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ON SOME EQUIVALENT PROBLEMS OF STOCHASTIC DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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Abstract. In this paper we study the existence of a unique mean square (m.s) continuous solution of the stochastic functional integral equation of fractional order $\beta \in (0,1]$

$$X(t) = P(t) + I^{\beta} f(t, X(\phi(t))), \ t \in [0,T].$$

As an application we study the existence of mean square continuous solution of some Cauchy's type problems of stochastic fractional order functional differential equations.

1. Introduction

The definition and properties of the stochastic fractional calculus have been studied in [2]-[5].
Let P be a mean square continuous second order stochastic process, $\phi : [0,T] \longrightarrow [0,T]$ be continuous real valued function and $\beta \in (0,1]$. Here we study the existence of unique mean square continuous solution of the stochastic fractional order integral equation

$$X(t) = P(t) + I^{\beta} f(t, X(\varphi(t))), \ t \in [0,T]. \tag{1}$$

As an application we prove the existence of unique m.s continuous solution for each of the following problems

$$\begin{cases} \frac{dX}{dt} = f(t, X(\varphi(t))), \ t \in [0,T] \\ X(0) = X_o. \end{cases} \tag{2}$$

$$\begin{cases} {}^{R}D^{\alpha} X(t) = f(t, X(\varphi(t))), \ t \in [0,T] \\ X(0) = 0. \end{cases} \tag{3}$$

$$\begin{cases} {}^{R}D^{\alpha} X(t) = f(t, X(\varphi(t))), \ t \in [0,T] \\ I^{1-\alpha} X(t) \mid_{t=0} = 0. \end{cases} \tag{4}$$

$$\begin{cases} {}^{R}D^{\alpha} X(t) = f(t, X(\varphi(t))), \ t \in [0,T] \\ t^{1-\alpha} X(t) \mid_{t=0} = 0. \end{cases} \tag{5}$$

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$$\begin{cases} {}^{C-R}D^{\alpha}X(t) = f(t, X(\varphi(t))), \ t \in [0,T] \\ X(0) = X_0. \end{cases} \tag{6}$$

where X_0 is a second order random variable i.e., $E({X_0}^2) < \infty$.

2. Preliminaries

Let $I = [a, b]$. Let (Ω, F, P) be a fixed probability space, where Ω is a sample space, F is a $\sigma-$algebra and P is a probability measure. Let $X(t;\omega) = \{X(t), \ t \in I, \omega \in \Omega\}$ be a second order stochastic process, i.e., $E(X^2(t)) < \infty, \ t \in I$.
Let $C = C(I, L_2(\Omega))$ be the space of all second order stochastic processes which is mean square (m.s) continuous on I. This space is a Banach space endowed with the norm [6]-[8]

$$\|X\|_C = \max_{t \in I} \|X\|_2 \ \text{ where } \|X\|_2 = (E(X^2(t)))^{1/2}.$$

Let $\Re = \Re(I, L_2(\Omega))$ be the class of all second order stochastic processes which is mean square (m.s) Riemann integrable on I i.e.,

$$E(\int_a^b X^2(t) \ dt) < \infty$$

The norm of $X \in \Re(I, L_2(\Omega))$ is given by [6]-[8]

$$\|X\|_{\Re} =\mid E(\int_a^b X^2(t) \ dt) \mid^{1/2}$$

Definition 1. [2]-[5] Let $X \in C(I, L_2(\Omega))$ and $\beta \in (0, 1)$. The stochastic fractional order integral $I_a^{\beta} X(t)$ is defined by

$$I_a^{\beta} X(t) = \int_a^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} X(s) \ ds.$$

For the existence of the integral $I_a^{\beta} X(t)$ we have the following theorem.
Theorem 1. [2]-[5] Let $\alpha, \beta \in (0, 1)$. If $X \in C(I, L_2(\Omega))$, then $I_a^{\beta} X(t)$ exists in m.s sense as a second order m.s continuous stochastic process $I_a^{\beta} X \in C(I, L_2(\Omega))$ with the following properties

1. $I_a^{\beta} : C(I, L_2(\Omega)) \longrightarrow C(I, L_2(\Omega))$
2. $I_a^{\alpha} I_a^{\beta} X(t) = I_a^{\beta} I_a^{\alpha} X(t) = I_a^{\alpha+\beta} X(t)$
3. $I_a^{\beta} X(t) \mid_{t=a} = 0$
4. $L.i.m_{\beta \to 1} I_a^{\beta} X(t) = I_a X(t) = \int_a^t X(s) \ ds$
5. $X \in C^1(I, L_2(\Omega)), \Rightarrow$

$$L.i.m_{\beta \to \ 0} \ I_a^{\beta} X(t) = X(t).$$

Definition 2. [2]-[5] Let $X(t) \in \Re(I, L_2(\Omega))$ and $I_a^{1-\alpha} X(t)$ is m.s differentiable. Then the differintegral operator, Riemann-Liouville sense, of $X(t)$ of order $\alpha \in (0, 1)$ is defined by the second order process ([2]),

$${}^{R}D_a^{\alpha} X(t) = \frac{d}{dt} I_a^{1-\alpha} X(t).$$

Definition 3. [2]-[5] Let $X(t) \in C^1(I, L_2(\Omega))$ be a second order stochastic process which is m.s differentiable with m.s continuous derivative). The fractional-order derivative, Caputo sense, of $X(t)$ of order $\alpha \in (0, 1]$ is defined by the second order process ([2]),

$$^{C}D_a^{\alpha}X(t) = I_a^{1-\alpha}\frac{d}{dt}X(t).$$

Definition 4. [5] The Caputo fractional-order derivative via the Riemann-Liouville of the second order stochastic process $X(t)$ is defined by

$$^{C-(R-L)}D^{\alpha}X(t) = \frac{d}{dt}I^{1-\alpha}(X(t) - X(0)).$$

3. Existence of solution

Consider the stochastic fractional order functional integral equation (1) with the following assumptions

(i) $f : [0, T] \times L_2(\Omega) \longrightarrow L_2(\Omega)$ is m.s continuous and satisfies the Lipschitz condition

$$\|f(t, X(t)) - f(t, Y(t))\|_2 \leq K\|X(t) - Y(t)\|_2,$$

where K is constant

(ii) $P \in C(I, L_2(\Omega))$

(iii) $\varphi : [0, T] \longrightarrow [0, T]$ is continuous real valued function,

(iv) $f(t, 0) \neq 0$ is continuous, $\sup_t |f(t, 0) = ||f||$.

The following lemma can be proved.

Lemma 1. Let the assumptions (i) and (iv) be satisfied, then

$$\|f(t, X(t))\|_2 \leq K\|X(t)\|_2 + ||f||.$$

For the existence of solution of the stochastic fractional order integral equation (1) we have the following theorem.

Theorem 1. Let the assumptions (i)-(iv) be satisfied. If $\frac{KT^{\beta}}{\Gamma(\beta+1)} < 1$, then the stochastic fractional order integral equation (1) has a unique solution $X \in C(I, L_2(\Omega))$.

Proof. Define the operator

$$FX(t) = P(t) + I^{\beta}f(t, X(\varphi(t))).$$

Then we will prove that $F : C(I, L_2(\Omega)) \longrightarrow C(I, L_2(\Omega))$ and is contraction. Then applying the Banach fixed point theorem[1].

Firstly, let $t_1, t_2 \in [0, T]$ such that $| t_2 - t_1 |< \delta$ and let $X \in C(I, L_2(\Omega))$, then

$$FX(t_2)-FX(t_1) = (P(t_2)-P(t_1))+\int_0^{t_2} \frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)} f(s, X(\varphi(s)))\, ds-\int_0^{t_1} \frac{(t_1 - s)^{\beta-1}}{\Gamma(\beta)} f(s, X(\varphi(s)))\, ds$$

$$= (P(t_2)-P(t_1))+\int_0^{t_1} \frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)} f(s, X(\varphi(s)))\, ds+\int_{t_1}^{t_2} \frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)} f(s, X(\varphi(s)))\, ds- \int_0^{t_1} \frac{(t_1 - s)^{\beta-1}}{\Gamma(\beta)} f(s, X(\varphi(s)))\, ds$$

$$= (P(t_2)-P(t_1))+\int_{t_1}^{t_2} \frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)} f(s, X(\varphi(s)))\, ds+\int_0^{t_1} [\frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_1 - s)^{\beta-1}}{\Gamma(\beta)}] f(s, X(\varphi(s)))\, ds$$

$$\|FX(t_2)-FX(t_1)\|_2 \leq \|P(t_2)-P(t_1)\|_2+\int_0^{t_1} \mid \frac{(t_1 - s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)} \mid \|f(s, X(\varphi(s)))\|_2\, ds$$

$$+\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\|f(s,X(\varphi(s)))\|_2\ ds$$

$$\leq \|P(t_2)-P(t_1)\|_2+\int_0^{t_1}[\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}][K\|X\|_C+\|f\|]\ ds+$$

$$\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}[K\|X\|_C+\|f\|]\ ds$$

$$\leq \|P(t_2)-P(t_1)\|_2+[K\|X\|_C+\|f\|][\frac{-(t_1-s)^{\beta}}{\Gamma(\beta+1)}\ |_0^{t_1}+\frac{(t_2-s)^{\beta}}{\Gamma(\beta+1)}\ |_0^{t_1}-\frac{(t_2-s)^{\beta}}{\Gamma(\beta+1)}\ |_{t_1}^{t_2}]$$

$$=\|P(t_2)-P(t_1)\|_2+[K\|X\|_C+\|f\|][\frac{t_1^{\beta}}{\Gamma(\beta+1)}+\frac{(t_2-t_1)^{\beta}}{\Gamma(\beta+1)}-\frac{{t_2}^{\beta}}{\Gamma(\beta+1)}+\frac{(t_2-t_1)^{\beta}}{\Gamma(\beta+1)}]$$

$$=\|P(t_2)-P(t_1)\|_2+[K\|X\|_C+\|f\|][\frac{2(t_2-t_1)^{\beta}}{\Gamma(\beta+1)}-\frac{({t_2}^{\beta}-{t_1}^{\beta})}{\Gamma(\beta+1)}].$$

This proves that $F: C(I, L_2(\Omega)) \longrightarrow C(I, L_2(\Omega))$.

Secondly, for $X, Y \in C(I, L_2(\Omega))$ we have

$$\|FX(t)-FY(t)\|_2\leq\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\|f(s,X((\varphi(s)))-f(s,Y((\varphi(s)))\|_2\ ds$$

$$\leq K\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\|X((\varphi(s))-Y((\varphi(s))\|_2\ ds$$

$$\leq K\|X-Y\|_C\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ ds$$

$$\leq K\|X-Y\|_C\frac{t^{\beta}}{\Gamma(\beta+1)}.$$

Hence

$$\|FX-FY\|_C\leq\frac{KT^{\beta}}{\Gamma(\beta+1)}\|X-Y\|_C.$$

If $\frac{KT^{\beta}}{\Gamma(\beta+1)} < 1$, then F is contraction operator. By the Banach fixed point theorem [1], there exists a unique solution $X \in C(I, L_2(\Omega))$ of the integral equation (1).

Now, let $P(t) = 0,\ f(t, 0) = 0,\ t \in [0, T]$ and $\frac{KT^{\beta}}{\Gamma(\beta+1)} < 1$, then from Lemma 1 the solution of (1) satisfies the inequality

$$||x||_C\leq\frac{KT^{\beta}}{\Gamma(\beta+1)}\ ||x||_C$$

which implies that $x(t) = 0,\ t \in [0, T]$.
We have the following corollary.
Corollary 1. Let $P(t) = 0,\ f(t, 0) = 0,\ t \in [0, T]$ and $\frac{KT^{\beta}}{\Gamma(\beta+1)} < 1$, then the integral equation (1) has only the zero solution.

4. Applications

Consider the three Cauchy's type problems (3)-(5). Then we have the following theorems.

Theorem 2. Let the assumptions of Theorem 1 be satisfied. Then the three Cauchy's type problems (3)-(5) are equivalent to the stochastic fractional order integral equation

$$X(t) = I^{\alpha} f(t, X(\varphi(t))) \tag{7}$$

Proof. Firstly, consider the initial value problem (3). Integrating we obtain

$$I^{1-\alpha} X(t) - c = I f(t, X(\varphi(t))).$$

Operate with I^{α} we get

$$I X(t) = c \frac{t^{\alpha}}{\Gamma(1+\alpha)} + I^{\alpha+1} f(t, X(\varphi(t))).$$

Differentiating we obtain

$$X(t) = c \frac{t^{\alpha-1}}{\Gamma(\alpha)} + I^{\alpha} f(t, X(\varphi(t))),$$

then at $t = 0$ we deduce that $c = 0$ and obtain the integral equation (7).
Operate with $I^{1-\alpha}$ we get

$$I^{1-\alpha} X(t) = I f(t, X(\varphi(t))).$$

Differentiating we obtain

$$\frac{d}{dt} I^{1-\alpha} X(t) = f(t, X(\varphi(t))).$$

Also

$$X(0) = I^{\alpha} f(t, X(\varphi(t))) \mid_{t=0} = 0.$$

Hence the initial value problem (3) is equivalent to the the integral equation (7).
Secondly, consider the initial value problem (4). Integrating we obtain

$$I^{1-\alpha} X(t) - I^{1-\alpha} X(t) \mid_{t=0} = I f(t, X(\varphi(t))).$$

Operate with I^{α} we get

$$I X(t) = I^{\alpha+1} f(t, X(\varphi(t))).$$

Differentiating we obtain

$$X(t) = I^{\alpha} f(t, X(\varphi(t))).$$

Operate with $I^{1-\alpha}$ we get

$$I^{1-\alpha} X(t) = I f(t, X(\varphi(t))).$$

Differentiating we obtain

$$\frac{d}{dt} I^{1-\alpha} X(t) = f(t, X(\varphi(t))).$$

Also

$$I^{1-\alpha}X(t)\mid_{t=0}=\int_0^0 f(s,X(\varphi(s)))\;ds=0.$$

Hence the initial value problem (4) is equivalent to the the integral equation (7). Finally, consider the initial value problem (5). Integrating we obtain

$$I^{1-\alpha}X(t)-C=If(t,X(\varphi(t)))$$

Operate with I^α we get

$$IX(t)=I^\alpha C+I^{\alpha+1}f(t,X(\varphi(t))).$$

i.e.,

$$IX(t)=\frac{t^\alpha}{\Gamma(\alpha+1)}C+I^{\alpha+1}f(t,X(\varphi(t))).$$

Differentiating we obtain

$$X(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}C+I^\alpha f(t,X(\varphi(t))),$$

$$t^{1-\alpha}X(t)=\frac{C}{\Gamma(\alpha)}+t^{1-\alpha}I^\alpha f(t,X(\varphi(t))),$$

then

$$t^{1-\alpha}X(t)\mid_{t=0}=\frac{C}{\Gamma(\alpha)}+t^{1-\alpha}I^\alpha f(t,X(\varphi(t)))\mid_{t=0}.$$

i.e.,

$$t^{1-\alpha}X(t)\mid_{t=0}=\frac{C}{\Gamma(\alpha)}=0.$$

Hence

$$X(t)=I^\alpha f(t,X(\varphi(t))).$$

Operate with $I^{1-\alpha}$ we obtain

$$I^{1-\alpha}X(t)=If(t,X(\varphi(t))).$$

Differentiating we obtain

$$\frac{d}{dt}I^{1-\alpha}X(t)=f(t,X(\varphi(t))).$$

Also

$$t^{1-\alpha}X(t)\mid_{t=0}=t^{1-\alpha}I^\alpha f(t,X(\varphi(t)))\mid_{t=0},$$

then

$$t^{1-\alpha}X(t)\mid_{t=0}=0.$$

Hence the initial value problem (5) is equivalent to the the integral equation (7). Therefore, the three Cauchy's type problems (3)-(5) are equivalent to the stochastic fractional order integral equation (7).
Then from Theorem 1 there exists a unique solution $X\in C(I,L_2(\Omega))$ of each of the problems (3)-(5) which is the solution of the integral equation (7).

Now from corollary 1 and Theorem 2 we can prove the following corollary
Corollary 2. Let the assumptions of Theorem 1 be satisfied. If $p(t)=0,\ f(t,0)=0,\ t\in[0,T]$ and $\frac{KT^\beta}{\Gamma(\beta+1)}<1$, then the problem

$$\begin{cases} {}^RD^\alpha X(t)=f(t,X(\varphi(t))),\ t\in[0,T] \\ X(0)=0. \end{cases} \tag{8}$$

has only the zero solution $X(t) = 0,\ t \in [0,T]$.
Consider now the initial value problem (6)
Theorem 3. Let the assumptions of Theorem 1 be satisfied. Then the initial value problem (6) has a unique solution $X(t) \in C(I, L_2(\Omega))$. This solution is the solution of the integral equation

$$X(t) = X_0 + I^{\alpha} f(t, X((\varphi(t))) \tag{9}$$

Proof. Consider the initial value problem (6). Integrating equation (6) we obtain

$$I^{1-\alpha}[X(t) - X(0)] - I^{1-\alpha}[X(t) - X(0)] \mid_{t=0} = If(t, X((\varphi(t)))$$

i.e.,

$$I^{1-\alpha}[X(t) - X(0)] = If(t, X((\varphi(t))).$$

Operate with I^{α} we get

$$I[X(t) - X(0)] = I^{1+\alpha} f(t, X((\varphi(t))).$$

Differentiating we obtain

$$X(t) - X(0) = I^{\alpha} f(t, X((\varphi(t))),$$

then

$$X(t) = X_0 + I^{\alpha} f(t, X((\varphi(t))).$$

Now, from (9) we have

$$X(t) - X(0) = I^{\alpha} f(t, X((\varphi(t))).$$

Operate with $I^{1-\alpha}$ we obtain

$$I^{1-\alpha}[X(t) - X(0)] = If(t, X((\varphi(t))).$$

Differentiating we obtain

$$\frac{d}{dt} I^{1-\alpha}[X(t) - X(0)] = f(t, X((\varphi(t))).$$

Also, from (9) we have

$$X(0) = X_0 + \int_0^0 \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, X((\varphi(s))) \, ds = X_0.$$

Hence the initial value problem (6) is equivalent to the integral equation (9).
Applying Theorem 1 we deduce that the problem (6) has a unique solution $X(t) \in C(I, L_2(\Omega))$. This solution is the solution of the integral equation (9).
Finally consider the initial value problem (2).
Theorem 4. Let the assumptions of Theorem 1 be satisfied. Then the initial value problem (2) has a unique solution $X(t) \in C(I, L_2(\Omega))$.
Proof. Letting $\beta \longrightarrow 1$ in (1) we obtain the stochastic integral equation

$$X(t) = X_0 + \int_0^t f(s, X(\varphi(s))) \, ds.$$

Which is equivalent to the initial value problem (2). Then applying Theorem 1 we obtain the results.

5. Conclusion

Our results show the richness of the applications of the definition of Riemann-Liouville fractional order derivative in Cauchy's type problems of differential equations.

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