\big(k_1+\frac{2}{\Gamma(\alpha+1)}(c_0R^{\gamma_0}+c_1R^{\gamma_1})\big) \\
\leq\ & \frac{3M}{\Gamma(2-\beta)}(c_0R^{\gamma_0}+c_1R^{\gamma_1})+\frac{2\lambda k_1}{\Gamma(2-\beta)(1-\lambda\xi^2)}+\frac{k_2}{\Gamma(2-\beta)} \\
\leq\ & \frac{R}{2}.
\end{aligned}
$$

It is obvious that $Ty(t)$ and $D_*^\beta(Ty)(t)$ are continuous in I. Therefore $T:\mathcal{A}\longrightarrow\mathcal{A}$. Now, suppose that condition (**H2**) holds. Choose

$$
0<R\leq\min\big\{(\frac{1}{4Mc_0})^{\frac{1}{1-\gamma_0}},(\frac{1}{4Mc_1})^{\frac{1}{1-\gamma_1}},(\frac{\Gamma(2-\beta)}{6Mc_0})^{\frac{1}{1-\gamma_0}},(\frac{\Gamma(2-\beta)}{6Mc_1})^{\frac{1}{1-\gamma_1}}\big\}.
$$

By similar process, we obtain $\|Ty\|\leq R$ and therefore in this case, $T:\mathcal{A}\longrightarrow\mathcal{A}$. Here, we need to show that T is completely continuous operator. First, equicontinuity of T will be shown as follow.

Let $t_1, t_2 \in I$ such that $t_1 < t_2$ and $N = \max_{t\in I, y\in\mathcal{A}} |f(t, y(t), D_*^\beta y(t))| + 1$. Then

$$\begin{aligned}
|Ty(t_2) \quad - \quad & Ty(t_1)| \\
= \quad & \Big| \int_0^1 \big(G(t_2,s) - G(t_1,s)\big) f(y(s), D_*^\beta y(s)) ds \\
& + \frac{\lambda(t_2^2 - t_1^2)}{1-\lambda\xi^2} \int_0^1 G(\xi,s) f(y(s), D_*^\beta y(s)) ds \Big| \\
\leq \quad & N \int_0^1 \big|\big(G(t_2,s) - G(t_1,s)\big| ds + \frac{2\lambda N}{1-\lambda\xi^2}(t_2^2 - t_1^2) \\
\leq \quad & \frac{2\lambda N}{1-\lambda\xi^2}(t_2^2 - t_1^2) \\
& + N\Big(\int_0^{t_1} \frac{(t_2^2 - t_1^2)(1-s)^{\alpha-1} + (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
& + \int_{t_1}^{t_2} \frac{(t_2^2 - t_1^2)(1-s)^{\alpha-1} + (t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds + \int_{t_2}^1 \frac{(t_2^2 - t_1^2)(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds \Big) \\
\leq \quad & \frac{2\lambda N}{1-\lambda\xi^2}(t_2^2 - t_1^2) + N\Big((t_2^2 - t_1^2) \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
& + \int_0^{t_2} \frac{(t_2^2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds - \int_0^{t_1} \frac{(t_1^2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \Big) \\
\leq \quad & N\Big(\frac{t_2^2 - t_1^2}{\Gamma(\alpha+1)} + \frac{t_2^\alpha - t_1^\alpha}{\Gamma(\alpha+1)} + \frac{2\lambda(t_2^2 - t_1^2)}{(1-\lambda\xi^2)\Gamma(\alpha+1)} \Big) \\
\leq \quad & N\Big(M(t_2^2 - t_1^2) + \frac{t_2^\alpha - t_1^\alpha}{\Gamma(\alpha+1)} \Big),
\end{aligned}$$

and,

$$\begin{aligned}
|D_*^\beta(Ty)(t_2) \quad - \quad & D_*^\beta(Ty)(t_1)| \\
= \quad & \frac{1}{\Gamma(1-\beta)} \Big| \int_0^{t_2} (t_2 - s)^{-\beta} \Big(\int_0^1 \frac{\partial}{\partial s} G(s,\tau)\big) f(y(\tau), D_*^\beta y(\tau)) d\tau \\
& + \frac{2\lambda s}{1-\lambda\xi^2} \int_0^1 G(\xi,\tau) f(y(\tau), D_*^\beta y(\tau)) d\tau \Big) ds \\
& - \int_0^{t_1} (t_1 - s)^{-\beta} \Big(\int_0^1 \frac{\partial}{\partial s} G(s,\tau) f(y(\tau), D_*^\beta y(\tau)) d\tau \\
& + \frac{2\lambda s}{1-\lambda\xi^2} \int_0^1 G(\xi,\tau) f(y(\tau), D_*^\beta y(\tau)) d\tau \Big) ds \Big| \\
\leq \quad & \frac{3N}{\Gamma(1-\beta)\Gamma(\alpha)} \Big| \int_0^{t_2} (t_2 - s)^{-\beta} ds - \int_0^{t_1} (t_1 - s)^{-\beta} ds \Big| \\
& + \frac{6\lambda N}{\Gamma(1-\beta)\Gamma(\alpha)(1-\lambda\xi^2)} \Big| \int_0^{t_2} s(t_2 - s)^{-\beta} ds - \int_0^{t_1} s(t_1 - s)^{-\beta} ds \Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{3N}{\Gamma(1-\beta)\Gamma(\alpha)}\Big|\int_0^{t_1}\big((t_2-s)^{-\beta}-(t_1-s)^{-\beta}\big)ds+\int_{t_1}^{t_2}(t_2-s)^{-\beta}ds\Big| \\
&\quad+\frac{6\lambda N}{\Gamma(1-\beta)\Gamma(\alpha)(1-\lambda\xi^2)}\Big|\int_0^{t_1}\big(s(t_2-s)^{-\beta}-s(t_1-s)^{-\beta}\big)ds+\int_{t_1}^{t_2}s(t_2-s)^{-\beta}ds\Big| \\
&\leq \frac{3N}{\Gamma(2-\beta)\Gamma(\alpha)}\Big(t_2^{1-\beta}-t_1^{1-\beta}+2(t_2-t_1)^{1-\beta}\Big) \\
&\quad+\frac{6\lambda N}{\Gamma(\alpha)(1-\lambda\xi^2)}\Big(\frac{2t_1(t_2-t_1)^{1-\beta}}{\Gamma(2-\beta)}+\frac{t_2^2-t_1^2}{\Gamma(3-\beta)}+\frac{2(t_2-t_1)^{2-\beta}}{\Gamma(3-\beta)}\Big).
\end{aligned}
$$

Since the functions $t_2^2-t_1^2, t_2^\alpha-t_1^\alpha, (t_2-t_1)^{2-\beta}$ and $t_1(t_2-t_1)^{1-\beta}$ are continuous, we conclude that, Ty is an equicontinuous set. Obviously, Ty is uniformly bounded because $T\mathcal{A}\subseteq\mathcal{A}$. By means of Arzela-Ascoli theorem, T is a compact operator. Therefore, from Schauder fixed point theorem, the operator T has a fixed point i.e., the fractional boundary value problem (1) has a solution. □

In what follows, we prove the uniqueness of solution for (1) based on application of Banach fixed point theorem.

Theorem 3.2. *Let $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ be continuous function and it fulfill a Lipschitz condition with respect to the first and second variables with Lipschitz constant $0<L<\frac{\Gamma(2-\beta)}{M(3+\Gamma(2-\beta))}$; i.e. $|f(x_1,y_1)-f(x_2,y_2)|\leq L(|x_1-x_2|+|y_1-y_2|)$. Then problem (1) has a unique solution.*

Proof. In Theorem 3.1 we have shown that T is continuous operator and $T:\mathcal{A}\longrightarrow\mathcal{A}$.Therefore, using Banach fixed point theorem, it is sufficient to show that T is a contraction mapping. For any $y_1,y_2\in X$

$$
\begin{aligned}
|Ty_1(t) &- Ty_2(t)| \\
&\leq \Big|\int_0^1 G(t,s)\Big(f(y_1(s),D_*^\beta y_1(s))-f(y_2(s),D_*^\beta y_2(s))\Big)ds\Big| \\
&\quad+\frac{\lambda t^2}{1-\lambda\xi^2}\Big|\int_0^1 G(\xi,s)\Big(f(y_1(s),D_*^\beta y_1(s))-f(y_2(s),D_*^\beta y_2(s))\Big)ds\Big| \\
&\leq L\|y_1-y_2\|\Big(\int_0^1|G(t,s)|ds+\frac{\lambda t^2}{1-\lambda\xi^2}\int_0^1|G(\xi,s)|ds\Big) \\
&\leq LM\|y_1-y_2\|,
\end{aligned}
$$

$$
\begin{aligned}
|D_*^\beta(Ty_1)(t) &- D_*^\beta(Ty_2)(t)| \\
&= \Big|\frac{1}{\Gamma(1-\beta)}\int_0^t(t-s)^{-\beta}\big((Ty_1)'(s)-(Ty_2)'(s)\big)ds\Big| \\
&\leq \frac{1}{\Gamma(1-\beta)}\Big|\int_0^t(t-s)^{-\beta}\Big(\int_0^1\frac{\partial}{\partial s}G(s,\tau)\big(f(y_1(\tau),D_*^\beta y_1(\tau))-f(y_2(\tau),D_*^\beta y_2(\tau))\big)d\tau \\
&\quad+\frac{2\lambda s}{1-\lambda\xi^2}\int_0^1 G(\xi,\tau)\big(f(y_1(\tau),D_*^\beta y_1(\tau))-f(y_2(\tau),D_*^\beta y_2(\tau))\big)d\tau\Big)ds\Big| \\
&\leq \frac{3L}{\Gamma(\alpha)\Gamma(1-\beta)}\|y_1-y_2\|\Big(\int_0^t(t-s)^{-\beta}ds+\frac{2\lambda}{1-\lambda\xi^2}\int_0^t s(t-s)^{-\beta}ds\Big) \\
&\leq \frac{3LM}{\Gamma(2-\beta)}\|y_1-y_2\|.
\end{aligned}
$$

Therefore $\|Ty_1 - Ty_2\| \leq \big(ML + \frac{3LM}{\Gamma(2-\beta)}\big)\|y_1 - y_2\|$. Hence by the Banach fixed point theorem, T has a unique fixed point which is a solution of problem (1). $\square$

4. Stability of solution

In this section, we study the stability analysis of problem (1) under various perturbations. Dependence solution on boundary value condition is discussed in Theorem 4.1. Stability of solution with respect to the perturbation of f is analyzed in Theorem 4.2. Finally, the perturbation effect of fractional order derivative on solution is studied in Lemma 4.3 and Theorem 4.4.

Theorem 4.1. *Suppose function f fulfill the conditions of Theorem 3.2 and let $\tilde{y}(t)$ be the solution of the following perturbed problem on boundary value conditions*

$$D_*^{\alpha} y(t) = f(y(t), D_*^{\beta} y(t)), \quad 2 < \alpha \leq 3,\ 0 < \beta < 1, \tag{11}$$

$$y(0) = \varepsilon_1, \quad y'(0) = \varepsilon_2,$$

$$y(1) = \lambda y(\xi) + \varepsilon_3, \qquad \xi \in (0,1), \quad 0 \leq \lambda < \frac{1}{\xi^2}. \tag{12}$$

Then $\|y - \tilde{y}\| = O(\varepsilon)$, where $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.

Proof. Similar to Lemma 2.5 the solution of problem (11) is

$$\tilde{y}(t) = \int_0^1 G(t,s) f(\tilde{y}(s), D_*^{\beta}\tilde{y}(s))ds + \frac{\lambda t^2}{(1-\lambda\xi^2)}\int_0^1 G(\xi,s) f(\tilde{y}(s), D_*^{\beta}\tilde{y}(s))ds + p(t), \tag{13}$$

where

$$p(t) = \frac{t^2}{1-\lambda\xi^2}\big(\varepsilon_1(\lambda - 1) + \varepsilon_2(\lambda\xi - 1)\big) + \varepsilon_2 t + \varepsilon_1.$$

Thus,

$$\begin{aligned}
|y - \tilde{y}| \quad &\leq \quad \Big|\int_0^1 G(t,s)\big(f(y(s), D_*^{\beta}y(s)) - f(\tilde{y}(s), D_*^{\beta}\tilde{y}(s))\big)ds\Big| \\
&\quad + \frac{\lambda t^2}{(1-\lambda\xi^2)}\Big|\int_0^1 G(\xi,s)\big(f(y(s), D_*^{\beta}y(s)) - f(\tilde{y}(s), D_*^{\beta}\tilde{y}(s))\big)ds\Big| + |p(t)| \\
&\leq \quad L\|y - \tilde{y}\|\Big(\int_0^1 G(t,s)ds + \frac{\lambda t^2}{(1-\lambda\xi^2)}\int_0^1 G(t,s)ds\Big) + |p(t)| \\
&\leq \quad LM\|y - \tilde{y}\| + |p(t)|,
\end{aligned}$$

$$\begin{aligned}
|D_*^{\beta}y(t) \quad &- \quad D_*^{\beta}\tilde{y}(t)| \\
&\leq \quad \frac{1}{\Gamma(1-\beta)}\Big|\int_0^t (t-s)^{-\beta}\Big(\int_0^1 \frac{\partial}{\partial s}G(s,\tau)\big(f(y(\tau), D_*^{\beta}y(\tau)) - f(\tilde{y}(\tau), D_*^{\beta}\tilde{y}(\tau))\big)d\tau \\
&\quad + \frac{2\lambda s}{1-\lambda\xi^2}\int_0^1 G(\xi,\tau)\big(f(y(\tau), D_*^{\beta}y(\tau)) - f(\tilde{y}(\tau), D_*^{\beta}\tilde{y}(\tau))\big)d\tau\Big)ds\Big| + |D_*^{\beta}p(t)| \\
&\leq \quad \frac{3L}{\Gamma(\alpha)\Gamma(1-\beta)}\|y - \tilde{y}\|\Big(\int_0^t (t-s)^{-\beta}ds + \frac{2\lambda}{1-\lambda\xi^2}\int_0^t s(t-s)^{-\beta}ds\Big) + |D_*^{\beta}p(t)| \\
&\leq \quad \frac{3LM}{\Gamma(2-\beta)}\|y - \tilde{y}\| + |D_*^{\beta}p(t)|.
\end{aligned}$$

Therefore,

$$\begin{aligned}\|y-\tilde{y}\| &\le \frac{1}{1-\big(LM+\frac{3LM}{\Gamma(2-\beta)}\big)}\Big(\big|\frac{t^2}{1-\lambda\xi^2}\big(\varepsilon_1(\lambda-1)+\varepsilon_2(\lambda\xi-1)\big)+\varepsilon_2 t+\varepsilon_1\big| \\ &\quad +\big|\frac{2t^{2-\beta}}{(1-\lambda\xi^2)\Gamma(3-\beta)}\big(\varepsilon_1(\lambda-1)+\varepsilon_2(\lambda\xi-1)\big)+\frac{\varepsilon_2}{\Gamma(2-\beta)}t^{1-\beta}\big|\Big) \\ &\le \frac{\varepsilon}{1-\big(LM+\frac{3LM}{\Gamma(2-\beta)}\big)}\Big(\big|\frac{1}{1-\lambda\xi^2}(1+\frac{2}{\Gamma(3-\beta)})\big(\lambda+2+\lambda\xi\big)+2+\frac{1}{\Gamma(2-\beta)}\big|\Big).\end{aligned}$$

This is complete the proof. □

Theorem 4.2. *Suppose that the conditions of Theorem 3.2 hold and Let $\tilde{y}(t)$ be the solution of the following perturbed problem on function f*

$$D_*^{\alpha}y(t)=f(y(t),D_*^{\beta}y(t))+\varepsilon,\quad 2<\alpha\le 3,\ 0<\beta<1, \tag{14}$$

$$y(0)=y'(0)=0,$$

$$y(1)=\lambda y(\xi),\qquad \xi\in(0,1),\quad 0\le\lambda<\frac{1}{\xi^2}. \tag{15}$$

Then $\|y-\tilde{y}\|=O(\varepsilon)$.

Proof. The solution of problem (14) is

$$\begin{aligned}\tilde{y}(t) &= \int_0^1 G(t,s)\big(f(\tilde{y}(s),D_*^{\beta}\tilde{y}(s))+\varepsilon\big)ds \\ &\quad +\frac{\lambda t^2}{(1-\lambda\xi^2)}\int_0^1 G(\xi,s)\big(f(\tilde{y}(s),D_*^{\beta}\tilde{y}(s))+\varepsilon\big)ds.\end{aligned} \tag{16}$$

Then, similar to the proof of the previous theorem

$$\begin{aligned}|y-\tilde{y}| &\le LM\|y-\tilde{y}\|+\varepsilon\Big(\int_0^1 G(t,s)ds+\frac{\lambda t^2}{(1-\lambda\xi^2)}\int_0^1 G(t,s)ds\Big) \\ &\le LM\|y-\tilde{y}\|+\varepsilon M,\end{aligned}$$

and

$$\begin{aligned}|D_*^{\beta}y(t) &- D_*^{\beta}\tilde{y}(t)| \\ &\le \frac{3LM}{\Gamma(\alpha)\Gamma(1-\beta)}\|y-\tilde{y}\|+\varepsilon\Big(\int_0^t (t-s)^{-\beta}ds+\frac{2\lambda}{1-\lambda\xi^2}\int_0^t s(t-s)^{-\beta}ds\Big) \\ &\le \frac{3LM}{\Gamma(2-\beta)}\|y-\tilde{y}\|+\frac{3\varepsilon M}{\Gamma(2-\beta)}.\end{aligned}$$

Therefore,

$$\|y-\tilde{y}\|\le\frac{\varepsilon}{1-\big(LM+\frac{3LM}{\Gamma(2-\beta)}\big)}\Big(M+\frac{3M}{\Gamma(2-\beta)}\Big).$$

□

For perturbation analysis on the fractional order of the derivative, we first state and prove the following lemma and then the main theorem will be discussed.

Lemma 4.3. *Let $t,x\in I$ and $\alpha>\alpha-\varepsilon>2$, then*

$$\int_0^t\Big|\frac{x^{\alpha-1}}{\Gamma(\alpha)}-\frac{x^{\alpha-\varepsilon-1}}{\Gamma(\alpha-\varepsilon)}\Big|dx=O(\varepsilon). \tag{17}$$

Proof. We estimate the integral as follows

$$\begin{aligned}\int_0^t \big|\frac{x^{\alpha-1}}{\Gamma(\alpha)}-\frac{x^{\alpha-\varepsilon-1}}{\Gamma(\alpha-\varepsilon)}\big|dx &\leq \int_0^t \big|\frac{x^{\alpha-1}}{\Gamma(\alpha)}-\frac{x^{\alpha-\varepsilon-1}}{\Gamma(\alpha)}\big|dx+\int_0^t \big|\frac{x^{\alpha-\varepsilon-1}}{\Gamma(\alpha)}-\frac{x^{\alpha-\varepsilon-1}}{\Gamma(\alpha-\varepsilon)}\big|dx\\ &\leq \frac{1}{\Gamma(\alpha)}\big(\frac{1}{\alpha}-\frac{1}{\alpha-\varepsilon}\big)+\frac{1}{\alpha-\varepsilon}\big(\frac{1}{\Gamma(\alpha)}-\frac{1}{\Gamma(\alpha-\varepsilon)}\big)\\ &\leq \varepsilon\big(\frac{1}{\alpha(\alpha-\varepsilon)\Gamma(\alpha)}+\frac{|\Gamma'(\eta)|}{(\alpha-\varepsilon)\Gamma(\alpha)\Gamma(\alpha-\varepsilon)}\big),\end{aligned}$$

where $\alpha-\varepsilon<\eta<\alpha$ and $\Gamma'(\eta)$ is derivative of the gamma function. □

Theorem 4.4. *Suppose that the conditions of Theorem 3.2 hold and Let $\tilde{y}(t)$ be the solution of the following perturbed problem on fractional order derivative α*

$$D_*^{\alpha-\varepsilon}y(t)=f(y(t),D_*^{\beta}y(t)),\quad 2<\alpha\leq 3,\ 0<\beta<1, \tag{18}$$

$$y(0)=y'(0)=0,$$

$$y(1)=\lambda y(\xi),\qquad \xi\in(0,1),\quad 0\leq\lambda<\frac{1}{\xi^2}, \tag{19}$$

where $2<\alpha-\varepsilon<\alpha\leq 3$. Then $\|y-\tilde{y}\|=O(\varepsilon)$.

Proof. According to the above discussion, the solution of problem (18) is given by

$$\tilde{y}(t)=\int_0^1 \widetilde{G}(t,s)f(\tilde{y}(s),D_*^{\beta}\tilde{y}(s))ds+\frac{\lambda t^2}{(1-\lambda\xi^2)}\int_0^1 \widetilde{G}(\xi,s)f(\tilde{y}(s),D_*^{\beta}\tilde{y}(s))ds, \tag{20}$$

where

$$\widetilde{G}(t,s)=\begin{cases}\frac{(t-s)^{\alpha-\varepsilon-1}-t^2(1-s)^{\alpha-\varepsilon-1}}{\Gamma(\alpha-\varepsilon)}, & 0\leq s<t\leq 1,\\ \frac{-t^2(1-s)^{\alpha-\varepsilon-1}}{\Gamma(\alpha-\varepsilon)}, & 0\leq t<s\leq 1.\end{cases} \tag{21}$$

Then,

$$\begin{aligned}|y-\tilde{y}| &\leq \Big|\int_0^1 G(t,s)f(y(s),D_*^{\beta}y(s))ds-\int_0^1 \widetilde{G}(t,s)f(\tilde{y}(s),D_*^{\beta}\tilde{y}(s))ds\Big|\\ &\quad+\frac{\lambda t^2}{1-\lambda\xi^2}\Big|\int_0^1 G(\xi,s)f(y(s),D_*^{\beta}y(s))ds-\int_0^1 \widetilde{G}(\xi,s)f(\tilde{y}(s),D_*^{\beta}\tilde{y}(s))ds\Big|\\ &\leq \Big|\int_0^1 G(t,s)\big(f(y(s),D_*^{\beta}y(s))-f(\tilde{y}(s),D_*^{\beta}\tilde{y}(s))\big)ds\Big|\\ &\quad+\Big|\int_0^1 \big(G(t,s)-\widetilde{G}(t,s)\big)f(\tilde{y}(s),D_*^{\beta}\tilde{y}(s))ds\Big|\\ &\quad+\frac{\lambda t^2}{1-\lambda\xi^2}\Big(\Big|\int_0^1 G(\xi,s)\big(f(y(s),D_*^{\beta}y(s))-f(\tilde{y}(s),D_*^{\beta}\tilde{y}(s))\big)ds\Big|\\ &\quad+\Big|\int_0^1 \big(G(\xi,s)-\widetilde{G}(\xi,s)\big)f(\tilde{y}(s),D_*^{\beta}\tilde{y}(s))ds\Big|\Big)\\ &\leq L\|y-\tilde{y}\|\int_0^1 |G(t,s)|ds+\|f\|_{\varepsilon}\int_0^1 \big|G(t,s)-\widetilde{G}(t,s)\big|ds\\ &\quad+\frac{\lambda}{1-\lambda\xi^2}\Big(L\|y-\tilde{y}\|\int_0^1 |G(\xi,s)|ds+\|f\|_{\varepsilon}\int_0^1 \big|G(\xi,s)-\widetilde{G}(\xi,s)\big|ds\Big)\\ &\leq LM\|y-\tilde{y}\|+\|f\|_{\varepsilon}\Big(\int_0^1 \big|G(t,s)-\widetilde{G}(t,s)\big|ds+\frac{\lambda}{1-\lambda\xi^2}\int_0^1 \big|G(\xi,s)-\widetilde{G}(\xi,s)\big|ds\Big)\end{aligned}$$

where $\|f\|_\varepsilon = \sup_{0<\varepsilon<\alpha-2} |(f(\tilde{y}(t), D_*^\beta \tilde{y}(t))|$,

$$
\begin{aligned}
|D_*^\beta y(t) \; - \; & D_*^\beta \tilde{y}(t)| \\
\leq \; & \frac{1}{\Gamma(1-\beta)}\Big|\int_0^t (t-s)^{-\beta}\Big(\int_0^1 \frac{\partial}{\partial s}G(s,\tau)f(y(\tau), D_*^\beta y(\tau)) \\
& -\int_0^1 \frac{\partial}{\partial s}\widetilde{G}(s,\tau)f(\tilde{y}(\tau), D_*^\beta \tilde{y}(\tau)))d\tau\Big)ds\Big| + \frac{2\lambda}{\Gamma(1-\beta)(1-\lambda\xi^2)} \\
& \Big|\int_0^t (t-s)^{-\beta}\Big(\int_0^1 sG(\xi,\tau)f(y(\tau), D_*^\beta y(\tau)) - \int_0^1 s\widetilde{G}(\xi,\tau)f(\tilde{y}(\tau), D_*^\beta \tilde{y}(\tau)))d\tau\Big)ds\Big| \\
\leq \; & \frac{L\|y-\tilde{y}\|}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big(\int_0^1 \big|\frac{\partial}{\partial s}G(s,\tau)\big|d\tau\Big)ds \\
& +\|f\|_\varepsilon \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\Big(\int_0^1 \big|\frac{\partial}{\partial s}G(s,\tau) - \frac{\partial}{\partial s}\widetilde{G}(t,\tau)\big|d\tau\Big)ds \\
& +\frac{2\lambda}{\Gamma(1-\beta)(1-\lambda\xi^2)}\Big(L\|y-\tilde{y}\|\int_0^t s(t-s)^{-\beta}\Big(\int_0^1 \big|G(\xi,\tau)\big|d\tau\Big)ds \\
& +\|f\|_\varepsilon \int_0^t s(t-s)^{-\beta}\int_0^1 \big|G(\xi,\tau)-\widetilde{G}(\xi,\tau)\big|d\tau ds\Big) \\
\leq \; & \frac{3LM}{\Gamma(2-\beta)}\|y-\tilde{y}\| + \|f\|_\varepsilon \frac{1}{\Gamma(1-\beta)}\int_0^t s(t-s)^{-\beta}\Big(\int_0^1 \big|\frac{\partial}{\partial s}G(s,\tau) - \frac{\partial}{\partial s}\widetilde{G}(t,\tau)\big|d\tau\Big)ds \\
& +\frac{2\lambda}{\Gamma(1-\beta)(1-\lambda\xi^2)}\|f\|_\varepsilon \int_0^t s(t-s)^{-\beta}\int_0^1 \big|G(\xi,\tau)-\widetilde{G}(\xi,\tau)\big|d\tau ds\Big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|y-\tilde{y}\| \;\; \leq \; & \frac{1}{1-\big(LM+\frac{3LM}{\Gamma(2-\beta)}\big)}\Big(\int_0^1 \big|G(t,s)-\widetilde{G}(t,s)\big|ds + \frac{\lambda}{1-\lambda\xi^2}\int_0^1 \big|G(\xi,s)-\widetilde{G}(\xi,s)\big|ds \\
& +\frac{1}{\Gamma(1-\beta)}\int_0^t s(t-s)^{-\beta}\Big(\int_0^1 \big|\frac{\partial}{\partial s}G(s,\tau) - \frac{\partial}{\partial s}\widetilde{G}(t,\tau)\big|d\tau\Big)ds \\
& +\frac{2\lambda}{\Gamma(1-\beta)(1-\lambda\xi^2)}\int_0^t s(t-s)^{-\beta}\int_0^1 \big|G(\xi,\tau)-\widetilde{G}(\xi,\tau)\big|d\tau ds\Big).
\end{aligned}
$$

According to structure of $G(t,s)$, we know that every term of $|G(t,s)-\widetilde{G}(t,s)|$ and $|\frac{\partial}{\partial t}G(t,s) - \frac{\partial}{\partial t}\widetilde{G}(t,s)|$ is in the form of (17). Hence, Lemma 4.3 obtains

$$
\int_0^1 \big|G(t,s)-\widetilde{G}(t,s)\big|ds = O(\varepsilon), \quad \int_0^1 \big|\frac{\partial}{\partial t}G(t,s) - \frac{\partial}{\partial t}\widetilde{G}(t,s)\big|ds = O(\varepsilon). \tag{22}
$$

Therefore, $\|y-\tilde{y}\| = O(\varepsilon)$ and the proof is complete. □

5. Some Examples

In this section, we have discussed some examples to illustrate our results.

Example 5.1. *Consider the problem*

$$
D_*^{5/2}y(t) = \frac{1}{2}\big(y(t)\big)^{\frac{1}{2}} + \frac{3}{10}\big(D_*^{\frac{1}{2}}y(t)\big)^{\frac{1}{4}}, \tag{23}
$$

$$
y(0) = y'(0) = 0, \quad y(1) = \frac{1}{2}y(\frac{1}{\sqrt{2}}). \tag{24}
$$

Then,

$$f(y(t), D_*^{\frac{1}{2}} y(t)) \le e^t + \frac{1}{2}\big|y(t)\big|^{\frac{1}{2}} + \frac{3}{10}\big|D_*^{\frac{1}{2}} y(t)\big|^{\frac{1}{4}}.$$

*Therefore, the condition (**H1**) in Theorem 3.1 holds, and hence this problem has a solution.*

Example 5.2. *Consider the following problem*

$$D_*^{2.7} y(t) = 4\big(y(t)\big)^3 + 2\big(D_*^{0.4} y(t)\big)^5, \tag{25}$$

$$y(0) = y'(0) = 0, \quad y(1) = \frac{1}{2} y(\frac{1}{3}). \tag{26}$$

Then,

$$f(y(t), D_*^{0.4} y(t)) \le 4\big|y(t)\big|^3 + 2\big|D_*^{0.4} y(t)\big|^5.$$

*Therefore, the condition (**H2**) in Theorem 3.1 holds, and hence this problem has a solution.*

Example 5.3. *Consider the problem*

$$D_*^{5/2} y(t) = \frac{1}{18}\big(y(t)\big) + \frac{1}{27}\sin\big(D_*^{\frac{1}{2}} y(t)\big) = 0, \tag{27}$$

$$y(0) = y'(0) = 0, \quad y(1) = \frac{1}{3} y(\frac{1}{\sqrt{2}}). \tag{28}$$

Then, a simple computation shows that

$$\big|f(y(t), D_*^{\frac{1}{2}} y(t)) - f(t, \tilde{y}(t), D_*^{\frac{1}{2}} \tilde{y}(t))\big| \le \frac{1}{9}\big(|y - \tilde{y}| + |D_*^{\frac{1}{2}} y(t) - D_*^{\frac{1}{2}} \tilde{y}(t)|\big).$$

Since $0 < L = \frac{1}{9} < 0.263$*, Theorem 3.2 implies that this problem has a unique solution.*

6. Conclusion

In this paper we have discussed the well-posed conditions for a class of fractional order boundary value problems. We have proved the existence and uniqueness of solution by means of Schauder fixed point and Banach contraction map theorems on the interval $[0, 1]$. We have also studied the perturbation on boundary condition, on the function exists in the right hand side of the problem and on the fractional order α. In other words, we have shown that the solution of problem is stable under the small perturbation.

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