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SYMMETRY RESULTS FOR SOLUTIONS OF AN INTEGRAL SYSTEM

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Abstract. In this paper, we study the symmetry property of positive solutions for system

$$\begin{cases} \mathcal{L}_{K_1}u(x) = v(x)^p + g_1(x), & x \in B_1, \\ \mathcal{L}_{K_2}v(x) = u(x)^q + g_2(x), & x \in B_1, \\ u(x) = v(x) = 0, & x \in B_1^c, \end{cases} \tag{1}$$

where $p, q > 1$, the domain B_1 denotes the open unit ball centered at the origin in $\mathbb{R}^N (N \geq 2)$ and the operator $\mathcal{L}_{K_i}$ is a nonlocal operator defined by

$$\mathcal{L}_{K_i}u(x) = P.V. \int_{\mathbb{R}^N} (u(x) - u(y))K_i(x - y)dy, \tag{2}$$

for $i = 1, 2$, where the kernel K_i satisfies that

$$K_i(x) = \begin{cases} |x|^{-N-2\alpha_i}, & x \in B_r, \\ \theta_i(x), & x \in B_r^c, \end{cases} \tag{3}$$

with $r > 0$, $\alpha_i \in (0, 1)$ and $\theta_i \in L^1(B_r^c)$ being a nonnegative and radially symmetric function such that K_i is decreasing. The functions g_1 and g_2 are radially symmetric and decreasing in $|x|$.

1. Introduction

The purpose of this paper is to study symmetry results of positive solutions for the system

$$\begin{cases} \mathcal{L}_{K_1}u(x) = v(x)^p + g_1(x), & x \in B_1, \\ \mathcal{L}_{K_2}v(x) = u(x)^q + g_2(x), & x \in B_1, \\ u(x) = v(x) = 0, & x \in B_1^c, \end{cases} \tag{4}$$

where $p, q > 1$ and B_1 is the open unit ball centered at the origin in $\mathbb{R}^N$ with $N \geq 2$. For $i = 1, 2$, the operator $\mathcal{L}_{K_i}$ is a nonlocal operator defined by

$$\mathcal{L}_{K_i}u(x) = P.V. \int_{\mathbb{R}^N} (u(x) - u(y))K_i(x - y)dy, \tag{5}$$

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where the kernel K_i satisfies that

$$K_i(x) = \begin{cases} |x|^{-N-2\alpha_i}, & x \in B_r, \\ \theta_i(x), & x \in B_r^c, \end{cases} \tag{6}$$

with $r > 0$, $\alpha_i \in (0,1)$ and $\theta_i \in L^1(B_r^c)$ being a nonnegative and radially symmetric function such that K_i is decreasing. We remark that the operator $\mathcal{L}_{K_i}$ is the fractional Laplacian $(-\Delta)^{\alpha_i}$ when $\theta_i(x) = |x|^{-N-2\alpha_i}$. The functions g_1 and g_2 satisfy that

(G) For $i = 1, 2$, the function $g_i : B_1 \to \mathbb{R}$ is radially symmetric and decreasing in $|x|$.

It is of interest to study symmetry property of positive solutions for nonlinear elliptic equations by the method of moving planes. For the problem in bounded domain, radial symmetry of positive solutions has been extensively studied by numerous authors using the method of moving planes based on the Maximum Principle for small domain which is derived by the Aleksandrov-Bakelman-Pucci (ABP) estimate. We mentioned the initiated work by Serrin [15], Gidas-Ni-Nirenberg [6] and Beresticky-Nirenberg [1]. For the problem in the whole space, the Maximum Principle for small domain is not available and then the procedure of moving planes started in a different way. We refer to the work by Li [9], Gidas-Ni-Nirenberg [7], Li-Ni [10] and Pacella-Ramaswamy [14]. Recently, a great attention has been focused on the study of radial symmetry of positive solutions to equations involving fractional Laplacian or general integro-differential operator, see the results obtained by Li [11], Chen-Li-Ou [2], Fall-Jarohs [3], Ma-Chen [12, 13], Felmer-Quaas-Tan [4] and Sire-Valdinoci [16]. They obtained the symmetry results by the method of moving planes in integral form where the representation formula for solution is given by the kernel plays a key role in the procedure. More recently, Felmer-Wang [5] applied ABP estimate which has been proved by Guillen-Schwab [8] with the method of moving planes as in [1] to obtain radial symmetry and monotonicity properties of positive solutions for system involving the fractional Laplacian.

Motivated by the work mentioned above, we study the radial symmetry results of positive solutions for the system (4) involving nonlocal operators by the method of moving planes in the present paper. Before stating our theorem we make precise the notion of solution that we use in this article. We say that a pair $(u, v) \in C(\mathbb{R}^N) \times C(\mathbb{R}^N)$ is a classical solution of system (4) if $\mathcal{L}_{K_1} u$ and $\mathcal{L}_{K_2} v$ are well defined at any point of B_1, according to the definition given in (5) and if (u, v) satisfies the system (4) in a pointwise sense. Now we state our main theorem as follows

Theorem 1 Suppose that $p, q > 1$ and the functions g_1, g_2 satisfy (G). If (u, v) is a positive classical solution of system (4), then u and v are radially symmetric and strictly decreasing in $r = |x|$ for $r \in (0, 1)$.

We prove Theorem 1 by the method of moving planes. Our main idea is to transform the nonlocal operator $\mathcal{L}_{K_i}$ into the fractional Laplacian $(-\Delta)^{\alpha_i}$ and then using the ABP estimate for the equations involving the fractional Laplacian to start the moving planes. The difficulty is to combine the transforming above with a truncation technique in the procedure of moving planes.

2. Proof of Theorem 1

In this section, we prove symmetry property of positive solutions of (4) by the method of moving planes. First, we recall the ABP estimate for the equations involving the fractional Laplacian which plays a key role in the procedure of moving planes.

Proposition 1 Let Ω be an open bounded domain of $\mathbb{R}^N$ and $\alpha \in (0,1)$. Suppose that $h : \Omega \to \mathbb{R}$ is in $L^\infty(\Omega)$ and $w \in L^\infty(\mathbb{R}^N)$ satisfies

$$\begin{cases} -(-\Delta)^\alpha w(x) \leq h(x), & x \in \Omega, \\ w(x) \geq 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{7}$$

Then there exists $C > 0$ such that

$$-\inf_{\Omega} w \leq C\|h_+\|_{L^\infty(\Omega)}|\Omega|^{\frac{\alpha}{N}}, \tag{8}$$

where $h_+(x) = \max\{h(x), 0\}$.

Proof. By Theorem 9.1 in [8], there exists $C_0 > 0$ such that

$$\begin{aligned} -\inf_{\Omega} w &\leq C_0 d^\alpha \|h_+\|^{1-\alpha}_{L^\infty(\Omega)}\|h_+\|^{\alpha}_{L^N(\Omega)} \\ &\leq C_0 d^\alpha \|h_+\|_{L^\infty(\Omega)}|\Omega|^{\frac{\alpha}{N}}, \end{aligned}$$

where $d = \mathrm{diam}(\Omega)$. Taking $C = C_0 d^\alpha$, we complete the proof. □

Now we give the proof of Theorem 1 by the method of moving planes. To this end, we denote $\lambda \in (0,1)$,

$$\Sigma_\lambda = \{x = (x_1, x') \in B_1 \mid x_1 > \lambda\},$$

$$T_\lambda = \{x = (x_1, x') \in B_1 \mid x_1 = \lambda\},$$

$$w_{\lambda,u}(x) = u(x_\lambda) - u(x) \quad \text{and} \quad w_{\lambda,v}(x) = v(x_\lambda) - v(x),$$

where $x_\lambda = (2\lambda - x_1, x')$ for $x = (x_1, x') \in \mathbb{R}^N$.

Proof of Theorem 1. We will divide this proof into four steps.

Step 1. To prove that if λ is close to 1^-, then $w_{\lambda,u}, w_{\lambda,v} > 0$ in Σ_λ.

We first claim that $w_{\lambda,u}, w_{\lambda,v} \geq 0$ in Σ_λ when λ is close to 1. In fact, let us define

$$\Sigma^-_{\lambda,u} = \{x \in \Sigma_\lambda \mid w_{\lambda,u}(x) < 0\}, \qquad \Sigma^-_{\lambda,v} = \{x \in \Sigma_\lambda \mid w_{\lambda,v}(x) < 0\},$$

$$w^+_{\lambda,u}(x) = \begin{cases} w_{\lambda,u}(x), & x \in \Sigma^-_{\lambda,u}, \\ 0, & x \in \mathbb{R}^N \setminus \Sigma^-_{\lambda,u}, \end{cases} \qquad w^+_{\lambda,v}(x) = \begin{cases} w_{\lambda,v}(x), & x \in \Sigma^-_{\lambda,v}, \\ 0, & x \in \mathbb{R}^N \setminus \Sigma^-_{\lambda,v}, \end{cases}$$

and

$$w^-_{\lambda,u}(x) = w_{\lambda,u}(x) - w^+_{\lambda,u}(x), \qquad w^-_{\lambda,v}(x) = w_{\lambda,v}(x) - w^+_{\lambda,v}(x).$$

We observe that $w^-_{\lambda,u}=0$ in $\Sigma^-_{\lambda,u}$, then for $x\in\Sigma^-_{\lambda,u}$, we have that

$$\begin{aligned}\mathcal{L}_{K_1}w^-_{\lambda,u}(x) &= \int_{\mathbb{R}^N}(w^-_{\lambda,u}(x)-w^-_{\lambda,u}(z))K_1(x-z)dz\\ &= -\int_{\mathbb{R}^N\setminus\Sigma^-_{\lambda,u}} w^-_{\lambda,u}(z)K_1(x-z)dz\\ &= -\int_{(B_1\setminus(B_1)_\lambda)\cup((B_1)_\lambda\setminus B_1)} w_{\lambda,u}(z)K_1(x-z)dz\\ &\quad -\int_{(\Sigma_\lambda\setminus\Sigma^-_{\lambda,u})\cup(\Sigma_\lambda\setminus\Sigma^-_{\lambda,u})_\lambda} w_{\lambda,u}(z)K_1(x-z)dz\\ &\quad -\int_{(\Sigma^-_{\lambda,u})_\lambda} w_{\lambda,u}(z)K_1(x-z)dz\\ &= -I_1-I_2-I_3,\end{aligned}$$

where, for any subset A of $\mathbb{R}^N$, $A_\lambda:=\{x_\lambda : x\in A\}$, the reflection of A with regard to T_λ. Since $u(z)=0$ for $z\in(B_1)_\lambda\setminus B_1$ and $u(z_\lambda)=0$ for $z\in B_1\setminus(B_1)_\lambda$, we have that

$$\begin{aligned}I_1 &= \int_{(B_1\setminus(B_1)_\lambda)\cup((B_1)_\lambda\setminus B_1)}(u(z_\lambda)-u(z))K_1(x-z)dz\\ &= \int_{(B_1)_\lambda\setminus B_1} u(z_\lambda)K_1(x-z)dz-\int_{B_1\setminus(B_1)_\lambda} u(z)K_1(x-z)dz\\ &= \int_{(B_1)_\lambda\setminus B_1} u(z_\lambda)(K_1(x-z)-K_1(x-z_\lambda))dz\ge 0,\end{aligned}$$

the last inequality holds, since $u(z_\lambda)\ge 0$ and $|x-z_\lambda|>|x-z|$ for all $x\in\Sigma^-_\lambda$ and $z\in(B_1)_\lambda\setminus B_1$. Now we study the sign of I_2, we first observe that $w_{\lambda,u}(z_\lambda)=-w_{\lambda,u}(z)$ for any $z\in\mathbb{R}^N$, then

$$\begin{aligned}I_2 &= \int_{\Sigma_\lambda\setminus\Sigma^-_{\lambda,u}} w_{\lambda,u}(z)K_1(x-z)dz+\int_{(\Sigma_\lambda\setminus\Sigma^-_{\lambda,u})_\lambda} w_{\lambda,u}(z)K_1(x-z)dz\\ &= \int_{\Sigma_\lambda\setminus\Sigma^-_{\lambda,u}} w_{\lambda,u}(z)(K_1(x-z)-K_1(x-z_\lambda))dz\ge 0,\end{aligned}$$

the last inequality holds, since $w_{\lambda,u}\ge 0$ in $\Sigma_\lambda\setminus\Sigma^-_{\lambda,u}$ and $|x-z_\lambda|>|x-z|$ for all $x\in\Sigma^-_{\lambda,u}$ and $z\in\Sigma_\lambda\setminus\Sigma^-_{\lambda,u}$. Finally, by the fact that $w_{\lambda,u}(z)<0$ for $z\in\Sigma^-_{\lambda,u}$, we have that

$$I_3=\int_{\Sigma^-_{\lambda,u}} w_{\lambda,u}(z_\lambda)K_1(x-z_\lambda)dz=-\int_{\Sigma^-_{\lambda,u}} w_{\lambda,u}(z)K_1(x-z_\lambda)dz\ge 0.$$

As a consequence, we have that $\mathcal{L}_{K_1}w^-_{\lambda,u}(x)\le 0$ for any $x\in\Sigma^-_{\lambda,u}$. Since the operator $\mathcal{L}_{K_1}$ is linearity, we have that

$$\mathcal{L}_{K_1}w^+_{\lambda,u}(x)=\mathcal{L}_{K_1}w_{\lambda,u}(x)-\mathcal{L}_{K_1}w^-_{\lambda,u}(x)\ge\mathcal{L}_{K_1}w_{\lambda,u}(x),\quad x\in\Sigma^-_{\lambda,u}. \tag{9}$$

Combining with (9) and (4), then for $x\in\Sigma^-_{\lambda,u}$,

$$\begin{aligned}\mathcal{L}_{K_1}w^+_{\lambda,u}(x) &\ge \mathcal{L}_{K_1}u(x_\lambda)-\mathcal{L}_{K_1}u(x)\\ &= v(x_\lambda)^p+g_1(x_\lambda)-v(x)^p-g_1(x)\\ &\ge v(x_\lambda)^p-v(x)^p,\end{aligned} \tag{10}$$

where the last inequality holds by the condition (G).

On the other hand, by directly computation, we have that for $x \in \Sigma^-_{\lambda,u}$,

$$\begin{aligned} &\mathcal{L}_{K_1} w^+_{\lambda,u}(x) = \int_{\mathbb{R}^N} (w^+_{\lambda,u}(x) - w^+_{\lambda,u}(z))K_1(x-z)dz \\ = \;& (-\Delta)^{\alpha_1} w^+_{\lambda,u}(x) + \int_{B^c_r(x)} (w^+_{\lambda,u}(x) - w^+_{\lambda,u}(z))[\theta_1(x-z) - |x-z|^{-N-2\alpha_1}]dz \\ \le \;& (-\Delta)^{\alpha_1} w^+_{\lambda,u}(x) + 2C\|w^+_{\lambda,u}\|_{L^\infty(\Sigma^-_{\lambda,u})}, \end{aligned}$$

where $C = \int_{B^c_r} |\theta_1(y) - |y|^{-N-2\alpha_1}|dy$. Together with (10), for $x \in \Sigma^-_{\lambda,u}$, we have that

$$\begin{aligned} -(-\Delta)^{\alpha_1} w^+_{\lambda,u}(x) &\le 2C\|w^+_{\lambda,u}\|_{L^\infty(\Sigma^-_{\lambda,u})} - v(x_\lambda)^p + v(x)^p \\ &= 2C\|w^+_{\lambda,u}\|_{L^\infty(\Sigma^-_{\lambda,u})} - \varphi(x)w_{\lambda,v}(x), \end{aligned}$$

where $\varphi(x) = \frac{v(x_\lambda)^p - v(x)^p}{v(x_\lambda) - v(x)}$ if $v(x_\lambda) \ne v(x)$ and $\varphi(x) = 0$ if $v(x_\lambda) = v(x)$. We observe that φ is bounded. Moreover, by the definition of $w^+_{\lambda,v}$, we have that for $x \in \Sigma^-_{\lambda,u}$,

$$\begin{aligned} -(-\Delta)^{\alpha_1} w^+_{\lambda,u}(x) &\le 2C\|w^+_{\lambda,u}\|_{L^\infty(\Sigma^-_{\lambda,u})} - \varphi(x)w^+_{\lambda,v}(x) \\ &\le \bar{C}(\|w^+_{\lambda,u}\|_{L^\infty(\Sigma^-_{\lambda,u})} + \|w^+_{\lambda,v}\|_{L^\infty(\Sigma^-_{\lambda,v})}), \end{aligned}$$

where $\bar{C} = 2C + \|\varphi\|_{L^\infty(B_1)}$. By the definition of $w^+_{\lambda,u}$, we observe that $w^+_{\lambda,u} = 0$ in $\mathbb{R}^N \setminus \Sigma^-_{\lambda,u}$ and then use Proposition 1, there exists $C_1 > 0$ such that

$$\|w^+_{\lambda,u}\|_{L^\infty(\Sigma^-_{\lambda,u})} \le C_1|\Sigma^-_{\lambda,u}|^{\frac{\alpha_1}{N}}(\|w^+_{\lambda,u}\|_{L^\infty(\Sigma^-_{\lambda,u})} + \|w^+_{\lambda,v}\|_{L^\infty(\Sigma^-_{\lambda,v})}). \tag{11}$$

Similarly, we can obtain that

$$\|w^+_{\lambda,v}\|_{L^\infty(\Sigma^-_{\lambda,v})} \le C_1|\Sigma^-_{\lambda,v}|^{\frac{\alpha_2}{N}}(\|w^+_{\lambda,v}\|_{L^\infty(\Sigma^-_{\lambda,v})} + \|w^+_{\lambda,u}\|_{L^\infty(\Sigma^-_{\lambda,u})}). \tag{12}$$

Therefore, by (11) and (12), we have that

$$\begin{aligned} &\|w^+_{\lambda,u}\|_{L^\infty(\Sigma^-_{\lambda,u})} + \|w^+_{\lambda,v}\|_{L^\infty(\Sigma^-_{\lambda,v})} \\ \le\;& C_1(|\Sigma^-_{\lambda,u}|^{\frac{\alpha_1}{N}} + |\Sigma^-_{\lambda,v}|^{\frac{\alpha_2}{N}})(\|w^+_{\lambda,u}\|_{L^\infty(\Sigma^-_{\lambda,u})} + \|w^+_{\lambda,v}\|_{L^\infty(\Sigma^-_{\lambda,v})}). \end{aligned}$$

Now we take λ close enough to 1 such that $C_1(|\Sigma^-_{\lambda,u}|^{\frac{\alpha_1}{N}} + |\Sigma^-_{\lambda,v}|^{\frac{\alpha_2}{N}}) < \frac{1}{2}$, then $\|w^+_{\lambda,u}\|_{L^\infty(\Sigma^-_{\lambda,u})} = \|w^+_{\lambda,v}\|_{L^\infty(\Sigma^-_{\lambda,v})} = 0$, that is, $|\Sigma^-_{\lambda,u}| = |\Sigma^-_{\lambda,v}| = 0$. Since $\Sigma^-_{\lambda,u}$ and $\Sigma^-_{\lambda,v}$ are open, then $\Sigma^-_{\lambda,u}$ and $\Sigma^-_{\lambda,v}$ are empty. Thus, $w_{\lambda,u}, w_{\lambda,v} \ge 0$ in Σ_λ if λ close to 1.

Next we continue to prove that $w_{\lambda,u}, w_{\lambda,v} > 0$ in Σ_λ for λ close to 1. If this conclusion is not true, we may assume that there exists $x_0 \in \Sigma_\lambda$ such that $w_{\lambda,u}(x_0) = 0$. We denote $A_\lambda = \{(x_1, x') \in \mathbb{R}^N \mid x_1 > \lambda\}$. One hand, by directly computation, we obtain that

$$\begin{aligned} \mathcal{L}_{K_1} w_{\lambda,u}(x_0) &= -\int_{\mathbb{R}^N} w_{\lambda,u}(y)K_1(x_0-y)dy \\ &= -\int_{A_\lambda} w_{\lambda,u}(y)(K_1(x_0-y) - K_1(x_0-y_\lambda))dy < 0, \end{aligned} \tag{13}$$

where the last inequality holds since K_1 is decreasing and $w_{\lambda,u} \geq 0$, $w_{\lambda,u} \not\equiv 0$ in A_λ for λ close to 1. On the other hand, by the linearity of the operator $\mathcal{L}_{K_1}$ and (4), we have that

$$\begin{aligned}\mathcal{L}_{K_1} w_{\lambda,u}(x_0) &= \mathcal{L}_{K_1} u((x_0)_\lambda) - \mathcal{L}_{K_1} u(x_0)\\ &= v((x_0)_\lambda)^p + g_1((x_0)_\lambda) - v(x_0)^p - g_1(x_0)\\ &\geq v((x_0)_\lambda)^p - v(x_0)^p \geq 0,\end{aligned}$$

where the first inequality holds by (G) and $x_0 \in \Sigma_\lambda$, the second inequality holds by $x_0 \in \Sigma_\lambda$ and $w_{\lambda,v} \geq 0$ in Σ_λ for λ close to 1. This is impossible with (13). Then $w_{\lambda,u} > 0$ in Σ_λ for λ close to 1. Similarly, we can obtain that $w_{\lambda,v} > 0$ in Σ_λ for λ close to 1.

Step 2. We prove that $\lambda_0 := \inf\{\lambda \in (0,1) \mid w_{\lambda,u}, w_{\lambda,v} > 0 \text{ in } \Sigma_\lambda\} = 0$.
Proceeding by contradiction, we may assume that $\lambda_0 > 0$, then $w_{\lambda_0,u}, w_{\lambda_0,v} \geq 0$ and $w_{\lambda_0,u}, w_{\lambda_0,v} \not\equiv 0$ in Σ_{λ_0}. Thus, by the same argument in Step 1, we obtain that $w_{\lambda_0,u}, w_{\lambda_0,v} > 0$ in Σ_{λ_0}.

Now we claim that if $w_{\lambda,u}, w_{\lambda,v} > 0$ for $\lambda \in (0,1)$, then there exists $\epsilon \in (0,\lambda)$ such that $w_{\lambda_\epsilon,u}, w_{\lambda_\epsilon,v} > 0$ in $\Sigma_{\lambda_\epsilon}$, where $\lambda_\epsilon = \lambda_0 - \epsilon$. Indeed, let $D_\mu = \{x \in \Sigma_\lambda \mid dist(x, \partial\Sigma_\lambda) \geq \mu\}$ for $\mu > 0$ small. Since $w_{\lambda,u}, w_{\lambda,v} > 0$ in Σ_λ and D_μ is compact, then there exists $\mu_0 > 0$ such that $w_{\lambda,u}, w_{\lambda,v} \geq \mu_0$ in D_μ. By continuity of $w_{\lambda,u}(x)$ and $w_{\lambda,v}(x)$ respect to λ, for $\epsilon > 0$ small enough, we have that

$$w_{\lambda_\epsilon,u},\ w_{\lambda_\epsilon,v} \geq 0 \ \text{ in } \ \mathrm{D}_\mu,$$

then $\Sigma^-_{\lambda_\epsilon,u}, \Sigma^-_{\lambda_\epsilon,v} \subset \Sigma_{\lambda_\epsilon} \setminus D_\mu$, $|\Sigma^-_{\lambda_\epsilon,u}|$ and $|\Sigma^-_{\lambda_\epsilon,v}|$ are small if ϵ and μ are small.

We repeat the arguments in Step 1 to obtain

$$\begin{aligned}&\|w^+_{\lambda,u}\|_{L^\infty(\Sigma^-_{\lambda_\epsilon,u})} + \|w^+_{\lambda_\epsilon,v}\|_{L^\infty(\Sigma^-_{\lambda_\epsilon,v})}\\ \leq\ & C_1(|\Sigma^-_{\lambda_\epsilon,u}|^{\frac{\alpha_1}{N}} + |\Sigma^-_{\lambda_\epsilon,v}|^{\frac{\alpha_2}{N}})(\|w^+_{\lambda_\epsilon,u}\|_{L^\infty(\Sigma^-_{\lambda_\epsilon,u})} + \|w^+_{\lambda_\epsilon,v}\|_{L^\infty(\Sigma^-_{\lambda_\epsilon,v})}),\end{aligned}$$

where $C_1 > 0$. We choose ϵ and μ small such that

$$C_1(|\Sigma^-_{\lambda_\epsilon,u}|^{\frac{\alpha_1}{N}} + |\Sigma^-_{\lambda_\epsilon,v}|^{\frac{\alpha_2}{N}}) < \frac{1}{2},$$

then $\|w^+_{\lambda_\epsilon,u}\|_{L^\infty(\Sigma^-_{\lambda_\epsilon,u})} = \|w^+_{\lambda_\epsilon,v}\|_{L^\infty(\Sigma^-_{\lambda_\epsilon,v})} = 0$. Therefore,

$$w_{\lambda_\epsilon,u}, w_{\lambda_\epsilon,v} \geq 0 \quad \text{in } \ \Sigma_{\lambda_\epsilon},$$

repeating the argument in Step 1, we obtain that $w_{\lambda_\epsilon,u}$ and $w_{\lambda_\epsilon,v}$ are positive in $\Sigma_{\lambda_\epsilon}$. This provides a contradiction with the definition of λ_0.

Step 3. From Step 2, we obtain that $u(-x_1, x') \geq u(x_1, x')$ for $x_1 \geq 0$. Using the same argument from the other side, we conclude that $u(-x_1, x') \leq u(x_1, x')$ for $x_1 \geq 0$. Thus, $u(-x_1, x') = u(x_1, x')$ for $x_1 \geq 0$. Repeating this procedure in all directions, we obtain symmetry property of u.

Step 4. To prove that $u(r)$ *and* $v(r)$ *are strictly decreasing in* $r \in (0,1)$.
Indeed, let $0 < x_1 < \widetilde{x}_1 < 1$ and $\lambda = \frac{x_1+\widetilde{x}_1}{2}$. By the same arguments above, we have that $w_{\lambda,u}$ and $w_{\lambda,v}$ are positive in Σ_λ. We observe that $(\widetilde{x}_1, 0, \cdots, 0) \in \Sigma_\lambda$ and $2\lambda - \widetilde{x}_1 = x_1$, then $u(x_1, 0, \cdots, 0) > u(\widetilde{x}_1, 0, \cdots, 0)$ and $v(x_1, 0, \cdots, 0) > v(\widetilde{x}_1, 0, \cdots, 0)$.

By the symmetry properties of u and v, then we conclude the monotonicity of u and v. □

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