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APPROXIMATION OF SOLUTIONS OF A STOCHASTIC FRACTIONAL DIFFERENTIAL EQUATION WITH DEVIATING ARGUMENT

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Abstract. The existence, uniqueness approximate solutions of a stochastic fractional differential equation with deviating argument is studied. Analytic semigroup theory and fixed point method is used to prove our results. Then we considered Faedo-Galerkin approximation of solution and proved some convergence results. We also studied an example to illustrate our result.

1. Introduction

The notion and methods of solving of differential equations involving fractional derivatives of the unknown function is a widely explored research field. The history of fractional calculus started almost at the same time when classical calculus was established. Fractional differential equations arise in the theory of fractals, visco-elasticity, seismology, polymers etc. Fractional derivatives depicts the memory and hereditary properties of various materials and processes that are mostly overlooked in integer-order models. We refer our readers to [6, 13, 14].

Random noise causes fluctuations in deterministic models. Stochastic problems are better than deterministic ones as these equations incorporate the randomness into the equations. Thus stochastic evolution equations are natural generalizations of ordinary differential equations. Lukasz Delong and Peter Imkeller [7] studied backward stochastic differential equations with time delayed generator. They proved the existence and uniqueness of a solution for a sufficiently small time horizon or for a sufficiently small Lipschitz constant of a generator. Bernt Oksendal et.al. [1] studied optimal control problems for time-delayed stochastic differential equations with jumps.

The approximation of a solution to a nonlinear Sobolev type evolution equation was studied by Bahuguna and Shukla [3] in a separable Hilbert space $(H, \|.\|, (.,.))$, where the linear operator A satisfies the assumption $(H1)$ stated in preliminaries so that A generates an analytic semigroup. The Faedo-Galerkin approximations of a solution to the particular determistic case of (1) where $\beta = 1$ and $f(t, u) = M(u)$

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has been considered by Milleta [11]. The more general case has been dealt with by D. Bahuguna, S.K. Srivastava and S. Singh [4].

Aftereffect or dead-time in the dynamical behavior of a system is studied through delay differential equations. Examples of such systems are hereditary systems, systems modeled by equations with deviating argument or differential-difference equations. They belong to the class of functional differential equations (FDEs) which are infinite dimensional, as against ordinary differential equations (ODEs). In the case of ODEs, the state is a n-vector $x(t)$ moving in Euclidean space $\mathbb{R}^n$. In order to consider an irreducible past effect, deviated time-argument is introduced. Then the state cannot be represented by a vector $x(t)$ defined at a discrete value of time t. Therefore, in FDEs the state must be a function x_t corresponding to a past time interval. In certain real world problems, delay depends not only on the time but also on the unknown quantity as we can see in [8]. [8, 9] can be referred for related work with deviated argument. In this case, the state function need not be simply a past action, but it can express a desired future goal or target.

By far the Faedo-Galerkin approximation of solution stochastic fractional differential equation with deviated argument is neglected in literature. In an attempt to fill this gap we study the following stochastic fractional differential equation with deviated argument in a separable Hilbert space $(H, (.,.))$.

$$\begin{aligned} {}^cD_t^\beta u(t) + Au(t) &= f(t, u(t), u(h(u(t), t)))\frac{dw(t)}{dt}, \ t \in [0, T] \\ u(0) &= u_0 \in H \end{aligned} \tag{1}$$

where $0 < \beta < 1$ and $0 < T < \infty$. ${}^cD_t^\beta$ denotes the Caputo fractional derivative of order β and $A : D(A) \subset X \to H$ is a linear operator. A and the functions f, h are defined in the hypotheses $(H1) - (H3)$ of section 2.

2. Preliminaries

In this section we recall a lemma, define the mild solution and few hypotheses. We deal with two separable Hilbert spaces H and K. We define the space H_α as $D(A^\alpha)$ endowed with the norm $\|.\|_\alpha$. Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space endowed with complete family of right continuous increasing sub $\sigma-$ algebras $\{\mathfrak{F}_t, t \in J\}$ such that $\mathfrak{F}_t \subset \mathfrak{F}$. A $H-$ valued random variable is a $\mathcal{F}-$ measurable process.We also assume that W is a Wiener process on K with covariance operator Q. Suppose Q is symmetric, positive, linear, and bounded operator with $TrQ < \infty$. Let $K_0 = Q^{\frac{1}{2}}(K)$. The space $L_2^0 = L_2(K_0, H_\alpha)$ is a separable Hilbert space with norm $\|\psi\|_{L_2^0} = \|\psi Q^{\frac{1}{2}}\|_{L_2(K,H_\alpha)}$. Let $L_2(\Omega, \mathfrak{F}, P; H_\alpha) \equiv L_2(\Omega; H_\alpha)$ be the Banach space of all strongly measurable, square integrable, $H_\alpha-$valued random variables equipped with the norm $\|u(.)\|_{L_2}^2 = E\|u(.;w)\|_{H_\alpha}^2$. C_T^α denotes the Banach space of all continuous maps from $J = (0, T]$ into $L_2(\Omega; H_\alpha)$ which satisfy $sup_{t\in J} E\|u(t)\|_{C^\alpha}^2 < \infty$. $L_2^0(\Omega, H_\alpha) = \{f \in L_2(\Omega, H_\alpha) : f \ is \ \mathcal{F}_0 - measurable\}$ denotes an important subspace. For $0 \le \alpha < 1$ define

$$C_T^{\alpha-1} = \{u \in C_T^\alpha : \|u(t) - u(s)\|_{\alpha-1} \le L|t - s|, \forall t, s \in [0, T]\}.$$

We assume the following hypotheses:

(H1) A is a closed, densely defined, self adjoint operator with pure point spectrum $0 \le \lambda_0 \le \lambda_1 \le \cdots \le \lambda_m \le \cdots$ with $\lambda_m \to \infty$ and $m \to \infty$ and corresponding complete orthonormal system of eigenfunctions ϕ_j such that

$$A\phi_j = \lambda_j \phi_j \ and \ < \phi_i, \phi_j >= \delta_{i,j}$$

(H2) The function $f : [O,T] \times H_\alpha \times H_{\alpha-1} \to L(K,H)$ is continuous and $\exists$ constant L_f such that
$\|f(s,u,u_1) - f(s,v,v_1)\|_Q^2 \leq L_f[\|t-s\|^{\theta_1} + \|u-v\|_\alpha + \|u_1 - v_1\|_{\alpha-1}]$

(H3) The map $h : H_\alpha \times \mathcal{R}_+ \to \mathcal{R}_+$ satisfies $\|h(u,t) - h(v,s)\| \leq L_h(\|u-v\|_\alpha + |t-s|^{\theta_2})$

If $(H1)$ is satisfied then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA} : t \geq 0\}$ in H. We also note that $\exists$ constant C such that $\|S(t)\| \leq Ce^{\omega t}$ and constants C_i's such that $\|\frac{d^i}{dt^i}S(t)\| \leq C_i, \ t > 0, \ i = 1,2$. Also $\|AS(t)\| \leq Ct^{-1}$ and $\|A^\alpha S(t)\| \leq C_\alpha t^{-\alpha}$.

Now let us define mild solution of (1) :

Definition 1 The mild solution of (1) is a continuous $\mathfrak{F}_t$ adapted stochastic process $u \in C_T^\alpha \cap C_T^{\alpha-1}$ which satisfies the following:

(1) $u(t) \in H_\alpha$ has *Càdlàg* paths on $t \in [0,T]$.
(2) $\forall t \in [0,T], \ u(t)$ is the solution of the integral equation

$$u(t) = T_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f(s, u(s), u(h(u(s),s)))dw(s), \ t \in [0,T] \quad (2)$$

where $S_\beta(t) = \int_0^\infty \zeta_\beta(\theta) S(t^\beta \theta) d\theta$; and $T_\beta(t) = q\int_0^\infty \theta \zeta_\beta(\theta) S(t^\beta \theta) d\theta$; ζ_β is a probability density function defined on $(0,\infty)$, i.e. $\zeta_\beta(\theta) \geq 0, \theta \in (0,\infty)$ and $\int_0^\infty \zeta_\beta(\theta) d\theta = 1$. Also $\|T_\beta(t)u\| \leq C\|u\|$, $\|S_\beta(t)u\| \leq \frac{\beta C}{\Gamma(1+\beta)}\|u\|$, $\|A^\alpha S_\beta(t)u\| \leq \frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} t^{-\alpha\beta}\|u\|$.

Lemma 2.1[5] Let $f : J \times \Omega \times \Omega \to L_2^0$ be a strongly measurable mapping with $\int_0^T E\|f(t)\|_{L_2^0}^p dt < \infty$. Then

$$E\|\int_0^t f(s)dw(s)\|^p \leq l_s \int_0^t E\|f(s)\|_{L_2^0}^p ds$$

$\forall t \in [0,T]$ and $p \geq 2$ where l_s is a constant containing p and T.
l_s is incorporated into the constants in the following sections.

3. Existence and Uniqueness of Approximate Solutions

In this section we consider a sequence of approximate integrals and establish the existence and uniqueness of solution for each of the approximate integral equations. For $0 \leq \alpha < 1$ and $u \in C_{T_0}^\alpha$, the hypotheses $(H2) - (H3)$, imply that $f(s, u(s), u(h(u(s),s)))$ is continuous on $[0,T_0]$. Therefore $\exists$ a positive constant

$$N = 2L_f[T_0^{\theta_1} + 2R(1 + LL_h) + LL_h T_0^{\theta_2}] + 2N_0, \ N_0 = E\|f(0,u_0,u_0)\|^2$$

such that $\|f(s,u(s),u(h(u(s),s)))\| \leq N, \ t \in [0,T]$. Choose T_0, $0 < T_0 \leq T$ such that

$$(\frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))})^2 N \frac{T_0^{\beta(1-\alpha)-1}}{\beta(1-\alpha)-1} \leq \frac{R}{4},$$

$$D = (\frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))})^2 2L_f \frac{T_0^{\beta(1-\alpha)-1}}{2\beta(1-\alpha)-1} \leq 1 \quad (3)$$

Let

$$B_R = \{u \in \mathcal{C}_{T_0}^\alpha \cap \mathcal{C}_{T_0}^{\alpha-1} : u(0) = u_0, \ \|u - u_0\|_{T_0,\alpha} \leq R\}$$

It is easy to see that B_R is a closed and bounded subset of $\mathcal{C}_{T_0}^{\alpha-1}$ and complete. Let us define the operator $\mathcal{F}_n : B_R :\to B_R$ by

$$(\mathcal{F}_n u)(t) = T_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s, u(s), u(h(u(s), s)))dw(s). \quad (4)$$

Theorem 3.1 If the hypotheses $(H1), (H2)$ *and* $(H3)$ are satisfied and $u_0 \in L_2^0(\Omega, X_\alpha)$, $0 \le \alpha < 1$, then $\exists$ a unique $u_n \in B_R$ such that $\mathcal{F}_n u_n = u_n$, $\forall$ $n = 0, 1, 2, \cdots$, i.e., u_n satisfies the approximate integral equation

$$u_n(t) = T_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s, u_n(s), u_n(h(u_n(s), s)))dw(s), \quad t \in [0, T] \quad (5)$$

Proof. Step1 : We need to show that $\mathcal{F}_n u \in \mathcal{C}_{T_0}^{\alpha-1}$, $\forall u \in \mathcal{C}_{T_0}^{\alpha-1}$. It is easy to check that $\mathcal{F}_n : \mathcal{C}_T^\alpha \to \mathcal{C}_T^\alpha$. If $u \in \mathcal{C}_{T_0}^{\alpha-1}$, $0 < t_1 < t_2 < T_0$ and $0 \le \alpha < 1$ then

$$\begin{aligned}
&E\|\mathcal{F}_n u(t_2) - \mathcal{F}_n u(t_1)\|_{\alpha-1}^2 \\
&\le 3E\|[T_\beta(t_2) - T_\beta(t_1)]u_0\|_{\alpha-1}^2 \\
&\quad + 3E\|\int_{t_1}^{t_2} (t_2-s)^{\beta-1} A^{\alpha-1} S_\beta(t_2-s) f_n(s, u(s), u(h(u(s), s)))dw(s)\|_Q^2 \\
&\quad + 3E\|\int_0^{t_1} A[(t_2-s)^{\beta-1} S_\beta(t_2-s) - (t_1-s)^{\beta-1} S_\beta(t_1-s)] \\
&\quad A^{\alpha-2} \times f_n(s, u(s), u(h(u(s), s)))dw(s)\|_Q \\
&\le 3E\|[T_\beta(t_2) - T_\beta(t_1)]u_0\|_{\alpha-1}^2 + 3\frac{\beta^2 C_\alpha^2 \Gamma^2(2-\alpha)}{\Gamma^2(1+\beta(1-\alpha))} \int_{t_1}^{t_2} \|(t_2-s)^{2\beta(1-\alpha)-2}\| \\
&\quad \times \|A^{-1}\|^2 E\|f_n(s, u(s), u(h(u(s), s)))\|^2 ds \\
&\quad + 3\int_0^{t_1} \|A[(t_2-s)^{\beta-1} S_\beta(t_2-s) - (t_1-s)^{\beta-1} S_\beta(t_1-s)] \\
&\quad \times \|A^{\alpha-2}\|^2 E\|f_n(s, u(s), u(h(u(s), s)))\|^2 ds
\end{aligned} \quad (6)$$

$\forall u \in H$, we can write

$$[S(t_2^\beta\theta) - S(t_1^\beta\theta)]u = \int_{t_1}^{t_2} \frac{d}{dt} S(t^\beta\theta)u dt = \int_{t_1}^{t_2} \theta\beta t^{\beta-1} A S(t^\beta\theta) dt.$$

The first term of (6) can be estimated as follows

$$\begin{aligned}
\|[T_\beta(t_2) - T_\beta(t_1)]u_0\|_{\alpha-1}^2 &\le (\int_0^\infty \zeta_\beta(\theta)\|S(t_2^\beta\theta) - S(t_1^\beta\theta)\|\|A^{\alpha-1}u_0\|d\theta)^2 \\
&\le (\int_0^\infty \zeta_\beta(\theta)[\int_{t_1}^{t_2} \|\frac{d}{dt} S(t^\beta\theta)\|dt]\|u_0\|_\alpha d\theta)^2 \\
&\le C_1^2 \|u_0\|_{\alpha-1}^2 (t_2-t_1)^2
\end{aligned} \quad (7)$$

For the second term of (6) we get the following estimate

$$\begin{aligned}
&\int_{t_1}^{t_2} (t_2-s)^{2\beta(1-\alpha)-2} E\|f_n(s, u(s), u(h(u(s), s)))\|^2 ds \\
&\quad \le \frac{N(t_2-t_1)^{2\beta(1-\alpha)-1}}{2\beta(1-\alpha)-1}
\end{aligned} \quad (8)$$

For the third term we will use the following estimate

$$\int_0^{t_1} \|A[(t_2-s)^{\beta-1}S_\beta(t_2-s)-(t_1-s)^{\beta-1}S_\beta(t_1-s)]\|^2$$
$$\times \|A^{\alpha-2}\|^2 E\|f_n(s,u(s),u(h(u(s),s)))\|^2 ds$$
$$\leq \int_0^{t_1}\Big(\int_0^\infty \zeta_\beta(\theta)\|[\frac{d}{dt}S((t-s)^\beta\theta)|_{t=t_2}-\frac{d}{dt}S((t-s)^\beta\theta)|_{t=t_1}]\|d\theta\Big)^2$$
$$\times E\|f(s,u(s),u(h(u(s),s)))\|^2 ds$$
$$\leq \int_0^{t_1}\Big(\int_0^\infty \zeta_\beta(\theta)[\int_{t_1}^{t_2}\|A^{\alpha-2}\frac{d^2}{dt^2}S((t-s)^\beta\theta)\|dt]d\theta\Big)^2 N ds$$
$$\leq C_2^2\|A^{\alpha-2}\|^2(t_2-t_1)^2 NT_0 \tag{9}$$

Hence from inequalities (7)-(9) we see that the map $\mathcal{F}_n : \mathcal{C}_{T_0}^{\alpha-1} \to \mathcal{C}_{T_0}^{\alpha-1}$ is well-defined. Now we prove that $\mathcal{F}_n : B_R \to B_R$. So for $t \in [0,T_0]$ and $u \in B_R$.

$$E\|(\mathcal{F}_n u)(t)-u_0\|_\alpha^2$$
$$\leq 2E\|(T_\beta(t)-I)u_0\|_\alpha^2$$
$$+2E\|\int_0^t (t-s)^{\beta-1}S_\beta(t-s)f(s,u(s),u(h(u(s),s)))dw(s)\|_Q^2$$
$$\leq 2E\|(T_\beta(t)-I)u_0\|_\alpha^2 + 2\Big(\frac{\beta C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\Big)^2\int_0^t\|(t_2-s)^{2\beta(1-\alpha)-2}\|^2$$
$$\times E\|f_n(s,u(s),u(h(u(s),s)))\|^2 ds$$
$$\leq \frac{R}{2}+2\Big(\frac{\beta C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\Big)^2 N\frac{T_0^{\beta(1-\alpha)-1}}{\beta(1-\alpha)-1}\leq\frac{R}{2}+\frac{R}{2}=R$$

Now we show that $\mathcal{F}_n$ is a contraction map by using (3) in last but one inequality. $\forall u,v\in B_R$

$$E\|(\mathcal{F}_n u)(t)-(\mathcal{F}_n v)(t)\|_\alpha^2 = E\|\int_0^t (t-s)^{\beta-1}A^\alpha S_\beta(t-s)$$
$$\times[f(s,u(s),u(h(u(s),s)))-f(s,v(s),v(h(v(s),s)))dw(s)]\|_Q^2$$
$$\leq\Big(\frac{\beta C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\Big)^2\int_0^t (t_2-s)^{2\beta(1-\alpha)-2}$$
$$\times E\|f(s,u(s),u(h(u(s),s)))-f(s,v(s),v(h(v(s),s)))\|^2 ds$$
$$\leq\Big(\frac{\beta C_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\Big)^2 2L_f(1+2LLh)\|u-v\|_\alpha^2\frac{T^2\beta(1-\alpha)-1}{2\beta(1-\alpha)-1}$$
$$\leq\|u-v\|_\alpha^2.$$

This implies that there exists a unique fixed point u_n of $\mathcal{F}_n$. Thus there a unique mild approximate solution of (1) □

Lemma 3.2 Let $(H1)-(H3)$ hold. If $u_0\in L_2^0(\Omega,D(A^\alpha))$, $\forall 0<\alpha<\eta<1$, then $u_n(t)\in D(A^\gamma)$ for all $t\in[0,T_0]$ with $0<\gamma<\eta<1$. Also if $u_0\in D(A)$, then $u_n(t)\in D(A^\gamma)$ $\forall t\in[0,T_0]$, where $0<\gamma<\eta<1$.

Proof. By Theorem 3.1 we get the existence of a unique $u_n\in B_R$, satisfying (5). Theorem 2.6.13 of [12] implies for $t>0$, $0\leq\gamma<1$, $S(t): H\to D(A^\gamma)$ and for

$0 \le \gamma < \eta < 1$, $D(A^\eta) \subset D(A^\gamma)$. It is easy to see that Holder continuity of u_n can be proved using the similar arguments from (6)-(9). Also from Theorem 1.2.4 in [12], we have $S(t)u \in D(A)$ if $u \in D(A)$. The result follows from these facts and that $D(A) \subset D(A^\gamma)$ for $0 \le \gamma < 1$. □

Lemma 3.3 Let $(H1)-(H3)$ hold and $u_0 \in L_2^0(\Omega, X_\alpha)$. Then for any $t_0 \in (0, T_0]$ $\exists$ a constant U_{t_0}, independent of n such that $E\|u_n(t)\|_\gamma^2 \le U_{t_0}\ \forall t \in [t_0, T_0],\ n = 1, 2, \cdots$. Also if $u_0 \in L_2^0(\Omega, D(A))$ then $\exists$ constant U_0 independent of n such that $E\|u_n(t)\|_\gamma^2 \le U_0\ \forall t \in [t_0, T_0],\ n = 1, 2, \cdots ,\ \forall\ 0 < \gamma \le 1$.

Proof. Let $u_0 \in L_2^0(\Omega, H_\alpha)$. Applying A^γ on both sides of (4)

$$\begin{aligned} &E\|u_n(t)\|_\gamma^2 \\ &\le 2E\|T_\beta(t)u_0\|_\gamma^2 + 2\|\int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s, u(s), u(h(u(s), s)))dw(s)\|_Q^2 \\ &\le 2C_\gamma^2 t_0^{-2\gamma\beta}\|u_0\|^2 + \left(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))}\right)^2 \frac{N(T_0)^{2\beta(1-\gamma)-1}}{2\beta(1-\gamma)-1} = U_{t_0}. \end{aligned}$$

Also if $u_0 \in L_2^0(\Omega, D(A))$, then we have that $u_0 \in L_2^0(\Omega, D(A^\gamma))$ for $0 \le \gamma < 1$. Hence,

$$\begin{aligned} &E\|u_n(t)\|_\gamma^2 \\ &\le 2E\|T_\beta(t)u_0\|_\gamma^2 + 2\|\int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s, u(s), u(h(u(s), s)))dw(s)\|_Q^2 \\ &\le 2C^2\|u_0\|^2 + \left(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))}\right)^2 \frac{N(T_0)^{2\beta(1-\gamma)-1}}{2\beta(1-\gamma)-1} = U_0. \end{aligned}$$

Hence proved. □

4. Convergence of Solutions

In this section the convergence of the solution $u_n \in H_\alpha$ of the approximate integral equation (5) to a unique solution u of (2), is discussed.

Theorem 4.1 Let the hypotheses $(H1) - (H3)$ hold and if $u_0 \in L_2^0(\Omega, H_\alpha)$ then $\forall t_0 \in (0, T]$,

$$\lim_{m\to\infty} \sup_{\{n\ge M, t_0\le t\le T_0\}} \|u_n(t) - u_m(t)\|_\alpha = 0.$$

Proof. Let $0 < \alpha < \gamma < \eta$. For $t_0 \in (0, T_0]$

$$\begin{aligned} &E\|f_n(t, u_n(t), u_n(h(u_n(t), t))) - f_m(t, u_m(t), u_m(h(u_m(t), t)))\|^2 \\ &\quad \le 2E\|f_n(t, u_n(t), u_n(h(u_n(t), t))) - f_n(t, u_m(t), u_m(h(u_m(t), t)))\|^2 \\ &\quad \le 2E\|f_n(t, u_m(t), u_m(h(u_m(t), t))) - f_m(t, u_m(t), u_m(h(u_m(t), t)))\|^2 \\ &\quad \le 2(2L_f(1 + 2LL_h)[E\|u_n - u_m\|_\alpha^2 + E\|(P^n - P^m)u_m(t)\|_\alpha^2]) \end{aligned} \tag{10}$$

Now,

$$E\|(P^n - P^m)u_m(t)\|^2 \le E\|A^{\alpha-\gamma}(P^n - P^m)A^\gamma u_m(t)\|^2 \le \frac{1}{\lambda_m^{2(\gamma-\alpha)}} E\|A^\gamma u_m(t)\|^2$$

Then we have

$$
\begin{aligned}
&E\|f_n(t,u_n(t),u_n(h(u_n(t),t))) - f_m(t,u_m(t),u_m(h(u_m(t),t)))\|^2 \\
&\le 2(2L_f(1+2LL_h)[E\|u_n-u_m\|_\alpha^2 + \frac{1}{\lambda_m^{2(\gamma-\alpha)}}E\|A^\gamma u_m(t)\|^2])
\end{aligned}
$$

For $0<t_0'<t_0$

$$
\begin{aligned}
E\|u_n(t)-u_m(t)\|_\alpha^2 &\le 2(\int_0^{t_0'} + \int_{t_0'}^t)\|(t-s)^{\beta-1}A^\alpha S_\beta(t-s)\|^2 \\
&\quad \times E\|f_n(t,u_n(t),u_n(h(u_n(t),t))) - f_m(t,u_m(t),u_m(h(u_m(t),t)))\|^2 ds
\end{aligned} \tag{11}
$$

The estimate of first integral of the above inequality is

$$
\begin{aligned}
&E\|u_n(t)-u_m(t)\|_\alpha^2 \\
&\le \int_0^{t_0'} \|(t-s)^{\beta-1}A^\alpha S_\beta(t-s)\|^2 \\
&\quad \times E\|f_n(t,u_n(t),u_n(h(u_n(t),t))) - f_m(t,u_m(t),u_m(h(u_m(t),t)))\|^2 ds \\
&\le (\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2 \frac{2N(t_0-\delta_1 t_0')^{2\beta(1-\gamma)-2}}{2\beta(1-\gamma)-1} t_0', \ 0<\delta<1
\end{aligned} \tag{12}
$$

The estimate of second integral is

$$
\begin{aligned}
E\|u_n(t)-u_m(t)\|_\alpha^2 &\le \int_{t_0'}^t \|(t-s)^{\beta-1}A^\alpha S_\beta(t-s)\|^2 \\
&\quad \times E\|f_n(t,u_n(t),u_n(h(u_n(t),t))) - f_m(t,u_m(t),u_m(h(u_m(t),t)))\|^2 ds \\
&\le (\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2 \int_{t_0'}^t (t-s)^{2\beta(\alpha-1)-2} \\
&\quad \times 4L_f(1+2LL_h)[E\|u_n-u_m\|_\alpha^2 + \frac{E\|A^\gamma u_m(s)\|^2}{\lambda^2(\gamma-\alpha)}]ds \\
&\le 4L_f(1+2LL_h)(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2[\int_{t_0'}^t (t-s)^{2\beta(\alpha-1)-2} \\
&\quad \times E\|u_n-u_m\|_\alpha^2 ds + \frac{U_{t_0}}{\lambda_m^{2(\gamma-\alpha)}} \frac{T_0^{2\beta(1-\alpha)-1}}{2\beta(1-\alpha)-1}]
\end{aligned} \tag{13}
$$

Substituting inequalities (12),(13) in (11) we get

$$
\begin{aligned}
&E\|u_n(t)-u_m(t)\|_\alpha^2 \\
&\le (\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2 \frac{4N(t_0-\delta_1 t_0')^{2\beta(1-\gamma)-2}}{2\beta(1-\gamma)-1} t_0' \\
&\quad + 8L_f(1+2LL_h)(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2[\int_{t_0'}^t (t-s)^{2\beta(\alpha-1)-2} \\
&\times E\|u_n-u_m\|_\alpha^2 ds + \frac{U_{t_0}}{\lambda_m^{2(\gamma-\alpha)}} \frac{T_0^{2\beta(1-\alpha)-1}}{2\beta(1-\alpha)-1}]
\end{aligned}
$$

By using Gronwall's inequality, there exists a constant D such that

$$E\|u_n(t)-u_m(t)\|_\alpha^2 \le [(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2 \frac{4N(t_0-\delta_1 t_0')^{2\beta(1-\gamma)-2}}{2\beta(1-\gamma)-1} t_0' + 8L_f(1+2LL_h)(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2 \frac{U_{t_0}}{\lambda_m^{2(\gamma-\alpha)}} \frac{T_0^{2\beta(1-\alpha)-1}}{2\beta(1-\alpha)-1}] \times D$$

Let $m\to\infty$. Taking supremum over $[t_0,T_0]$ we get the following inequality.

$$E\|u_n(t)-u_m(t)\|_\alpha^2 \le [(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))})^2 \frac{4N(t_0-\delta_1 t_0')^{2\beta(1-\gamma)-2}}{2\beta(1-\gamma)-1} t_0'] \times D$$

Since t_0' is arbitrary, the right hand side can be made infinitesimally small by choosing t_0' sufficiently small. Thus the lemma is proved. □

Corollary 4.2 If $u_0 \in D(A)$, then $\lim_{m\to\infty} \sup_{\{n\ge m,\ 0\le t\le T_0\}} E\|u_n(t)-u_m(t)\|_\alpha^2 = 0$

Proof. By using Lemma (3.2) and Lemma (3.3) we can take $t_0 = 0$ in the proof of Theorem 4.1 and hence the corollary follows. □

Theorem 4.3 Let us assume that $(H1)-(H3)$ are satisfied and suppose $u_0 \in L_2^0(\Omega, X_\alpha)$. Then for $t\in[0,T_0]$, there exists a unique function $u_n \in B_R$ where
$u_n(t) = T_\beta u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s,u_n(s),u_n(h_n(u_n(s),s)))dw(s)$,
and $u(t)\in B_R$, where
$u(t) = T_\beta u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f(s,u(s),u(h(u(s),s)))dw(s), t\in[0,T_0]$, such that $u_n \to u$ as $n\to\infty$ in B_R and u satisfies (2) on $[0,T_0]$.

Proof. By using above Corollary, Theorem 3.1 and Theorem 4.1 it is to see that $\exists\ u(t)\in B_R$ such that
$\lim_{n\to\infty} E\|u_n(t)-u(t)\|_\alpha^2 = 0$ on $[0,T_0]$. Now

$$\begin{aligned} E\|u_n(t) - T_\beta u_0 + \int_{t_0}^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s,u_n(s),u_n(h_n(u_n(s),s)))dw(s)\|^2 \\ \le E\|\int_0^{t_0} (t-s)^{\beta-1} S_\beta(t-s) f_n(s,u_n(s),u_n(h_n(u_n(s),s)))dw(s)\|^2 \\ \le (\frac{\beta C}{\Gamma(1+\beta)})^2 N \frac{T_0^{2\beta-2}}{2\beta-2} t_0 \end{aligned} \tag{14}$$

Let $n\to\infty$ then
$E\|u_n(t) - T_\beta u_0 + \int_{t_0}^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s,u_n(s),u_n(h_n(u_n(s),s)))dw(s)\|^2 \le (\frac{\beta C}{\Gamma(1+\beta)})^2 N \frac{T_0^{2\beta-2}}{2\beta-2} t_0$ and since t_0 is arbitrary we conclude $u(t)$ satisfies (2). Uniqueness follows easily from Theorem 3.1, Theorem 4.1 and Gronwall's inequality. □

5. Faedo-Galerkin Approximations

We know from the previous sections that for any $0\le T_0\le T$, we have a unique $u\in C_{T_0}^\alpha$ satisfying the integral equation
$u(t) = T_\beta u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f(s,u(s),u(h(u(s),s)))dw(s),\ t\in[0,T_0]$ Also, $\exists$ a unique solution $u_n \in C_{T_0}^\alpha$ of the approximate integral equation
$u_n(t) = T_\beta u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s,u_n(s),u_n(h(u_n(s),s)))dw(s),\ t\in[0,T_0]$.
Faedo-Galerkin approximation $\bar{u}_n = P^n u_n$ is given by

$P^n u_n(t) = \bar{u}_n(t) = T_\beta(t)P^n u_0$
$+\int_0^t (t-s)^{\beta-1} S_\beta(t-s)P^n f(s, u_n(s), u_n(h(u_n(s), s)))dw(s), t \in [0, T_0]$. If the solution $u(t)$ to (2) exists on $[0, T_0]$ then it has the representation
$u(t) = \sum_{i=0}^{\infty} \alpha_i(t)\phi_i$, where $\alpha_i(t) = (u(t), \phi_i)$ for $i = 0, 1, 2, 3, \cdots$ and
$\bar{u}_n(t) = \sum_{i=0}^{n} \alpha_i^n(t)\phi_i$, where $\alpha_i^n(t) = (\bar{u}_n(t), \phi_i)$ for $i = 0, 1, 2, 3, \cdots$.
As a consequence of Theorem 3.1 and Theorem 4.1, we have the following result.

Theorem 4.4 Let us assume that $(H1) - (H3)$ are satisfied and suppose $u_0 \in L_2^0(\Omega, X_\alpha)$. Then for $t \in [0, T_0]$, $\exists$ a unique function $u_n \in B_R$ where
$u_n(t) = T_\beta P^n u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s)P^n f_n(s, u_n(s), u_n(h(u_n(s), s)))dw(s)$,
and $u(t) \in B_R$, where
$u(t) = T_\beta u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s)f(s, u(s), u(h(u(s), s)))dw(s), t \in [0, T_0]$, such that $u_n \to u$ as $n \to \infty$ in B_R and u satisfies (2) on $[0, T_0]$.

Now the convergence of $\alpha_i^n(t) \to \alpha_i(t)$ is shown. It is easily seen that
$A^\alpha[u(t) - \bar{u}_n(t)] = A^\alpha\big[\sum_{i=0}^{n}\{\alpha_i(t) - \alpha_i^n(t)\}\phi_i\big] + A^\alpha \sum_{i=n+1}^{\infty} \alpha_i(t)\phi_i$
$= \sum_{i=0}^{n} \lambda_i^\alpha\{\alpha_i(t) - \alpha_i^n(t)\}\phi_i + \sum_{i=n+1}^{\infty} \lambda_i^\alpha \alpha_i(t)\phi_i$. Thus we have
$E\|A^\alpha[u(t) - \bar{u}_n(t)\|^2 \geq \sum_{i=0}^n \lambda_i^{2\alpha} E|\alpha_i(t) - \alpha_i^n(t)|^2$.

Theorem 4.5 Let us assume $(H1) - (H3)$ hold.
(i) If $u_0 \in L_2^0(\Omega, X_\alpha)$ then $\lim_{n\to\infty} \sup_{t\in[t_0,T_0]} [\sum_{i=0}^{n} \lambda_i(t)^{2\alpha} E\|\alpha_i(t) - \alpha_i^n(t)\|^2] = 0$
(ii) If $u_0 \in L_2^0(\Omega, D(A))$ then $\lim_{n\to\infty} \sup_{t\in[0,T_0]} [\sum_{i=0}^{n} \lambda_i(t)^{2\alpha} E\|\alpha_i(t) - \alpha_i^n(t)\|^2] = 0$. The theorem 4.5 follows from the facts mentioned above the theorem. **Corollary 4.6** Let us assume $(H1) - (H3)$ hold.
(i) If $u_0 \in L_2^0(\Omega, X_\alpha)$ then $\lim_{n\to\infty} \sup_{t\in[t_0,T_0], n\geq m} E\|A^\alpha[\bar{u}_n(t) - \bar{u}_m(t)]\|^2 = 0$
(ii) If $u_0 \in L_2^0(\Omega, D(A))$ then $\lim_{n\to\infty} \sup_{t\in[0,T_0], n\geq m} E\|A^\alpha[\bar{u}_n(t) - \bar{u}_m(t)]\|^2 = 0$

Proof.

$$\begin{aligned}
E\|A^\alpha[\bar{u}_n(t) - \bar{u}_m(t)]\|^2 &= E\|P^n u_n(t) - P^m u_m(t)\|_\alpha^2 \\
&\leq 2E\|P^n[u_n(t) - u_m(t)]\|_\alpha^2 + 2E\|(P^n - P^m)y_m(t)\|_\alpha^2 \\
&\leq 2E\|[u_n(t) - u_m(t)]\|_\alpha^2 + 2\frac{1}{\lambda_m^{\gamma-\alpha}} E\|A^\gamma u_m(t)\|^2
\end{aligned}$$

Then the result (i) follows from theorem 4.1 and result (ii) follows from corollary 4.2. □

6. Example

Consider the following stochastic fractional differential equation with deviating argument. Suppose for $t \geq 0,\ x \in (0,1), 0 < \beta \leq 1$

$$
\begin{aligned}
{}^{c}D^{\beta} v_t(t,x) &= v_{xx}(t,x) + F(t, v(t,x), v(h(t, v(t,x)))) \frac{dw(t)}{dt}, \\
v(t,x) &= v_0,\ t = 0,\ x \in (0,1) \ and \ v(t,0) = v(t,1) = 0,\ t \geq 0 \qquad (15)
\end{aligned}
$$

Let F is an appropriate Holder continuous function satisfying $(H2)$ in $L_2^0(K,(0,1))$. w is a standard $L_2(0,1)$ valued Weiner process.

Let us define $A = -\frac{d^2}{dx^2}$, $f := F$, $v(t,x) = u(t)$ and assume $\alpha = 1/2$. Let $D(A) = H_0^1(0,1) \cap H^2(0,1)$, $D(A^{1/2}) = H_0^1(0,1)$, i.e. the Banach space endowed with the norm

$$\|x\|_{1/2} := \|A^{1/2}x\|,\ x \in D(A^{1/2}).$$

We denote this space by $X_{1/2}$.

Also denote $C_t^{1/2} = C(t,0; D(A^{1/2}))$ endowed with sup norm

$$\|x\|_{t,1/2} := \sup_{0 \leq s \leq t} \|x(s)\|_{1/2},\ x \in C_t^{1/2}.$$

When $v \in D(A),\ \lambda \in \mathbf{R}$ with $Av = -v'' = \lambda v$ we have $< Av, v > = < \lambda v, v >$, i.e.

$$< -v'', v > = \|v'\|_{L^2}^2 = \lambda \|v\|_{L^2}^2.$$

Therefore the solution v of $Av = \lambda v$ is of the form

$$v(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x)$$

From the conditions $v(0) = v(1) = 0$ imply that $C = 0$ and $\lambda = \lambda_n = n^2\pi^2, n \in \mathbf{N}$. So, for each n the solution is

$$v_n(x) = D\sin(\sqrt{\lambda_n}x).$$

Also note that $< v_n, v_m > = 0$ for $n \neq m$ and $< v_n, v_n > = 1$. Therefore $D = \sqrt{2}$. For $v \in D(A),\ \exists$ a sequence of real numbers $\{a_n\}$ such that

$$v(x) = \sum_{n \in \mathbf{N}} a_n v_n(x),\ \sum_{n \in \mathbf{N}} (a_n)^2 < \infty,\ \sum_{n \in \mathbf{N}} (\lambda)^2 (a_n)^2.$$

So, $A^{1/2}v(x) = \sum_{n \in \mathbf{N}} \sqrt{\lambda}_n a_n v_n(x)$, with $v \in D(A^{1/2})$.
$X_{-1/2} = H^1(0,1)$ is a Sobolev space of negative index with equivalent norm $\|.\|_{-1/2} = \sum_{n=1}^{\infty} \| < ., v_n > \|^2$. Then (15) can be reformulated into (1). Now from Theorem 3.1 and Theorem 4.1 we can similarly prove the existence, uniqueness and approximation of the mild solution of (15).

7. Conclusion

Existence and uniqueness of approximate solutions of a prototype of stochastic fractional differential equation with deviating argument is established. By Faedo-Galerkin approximation of solution we proved some convergence results.

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