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BOUNDEDNESS OF A GENERALIZED FRACTIONAL INTEGRAL OPERATOR IN THE UNIT DISK

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ABSTRACT. We define a new fractional integral operator in the unit disk with rough kernel, which is a generalization of the Srivastava and Owa integral operator. We study the boundedness of this fractional integral operator on some spaces defined on the open unit disk. We obtain sufficient and necessary conditions on the parameters of these spaces.

1. Introduction

Multilinear analysis is considered as a very efficacious research area in studying harmonic analysis, geometric functions theory and univalent functions theory. Recently, fractional calculus in complex domains has confirmed delectable enforcements in the geometric function theory. The idiomatic of fractional operators and their generalizations have been applied in recognizing, for example, distortion inequalities, coefficient estimates, the characterization properties and convolution structures for different subclasses of analytic functions and the doings in the research monographs. All of these operators involve convolution with special functions such as Gauss hypergeometric function [1], the Meijer G- and Fox H-functions [2].

For the function f analytic in a simply-connected complex domain containing the origin, Srivastava and Owa [3] defined the following fractional integral operator

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \quad \alpha > 0,$$

where the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$. As a generalization to the Srivastava-Owa [3] fractional integral operator conditions are given by Ibrahim [4] for this fractional integral operator to be bounded in Bergman space;

$$I_z^{\alpha,\mu} f(z) = \frac{(\mu+1)^{1-\alpha}}{\Gamma(\alpha)} \int_0^z (z^{\mu+1} - \zeta^{\mu+1})^{\alpha-1} \zeta^\mu f(\zeta) d\zeta,$$

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where α and $\mu \neq -1$ are real numbers and the function $f(z)$ is analytic in a simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z^{\mu+1} - \zeta^{\mu+1})^{\alpha-1}$ is discarded by requiring $\log(z^{\mu+1} - \zeta^{\mu+1})$ to be real when $(z^{\mu+1} - \zeta^{\mu+1}) > 0$. When $\mu = 0$, we obtain the standard Srivastava-Owa fractional integral operator.

In this note, we further generalize these fractional integral operators in the unit disk based on rough kernel and multi linear distinct. We study the boundedness of this fractional integral operator on some spaces defined on the open unit disk such as Morrey space and its extension. We obtain sufficient and necessary conditions on the parameters of these spaces.

2. Main Results

For fixed complex number $\zeta \in U := \{z \in \mathbb{C}, |z| < 1\}$, nonzero real numbers $\lambda_1, ..., \lambda_n$ and $0 < \alpha \leq 1$, we define the n-linear fractional with rough kernel as follows:

$$I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1, ..., f_n)(z) = \int_0^z \prod_{j=1}^{n} f_j(\zeta) \frac{(z - \lambda_j \zeta)^{\alpha-1}}{\Gamma(\alpha)} d\zeta, \tag{1}$$

where the functions f_k, $k = 1, ..., n$ are analytic in a simply-connected region of the complex z-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z - \zeta)^{\alpha-1}$ is extracted by requiring $\log(z - \zeta)$ to be real when $(z - \zeta) > 0$.

Note that, when $j = n = 1$ and $\lambda = 1$, the operator (1) reduces to the Srivastava-Owa fractional integral operator.

In this section, we shall discuss the boundedness of the operator (1) in Morrey spaces [5]. The classical Morrey spaces $L^{p,\rho}$, the modified Morrey space and the center Morrey space can be viewed as generalized Lebesgue spaces L^p. These types of integral operators are utilized to study the behavior of solutions to the second order elliptic partial differential equations. For $1 \leq p < \infty$ and $\rho \geq 0$, Morrey space $L^{p,\rho}(\mathbb{R}^n)$ is defined by

$$L^{p,\rho}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : \|f\|_{L^{p,\rho}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{L^{p,\rho}(\mathbb{R}^n)} = \sup_{z\in\mathbb{R}^n, r>0} \Big(\frac{1}{r^\rho} \int_{\Omega\in\mathbb{R}^n} |f(z)|^p dz\Big)^{1/p}.$$

It is clear that

$$L^{p,0}(\mathbb{R}^n) \equiv L^p(\mathbb{R}^n), \quad L^{p,n}(\mathbb{R}^n) \equiv L^\infty(\mathbb{R}^n), \quad \text{and} \quad L^{p,\rho}(\mathbb{R}^n) \equiv \{0\}, \ \rho > n.$$

In our discussion, we shall use $L^{p,\rho}(U)$, $0 < \rho < 1$ where $U : \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk and we denote U_r the disk of radius $0 < r < 1$. We have the following auxiliary result:

Theorem 1 Let $p > 1$ and $0 < \rho < 1$. If $\frac{\rho}{p} \geq \frac{\rho_1}{p_1}$, $0 < \rho_1 < 1$ and $\frac{1}{p} \geq \frac{1}{p_1}$ then there exists a positive constant C such that

$$\|H(f)\|_{L^{p,\rho}} \leq C\|f\|_{L^{p_1,\rho_1}}, \tag{2}$$

where

$$H(f)=\sup_{0<r<1}\frac{1}{r^2}\int_{U_r}|f(\zeta)|d\zeta$$

is the Hardy-Littlewood maximal function.

Proof. Let $\varphi(z)$ be the characteristic function of the disk $U_r=\{z\in\mathbb{C}:|z|<r<1\}$. Then we have (see also [6])

$$\int_U \big(Hf(w)_{L^{p,\rho}}\big)^p\varphi(w)dw\le C_p\int_U |f|^pH\varphi(w)dw. \tag{3}$$

Taking $f\in L^{p,\rho}$, a calculation implies that

$$\int_{U_r}\big(H_\epsilon f(w)_{L^{p,\rho}}\big)^p dw\le C\|f\|^p_{L^{p,\rho}(U)}r^\rho, \tag{4}$$

where

$$H_\epsilon f(w)=(Hf^\epsilon)^{1/\epsilon}(w),\quad 1\le\epsilon<p.$$

Therefore, the proof is complete since for $\frac{\rho}{p}\ge\frac{\rho_1}{p_1}$, $0<\rho_1<1$ and $\frac{1}{p}\ge\frac{1}{p_1}$ we obtain

$$\|H(f)\|_{L^{p,\rho}}\le\|H_{p_1/p}(f)\|_{L^{p,\rho}}\le\|H_{p_1/p}(f)\|_{L^{p_1,\rho_1}}\le C\|f\|_{L^{p_1,\rho_1}}.$$

As an easy extension to Theorem 1 we obtain the following

Corollary 1 Let $p>1$, $0<\rho<1$ and $1/p=1/p_1+1/p_2+...+1/p_n$. If

$$\frac{\rho}{p}=\frac{\rho_1}{p_1}+...+\frac{\rho_n}{p_n}\text{ and }0<\rho_1,...,\rho_n<1$$

then there exists a positive constant C_n such that

$$\|H(f_1,...,f_n)(z)\|_{L^{p,\rho}}\le C_n\|f_1\|_{L^{p_1,\rho_1}}...\|f_n\|_{L^{p_n,\rho_n}}, \tag{5}$$

where

$$H(f_1,...,f_n)(z)=\sup_{0<r<1}\frac{1}{r^2}\int_{U_r}\prod_{j=1}^n|f_j(\zeta)|d\zeta.$$

Now we determine the sufficient conditions for boundedness of the integral operator (1).

Theorem 2 Suppose that $0<\alpha<1$, $0<\lambda_j<1$, $j=1,...,n,$ p is the harmonic mean of $p_1,...,p_n$, $1<p<\frac{1}{\alpha}$ and $0<\rho<1-\alpha p$. If

$$\frac{1}{q}:=\frac{1}{p}-\frac{\alpha}{1-\rho},\quad \frac{\rho}{p}=\frac{\rho_1}{p_1}+...+\frac{\rho_n}{p_n},\ \text{ and }\ 0<\rho_1,...,\rho_n<1, \tag{6}$$

then there exists a positive constant K such that

$$\|I^\alpha_{z,\lambda_1,...,\lambda_n}(f_1,...,f_n)\|_{L^{q,\rho}(U)}\le K\|f_1\|_{L^{p_1,\rho_1}(U)}...\|f_n\|_{L^{p_n,\rho_n}(U)}.$$

Proof. Let $f_1\in L^{p_1,\rho_1}(U),...,f_n\in L^{p_n,\rho_n}(U)$. For

$$\kappa = \frac{(1-\alpha\beta+\rho)}{2}, \quad \beta < p, \quad \text{and} \quad 0 < \rho < 1-\alpha p,$$

we receive

$$\rho < \kappa < 1-\alpha\beta, \quad \text{and} \quad \frac{(1-\rho)}{p} > \alpha > \frac{(1-\kappa)}{\beta}.$$

For $z \in U$, $|z-\zeta| = \varepsilon$ and $r < 1$ consider that

$$\begin{aligned} |I^{\alpha}_{z,\lambda_1,\ldots,\lambda_n}(f_1,...,f_n)(z)|_{L^{q,\rho}(U)} &= \Big(\int_{r\le\varepsilon} + \int_{r\ge\varepsilon}\Big) \prod_{j=1}^{n} |f_j(\zeta)| \frac{|(z-\lambda_j\zeta)|^{\alpha-1}}{\Gamma(\alpha)} d\zeta \\ &:= I_1(z) + I_2(z). \end{aligned}$$

The rest of the argument for the proof is given in the following three steps.

Step 1. *Estimate of* $I_1(z)$. By taking ζ along the negative real x and $\lambda_j < 1$, we have

$$\begin{aligned} I_1(z) &= \int_{r\le\varepsilon} \prod_{j=1}^{n} |f_j(\zeta)| \frac{|(z-\lambda_j\zeta)|^{\alpha-1}}{\Gamma(\alpha)} d\zeta \\ &\le \int_{r\le\varepsilon} \prod_{j=1}^{n} |f_j(\zeta)| \frac{|(z-\zeta)|^{\alpha-1}}{\Gamma(\alpha)} d\zeta \\ &\le \frac{\varepsilon^{\alpha-1}}{\Gamma(\alpha)} \int_{r\le\varepsilon} \prod_{j=1}^{n} |f_j(\zeta)| d\zeta \\ &\le \frac{\varepsilon^{\alpha-1}}{\Gamma(\alpha)} \int_{r\le\varepsilon} \prod_{j=1}^{n} |f_j(\zeta)| d\zeta \\ &\le C_1 H_\beta(f_1,...,f_n)(z), \end{aligned}$$

where C_1 is a positive constant depending on α, ε and

$$H_\beta(f_1,...,f_n)(z) = H(f_1^\beta,...,f_n^\beta)^{1/\beta}(z), \quad z \in U.$$

Step 2. *Estimate of* $I_2(z)$. By Hölder inequality, we have

$$\begin{aligned}
I_2(z) &= \int_{r\geq\varepsilon}\prod_{j=1}^{n}|f_j(\zeta)|\frac{|(z-\lambda_j\zeta)|^{\alpha-1}}{\Gamma(\alpha)}d\zeta\\
&\leq \frac{1}{\Gamma(\alpha)}\Big(\int_{r\geq\varepsilon}\prod_{j=1}^{n}\frac{|f_j^{\beta}(\zeta)|}{|(z-\lambda_j\zeta)|^{\kappa}}d\zeta\Big)^{1/\beta}\Big(\int_{r\geq\varepsilon}|(z-\lambda_j\zeta)|^{(\kappa/\beta+\alpha-1)^{\gamma}}d\zeta\Big)^{1/\gamma}\\
&\leq \ell\varepsilon\Big(\int_{r\geq\varepsilon}\prod_{j=1}^{n}\frac{|f_j^{\beta}(\zeta)|}{|(z-\lambda_j\zeta)|^{\kappa}}d\zeta\Big)^{1/\beta}\\
&\leq \ell\varepsilon^{\frac{\rho-1}{p+\alpha}}\Big(\int_U|f_1^{p_1}(z-\lambda_1\zeta)|d\zeta\Big)^{1/p_1}...\Big(\int_U|f_n^{p_n}(z-\lambda_n\zeta)|d\zeta\Big)^{1/p_n}\\
&:= C_2\Big(\int_U|f_1^{p_1}(z-\lambda_1\zeta)|d\zeta\Big)^{1/p_1}...\Big(\int_U|f_n^{p_n}(z-\lambda_n\zeta)|d\zeta\Big)^{1/p_n}\\
&\leq C_2\|f_1\|_{L^{p,\rho_1}(U)}...\|f_n\|_{L^{p,\rho_n}(U)}
\end{aligned}$$

where C_2 is a positive constant depending on $\alpha,\beta,\gamma,\kappa$ and ρ.

Step 3. For the estimate of $I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1,...,f_n)(z)$ we have

$$\begin{aligned}
|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1,...,f_n)(z)|_{L^{q,\rho}(U)} &= I_1(z)+I_2(z)\\
&\leq C_1H_\beta(f_1,...,f_n)(z)+C_2\|f_1\|_{L^{p,\rho_1}(U)}...\|f_n\|_{L^{p,\rho_n}(U)}\\
&\leq C_1(H_\beta(f_1,...,f_n)(z))^{p/q}\|f_1\|^{1-p/q}_{L^{p,\rho_1}(U)}...\|f_n\|^{1-p/q}_{L^{p,\rho_1}(U)}\\
&+C_2\|f_1\|_{L^{p,\rho_1}(U)}...\|f_n\|_{L^{p,\rho_n}(U)}\\
&\leq C_1\|(H_\beta(f_1,...,f_n)(z))^{p/q}\|\,\|f_1\|^{1-p/q}_{L^{p,\rho_1}(U)}...\|f_n\|^{1-p/q}_{L^{p,\rho_1}(U)}\\
&+C_2\|f_1\|_{L^{p,\rho_1}(U)}...\|f_n\|_{L^{p,\rho_n}(U)}\\
&\leq C_1\|f_1\|^{p/q}_{L^{p,\rho_1}(U)}...\|f_n\|^{p/q}_{L^{p,\rho_n}(U)}\,\|f_1\|^{1-p/q}_{L^{p,\rho_1}(U)}...\|f_n\|^{1-p/q}_{L^{p,\rho_n}(U)}\\
&+C_2\|f_1\|_{L^{p,\rho_1}(U)}...\|f_n\|_{L^{p,\rho_n}(U)}\\
&\leq C\|f_1\|_{L^{p,\rho_1}(U)}...\|f_n\|_{L^{p,\rho_n}(U)},
\end{aligned}$$

where $C:=max\{C_1,C_2\}$. This completes the proof.

Consequently, we obtain the following

Corollary 2 Let the assumptions of Theorem 2 hold. If (6) holds, then for some $J\in L^{(1-\rho)/\alpha,\rho}(U)$ there exists a positive constant $\mathfrak{K}$ such that

$$\|J.I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1,...,f_n)(z)\|_{L^{p,\rho}(U)}\leq\mathfrak{K}\|J\|_{L^{(1-\rho)/\alpha,\rho}(U)}\|f_1\|_{L^{p_1,\rho_1}(U)}...\|f_n\|_{L^{p_n,\rho_n}(U)}.$$

Next we determine the necessary conditions for boundedness of the integral operator (1).

Theorem 3 Let the operator $I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1,...,f_n)(z)$, $0<\lambda_1,...,\lambda_n<1$, be bounded from $L^{p,\rho_1}(U)\times...\times L^{p,\rho_n}(U)$ to $L^{p,q}(U)$. Then

$$\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{1-\rho},$$

where p is a harmonic mean of $p_1, ..., p_n,\ 1 < p < \frac{1}{\alpha}$, $0 < \rho < 1 - \alpha p$ and $p = \frac{\rho_1}{p_1} + ... + \frac{\rho_n}{p_n}$.

Proof. Set

$$f_{1t}(z) := f_1(tz), ..., f_{nt}(z) := f_n(tz).$$

Then for real positive t, we impose

$$\|f_{1t}\|_{L^{p_1,\rho_1}(U)} = t^{\frac{\rho_1}{p_1} - \frac{1}{p_1}} \|f_1\|_{L^{p_1,\rho_1}(U)}, ..., \|f_{nt}\|_{L^{p_n,\rho_n}(U)} = t^{\frac{\rho_n}{p_n} - \frac{1}{p_n}} \|f_n\|_{L^{p_n,\rho_n}(U)}$$

and so for $I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_{1t}, ..., f_{nt})(z) = t^{-\alpha} I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1, ..., f_n)(tz)$, we have

$$\|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_{1t}, ..., f_{nt})\|_{L^{q,\rho}(U)} = t^{\frac{\rho}{q} - \frac{1+\alpha}{q}} \|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1, ..., f_n)\|_{L^{q,\rho}(U)}.$$

Since $I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1, ..., f_n)(z)$ is bounded from $L^{p,\rho_1}(U) \times ... \times L^{p,\rho_n}(U)$ to $L^{p,q}(U)$, it implies that

$$\begin{aligned}
\|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1, ..., f_n)\|_{L^{q,\rho}(U)} &= t^{-\frac{\rho}{q} + \frac{1+\alpha}{q}} \|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_{1t}, ..., f_{nt})\|_{L^{q,\rho}(U)} \\
&\le C t^{-\frac{\rho}{q} + \frac{1+\alpha}{q}} \|f_{1t}\|_{L^{p_1,\rho_1}(U)}, ..., \|f_{nt}\|_{L^{p_n,\rho_n}(U)} \\
&\le C t^{-\frac{\rho}{q} + \frac{1+\alpha}{q} - \frac{1}{p} + \frac{\rho}{p}} \|f_1\|_{L^{p_1,\rho_1}(U)}, ..., \|f_n\|_{L^{p_n,\rho_n}(U)},
\end{aligned}$$

where C is a positive constant depending on p, q and ρ. Now if $\frac{1}{q} < \frac{1}{p} - \frac{\alpha}{1-\rho}$ then $\|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1, ..., f_n)(z)\|_{L^{q,\rho}(U)} = 0$ when $t \to 0$ and if $\frac{1}{q} > \frac{1}{p} - \frac{\alpha}{1-\rho}$ then $\|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1, ..., f_n)(z)\|_{L^{q,\rho}(U)} = 0$ when $t \to \infty$. Hence we must have $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{1-\rho}$ as required.

Now we are equipped to state and prove the following

Corollary 3 Let the assumptions of Theorem 2.4 hold. For $0 < \alpha < \frac{1}{p}$ and $0 < \mu < 1$ if

$$\frac{1}{q} = \frac{1}{p} - \alpha; \quad \frac{\mu}{q} = \frac{\rho}{p} = \frac{\rho_1}{p_1} + ... + \frac{\rho_n}{p_n}$$

then there exists a positive constant $\mathfrak{L}$ such that

$$\|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1, ..., f_n)\|_{L^{q,\mu}(U)} \le \mathfrak{L} \|f_1\|_{L^{p_1,\rho_1}(U)} ... \|f_n\|_{L^{p_n,\rho_n}(U)}.$$

Proof. By Hölder's inequality, we have

$$L^{\tau,\rho}(U) \subseteq L^{q,\mu}(U), \quad \tau := \frac{(1-\rho)q}{(1-\mu)}.$$

Therefore, we have the equality

$$\frac{1}{\tau} = \frac{1}{p} - \frac{\alpha}{(1-\rho)}.$$

So, as required, in view of Theorem 2.5 we have

$$\begin{aligned}
\|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1, ..., f_n)\|_{L^{q,\mu}(U)} &\le \|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1, ..., f_n)\|_{L^{\tau,\rho}(U)} \\
&\le \mathfrak{L} \|f_1\|_{L^{p_1,\rho_1}(U)} ... \|f_n\|_{L^{p_n,\rho_n}(U)}.
\end{aligned}$$

Finally, we study the boundedness of the integral operator (1) in the central Morrey space. For some related studies on this space see ([7]-[10]).

Definition 1 Let $\psi(r)$ be a positive measurable function on $\mathbb{R}_+$ and $1 \le p < \infty$. We denote by $Ł^{p,\psi}(U)$ the generalized central Morrey space, the space of all functions $f \in L^p(U)$ with finite quasinorm

$$\|f\|_{Ł^{p,\psi}(U)} = \sup_{0<r<1} \psi^{-1}(r)|U(0,r)|^{-1/p}\|f\|_{L^p(U(0,r))},$$

where $U_r := U(0,r)$ is a disk centered at 0 with radius $r < 1$ and $|U(0,r)|$ is the Lebesgue measure of the disk $U(0,r)$.

We note that $Ł^{p,\rho}(U)$ is a Banach space, $Ł^{p,\rho}(U) = \{0\}$ when $\rho < -1/p$ and $Ł^{p,-1/p}(U) = L_p(U)$.

Theorem 4 Assume $0 < \alpha < 1$, $1/p = 1/p_1 + ... + 1/p_n$, $1/q = 1/p - \alpha$ and $0 < \lambda_j < 1$, $j = 1, ..., n$. Then for $1 < p < 1/\alpha$, we have the inequality

$$\begin{aligned}&\|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1,...,f_n)\|_{L^q(U_r)}\\ &\qquad \le Cr^{1/q}\Big(\int_R^1 \|f_1\|^{p_1/p}_{L^{p_1}(U_\sigma)}\frac{d\sigma}{\sigma^{1/q+1}}\Big)^{p/p_1}...\Big(\int_R^1 \|f_n\|^{p_n/p}_{L^{p_n}(U_\sigma)}\frac{d\sigma}{\sigma^{1/q+1}}\Big)^{p/p_n}\end{aligned}$$

where $R = 2r$ holds for any disk $U_r, r < 1$ and for $f_1 \in L^{p_1}(U), ..., f_n \in L^{p_n}(U)$.

Proof. Set, in terms of the characteristic functions,

$$\begin{aligned}&f_1(z) = f_1(z)\varphi_1(z) + f_1(z)\varphi_2(z) := \widetilde{f}_1(z) + \widehat{f}_1(z)\\ &f_2(z) = f_2(z)\varphi_1(z) + f_2(z)\varphi_2(z) := \widetilde{f}_2(z) + \widehat{f}_2(z)\\ &\vdots\\ &f_n(z) = f_n(z)\varphi_1(z) + f_n(z)\varphi_2(z) := \widetilde{f}_n(z) + \widehat{f}_n(z).\end{aligned}$$

Thus, we have

$$\begin{aligned}\|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1,...,f_n)\|_{L^q(U_r)} &\le \|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(\widetilde{f}_1,...,\widetilde{f}_n)\|_{L^q(U_r)} + \|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(\widetilde{f}_1,\widehat{f}_2...,\widehat{f}_n)\|_{L^q(U_r)}\\ &+ \|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(\widehat{f}_1,\widetilde{f}_2...,\widetilde{f}_n)\|_{L^q(U_r)} + \|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(\widehat{f}_1,...,\widehat{f}_n)\|_{L^q(U_r)}.\end{aligned}$$

Now we proceed to find the upper bound of the above inequality and this can be held in two steps.

Step 1. *Estimate of* $I^{\alpha}_{z,\lambda_1,...,\lambda_n}(\widetilde{f}_1,...,\widetilde{f}_n)(z)$.

Since $I^{\alpha}_{z,\lambda_1,...,\lambda_n}(z)$ is bounded from $L^{p_1} \times ... \times L^{p_n}$ to L^q, we conclude that

$$\begin{aligned}\|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(\widetilde{f}_1,...,\widetilde{f}_n)\|_{L^q(U_r)} &\le \|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(\widetilde{f}_1,...,\widetilde{f}_n)\|_{L^q(U)}\\ &\le C\|\widetilde{f}_1\|_{L^{p_1}(U)}...\|\widetilde{f}_n\|_{L^{p_n}(U)}\\ &\le C\|f_1\|_{L^{p_1}(U)}...\|f_n\|_{L^{p_n}(U)},\end{aligned}$$

where $C > 0$ is independent of $f_1, ..., f_n$.

Step 2. *Estimate of* $I^{\alpha}_{z,\lambda_1,...,\lambda_n}(\widetilde{f}_1, \widehat{f}_2..., \widehat{f}_n)(z)$.

By letting ζ along the negative real axis, we obtain

$$\begin{aligned}
|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(\widetilde{f}_1, \widehat{f}_2..., \widehat{f}_n)(z)| &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^z \widetilde{f}_1(\zeta)\widehat{f}_2(\zeta)...\widehat{f}_n(\zeta)(z-\lambda_1\zeta)^{\alpha-1}...(z-\lambda_n\zeta)^{\alpha-1}d\zeta\Big| \\
&\le \frac{1}{\Gamma(\alpha)}\Big(\int_U |\widetilde{f}_1^{p_1/p}(\zeta)||(z-\lambda_1\zeta)^{(\alpha-1)p_1/p}|d\zeta\Big)^{p/p_1} \times ... \\
&\times \Big(\int_U |\widehat{f}_n(\zeta)^{p_n/p}||(z-\lambda_n\zeta)^{(\alpha-1)p_n/p}|d\zeta\Big)^{p/p_n} \\
&\le \frac{1}{\Gamma(\alpha)}\Big(\int_U |\widetilde{f}_1^{p_1/p}(\zeta)||(z-\zeta)^{(\alpha-1)p_1/p}|d\zeta\Big)^{p/p_1} \times ... \\
&\times \Big(\int_U |\widehat{f}_n(\zeta)^{p_n/p}||(z-\zeta)^{(\alpha-1)p_n/p}|d\zeta\Big)^{p/p_n} \\
&\le \frac{\varepsilon^{(\alpha-1)}}{\Gamma(\alpha)}\Big(\int_U |\widetilde{f}_1^{p_1/p}(\zeta)|d\zeta\Big)^{p/p_1} \times ... \times \Big(\int_U |\widehat{f}_n^{p_n/p}(\zeta)|d\zeta\Big)^{p/p_n} \\
&\le Cr^{1/q}\Big(\int_R^1 \|f_1\|^{p_1/p}_{L^{p_1}(U_\sigma)}\frac{d\sigma}{\sigma^{1/q+1}}\Big)^{p/p_1} ...\Big(\int_R^1 \|f_n\|^{p_n/p}_{L^{p_n}(U_\sigma)}\frac{d\sigma}{\sigma^{1/q+1}}\Big)^{p/p_n},
\end{aligned}$$

where C is a positive constant depending on α and ε.

In the same manner of Step 2 in Theorem 2, we may have

$$\begin{aligned}
&\|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(\widehat{f}_1, \widetilde{f}_2..., \widetilde{f}_n)\|_{L^q(U_r)} \\
&\qquad \le Cr^{1/q}\Big(\int_R^1 \|f_1\|^{p_1/p}_{L^{p_1}(U_\sigma)}\frac{d\sigma}{\sigma^{1/q+1}}\Big)^{p/p_1} ...\Big(\int_R^1 \|f_n\|^{p_n/p}_{L^{p_n}(U_\sigma)}\frac{d\sigma}{\sigma^{1/q+1}}\Big)^{p/p_n}
\end{aligned}$$

and

$$\begin{aligned}
&\|I^{\alpha}_{z,\lambda_1,...,\lambda_n}(\widehat{f}_1, ..., \widehat{f}_n)\|_{L^q(U_r)} \\
&\qquad \le Cr^{1/q}\Big(\int_R^1 \|f_1\|^{p_1/p}_{L^{p_1}(U_\sigma)}\frac{d\sigma}{\sigma^{1/q+1}}\Big)^{p/p_1} ...\Big(\int_R^1 \|f_n\|^{p_n/p}_{L^{p_n}(U_\sigma)}\frac{d\sigma}{\sigma^{1/q+1}}\Big)^{p/p_n}.
\end{aligned}$$

Hence, in general, we have the desired assertion.

Corollary 4 Assume $0 < \alpha < 1$, $1/p = 1/p_1 + ... + 1/p_n$, $1/q = 1/p - \alpha$ and $0 < \lambda_j < 1$, $j = 1, ..., n$. Then $I^{\alpha}_{z,\lambda_1,...,\lambda_n}(f_1, ..., f_n)$ is bounded from $Ł^{p_1,\rho_1}(U) \times ... \times Ł^{p_n,\rho_n}(U)$ to $Ł^{p,\rho}(U)$.

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