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APPLICATION OF A FAMILY OF INTEGRAL OPERATORS IN THE MAJORIZATION OF A CLASS OF $p-$ VALENTLY MEROMORPHIC FUNCTIONS OF COMPLEX ORDER

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Abstract. The main object of this present paper is to investigate the problem of majorization of certain class of meromorphic p-valent functions of complex order involving certain integral operator. Moreover we point out some new or known consequences of our main result.

1. Introduction and Preliminaries

Denote by $\mathcal{A}_p$ the class of analytic functions in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (p \in \mathbb{N} = \{1, 2, ...\}). \tag{1}$$

We note that $\mathcal{A}_1 \equiv \mathcal{A}$. For analytic functions $f, g \in \mathcal{A}$ are analytic functions in the unit disc Δ we say that f is majorized by g in Δ (see [7])and we write

$$f(z) \ll g(z), (z \in \Delta) \tag{2}$$

if there exists a function ϕ, analytic in Δ, such that $|\phi(z)| < 1$ and

$$f(z) = \phi(z)g(z), \quad z \in \Delta. \tag{3}$$

It may be noted here that (2) is closely related to the concept of quasi-subordination between analytic functions.

For $\gamma(\gamma \in \mathbb{C} \setminus \{0\})$, let $\mathcal{S}^*_\gamma$ be the class of starlike functions of complex order satisfying the condition

$$\frac{f(z)}{z} \neq 0 \text{ and } \Re\left(1 + \frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)} - 1\right]\right) > 0$$

and also let $\mathcal{C}_\gamma$ be the class of convex functions of complex order if

$$f'(z) \neq 0 \text{ and } \Re\left(1 + \frac{1}{\gamma}\left[\frac{zf''(z)}{f'(z)}\right]\right) > 0, (z \in \Delta).$$

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Further, $\mathcal{S}^*_{(1-\alpha)cos\beta\ e^{-i\beta}} = \mathcal{S}^*(\alpha, \beta),\ |\beta| < \frac{\pi}{2};\ 0 \le \alpha \le 1$ the class of $\beta-$ spiral-like function of order α investigated by Libera [5] and for $|\beta| = 0$, then $\mathcal{S}^*_{cos\beta\ e^{-i\beta}} = \mathcal{S}^*(\beta)$ the class of $\beta-$ spiral-like functions introduced by Spacek [10] (see[11]).

By using the Gaussian hypergeometric function

$$ {}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n\ n!} z^n, c \neq 0, -1, -2, -3 \dots $$

recently Srivastava et al.[12](see also [2]) defined Saigo-hypergeometric fractional integral and fractional derivative operators as given below.

Definition 1. *For real numbers $\lambda > 0$ and $\mu, \eta \in \mathbb{R}$ Saigo hypergeometric fractional integral operator $I_{0,z}^{\lambda,\mu,\eta}$ is defined by*

$$ I_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1}\ {}_2F_1\left(\lambda+\mu, -\eta; \lambda; 1 - \frac{t}{z}\right) f(t)dt, $$

where the function $f \in \mathcal{A}$ is analytic in a simply-connected region of the complex $z-$plane containing the origin, with the order

$$ f(z) = O(|z|^{\varepsilon}) \qquad (z \longrightarrow 0; \varepsilon > \max\{0, \mu - \eta\} - 1), $$

and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Definition 2. *Under the hypotheses of Definition 1, Saigo hypergeometric fractional derivative operator $\mathfrak{S}_{0,z}^{\lambda,\mu,\eta}$ is defined by*

$$ \mathfrak{S}_{0,z}^{\lambda,\mu,\eta} f(z) = \begin{cases} \dfrac{1}{\Gamma(1-\lambda)} \dfrac{d}{dz} \left\{ z^{\lambda-\mu} \int_0^z (z-t)^{-\lambda}\ {}_2F_1\left(\mu-\lambda, 1-\eta; 1-\lambda; 1-\dfrac{t}{z}\right) f(t)dt \right\} & \\ & (0 \le \lambda < 1); \\ \dfrac{d^n}{dz^n} \mathfrak{S}_{0,z}^{\lambda-n,\mu,\eta} f(z) & (n \le \lambda < n+1; n \in \mathbb{N}), \end{cases} $$

where the multiplicity of $(z-t)^{-\lambda}$ is removed as in Definition 1.

By Lemma 3 in [12], if $\lambda > 0$ and $n > \mu - \eta - 1$, then

$$ I_{0,z}^{\lambda,\mu,\eta} z^n = \frac{\Gamma(n+1)\Gamma(n-\mu+\eta+1)}{\Gamma(n-\mu+1)\Gamma(n+\lambda+\eta+1)} z^{n-\mu}. \tag{4} $$

It may be remarked that $I_{0,z}^{\lambda,-\lambda,\eta} f(z) = D_z^{-\lambda} f(z), (\lambda > 0)$ and $\mathfrak{S}_{0,z}^{\lambda,\lambda,\eta} f(z) = D_z^{\lambda} f(z), (0 \le \lambda < 1)$, where $D_z^{-\lambda}$ denotes fractional integral operator and D_z^{λ} denotes fractional derivative operator considered by Owa [9].

Recently Goyal and Prajapat [4] introduced the generalized fractional differintegral operator $\mathfrak{J}_{0,z}^{\lambda,\mu,\eta} : \mathcal{A}_p \longrightarrow \mathcal{A}_p$, by

$$ \mathfrak{J}_{0,z}^{\lambda,\mu,\eta} f(z) = \begin{cases} \dfrac{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu)} z^{\mu} \mathfrak{S}_{0,z}^{\lambda,\mu,\eta} f(z) & (0 \le \lambda < \eta+p+1,); \\ \dfrac{\Gamma(1+p-\mu)\Gamma(1+p+\eta-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\mu)} z^{\mu} I_{0,z}^{-\lambda,\mu,\eta} f(z) & (-\infty < \lambda < 0) \end{cases} \tag{5} $$

for $z \in \triangle$. It is easily seen from (5) that for a function $f \in \mathcal{A}_p$, we get

$$\begin{aligned}\mathfrak{J}_{0,z}^{\lambda,\mu,\eta} f(z) &= z^p + \sum_{n=1}^{\infty} \frac{(1+p)_n(1+p+\eta-\mu)_n}{(1+p-\mu)_n(1+p+\eta-\lambda)_n} a_{p+n} z^{p+n} \\ &= z^p\ {}_3F_2(1, 1+p, 1+p+\eta-\mu; 1+p-\mu, 1+p+\eta-\lambda; z) * f(z) \\ &\quad (z \in \triangle; p \in \mathbb{N}; \mu, \eta \in \mathbb{R}; \mu < p+1; -\infty < \lambda < \eta+p+1) \qquad (6)\end{aligned}$$

where $(a)_n$ denote the Pochhammer symbol (or the shifted factorial) defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & \text{for } n = 0 \\ a(a+1)(a+2)\dots(a+n-1) & \text{for } n \in \mathbb{N} = \{1, 2, \dots\}. \end{cases}$$

Let Σ_p be the class of $p-$valently meromorphic functions which are analytic and univalent in the punctured unit disk $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \Delta \setminus \{0\}$ of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_n z^{n-p}. \qquad (7)$$

with a simple pole at the origin.

Motivated by Lashin[6], by applying the operator $I_{0,z}^{\lambda,\mu,\eta}$ to the function $f \in \Sigma_p$ we define a new generalized Saigo integral operator for $p-$valently meromorphic functions as below:

$$\mathfrak{J}_{0,z,}^{p,\lambda,\mu,\eta} : \Sigma_p \longrightarrow \Sigma_p$$

given by

$$\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z) = \frac{\Gamma(1-p-\mu)\Gamma(1-p+\lambda+\eta)}{\Gamma(1-p)\Gamma(1-p-\mu+\eta)} z^{\mu} I_{0,z}^{\lambda,\mu,\eta} f(z). \qquad (8)$$

From (4), we get

$$I_{0,z}^{\lambda,\mu,\eta} z^{n-p} = \frac{\Gamma(n-p+1)\Gamma(n-p-\mu+\eta+1)}{\Gamma(n-p-\mu+1)\Gamma(n-p+\lambda+\eta+1)} z^{n-p-\mu}. \qquad (9)$$

Thus, equation (8) gives

$$\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z) = \frac{1}{z^p} + \frac{\Gamma(1-p-\mu)\Gamma(1-p+\lambda+\eta)}{\Gamma(1-p)\Gamma(1-p-\mu+\eta)} \sum_{n=1}^{\infty} C_n^p(\mu, \eta, \lambda) a_n z^{n-p} \qquad (10)$$

where $C_n^p(\mu, \eta, \lambda) = \frac{\Gamma(n-p+1)\Gamma(n-p-\mu+\eta+1)}{\Gamma(n-p-\mu+1)\Gamma(n-p+\lambda+\eta+1)}$. Further by simple computation, (8) yields following relation:

$$z(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))' = (\lambda - p + \eta)\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} f(z) - (\lambda+\eta)\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z). \qquad (11)$$

It is easy to verify from (11),that

$$z(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q+1)} = (\lambda - p + \eta)(\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} f(z))^{(q)} - (\lambda+\eta+q)(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q)}. \qquad (12)$$

A majorization problem for the class of analytic starlike functions have been investigated by MacGregor [7] and Altintas et al.[1]. Recently Goyal and Goswami [3] extended these results for the class of meromorphic functions making use of certain integral operator. In the present paper we investigate a majorization problem for the class of $p-$valently meromorphic starlike functions of complex order associated with the generalized Saigo-integral operator $\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta}$ defined in this paper.

Definition 3. *A function $f(z) \in \Sigma_p$ is said to in the class $\mathcal{M}_{0,z,q}^{p,\lambda,\mu,\eta}(\gamma, A, B)$ of meromorphic functions of complex order $\gamma \neq 0$ in Δ^* if and only if*

$$1 - \frac{1}{\gamma}\left[\frac{z(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q+1)}}{(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q)}} + p + q\right] \prec \frac{1 + Az}{1 + Bz}, \quad z \in \Delta^* \tag{13}$$

where $-1 \leq B < A \leq 1$, $\gamma \in \mathbb{C}\backslash\{0\}$, $\mu < p + 1$, $-\infty < \lambda < \eta + p + 1$, $p \in \mathbb{N}$ and $\mu, \eta \in \mathbb{R}$.

For simplicity, we put $\mathcal{M}_{0,z,q}^{p,\lambda,\mu,\eta}(\gamma, 1, -1) = \mathcal{M}_{0,z,q}^{p,\lambda,\mu,\eta}(\gamma)$, where $\mathcal{M}_{0,z,q}^{p,\lambda,\mu,\eta}(\gamma)$ denote the class of functions $f \in \Sigma_p$ satisfying the following inequality:

$$\Re\left(1 - \frac{1}{\gamma}\left[\frac{z(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q+1)}}{(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q)}} + p + q\right]\right) > 0, \quad z \in \Delta^* \tag{14}$$

where $\gamma \in \mathbb{C}\backslash\{0\}$, $\mu < p + 1$, $-\infty < \lambda < \eta + p + 1$, $p \in \mathbb{N}$ *and* $\mu, \eta \in \mathbb{R}$.

Example 1. *Putting $\gamma = (p - \alpha)cos\beta\ e^{-i\beta}$, $|\beta| < \frac{\pi}{2}$; $0 \leq \alpha < p$ the class $\mathcal{M}_{0,z,q}^{p,\lambda,\mu,\eta}(\gamma) = \mathcal{M}_{0,z,q}^{p,\lambda,\mu,\eta}((p - \alpha)cos\beta\ e^{-i\beta}) \equiv \mathcal{M}_{0,z,q}^{p,\lambda,\mu,\eta}(\alpha, \beta)$ called the generalized class of $\beta-$spiral-like functions of order $\alpha(0 \leq \alpha < p)$ if*

$$\Re\left(e^{i\beta}\left[\frac{z(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q+1)}}{(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q)}} + q\right]\right) < -\alpha\ cos\beta, \quad z \in \Delta^* \tag{15}$$

where $\mu < p + 1$, $-\infty < \lambda < \eta + p + 1$, $p \in \mathbb{N}$ and $\mu, \eta \in \mathbb{R}$.

Example 2. *Putting $\gamma = (p-\alpha); 0 \leq \alpha < p$ the class $\mathcal{M}_{0,z,q}^{p,\lambda,\mu,\eta}(p-\alpha) \equiv \mathcal{M}_{0,z,q}^{p,\lambda,\mu,\eta}(\alpha)$ the generalized class of $p-$valently meromorphic starlike functions of order $\alpha(0 \leq \alpha < p)$ if*

$$\Re\left(\frac{z(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q+1)}}{(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q)}} + q\right) < -\alpha \quad z \in \Delta^* \tag{16}$$

where $\mu < p + 1$, $-\infty < \lambda < \eta + p + 1$, $p \in \mathbb{N}$ and $\mu, \eta \in \mathbb{R}$.

By taking $q = 0$ we get $\mathcal{M}_{0,z,0}^{p,\lambda,\mu,\eta}(p - \alpha) \equiv \mathcal{M}_{0,z,0}^{p,\lambda,\mu,\eta}(\alpha)$ $p-$valently meromorphic starlike functions involving integral operator of order $\alpha(0 \leq \alpha < p)$ if

$$\Re\left(\frac{z(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))'}{(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))}\right) < -\alpha, (z \in \Delta^*)$$

2. A MAJORIZATION PROBLEM FOR THE CLASS $\mathcal{M}_{0,z,q}^{p,\lambda,\mu,\eta}(\gamma, A, B)$

Theorem 1. *Let the function $f(z) \in \Sigma_p$ and $g(z) \in \mathcal{M}_{0,z,q}^{p,\lambda,\mu,\eta}(\gamma, A, B)$ if $(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q)}$ is majorized by $(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} g(z))^{(q)}$ in Δ^* then*

$$|(\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} f(z))^{(q)}| \leq |(\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} g(z))^{(q)}|, \quad |z| \leq r_1, (z \in \Delta^*) \tag{17}$$

where $r_1 = r_1(A, B, \lambda, \eta, \mu, \rho)$ is the smallest positive root of the equation

$$\begin{aligned} &|(\lambda + \eta - p)B - \gamma(A - B)|r^3 - \{(\lambda + \eta - p) + 2\rho|B|\ \}r^2 \\ &\quad -\{|(\lambda + \eta - p)B - \gamma(A - B)| + 2\rho\}r + (\lambda + \eta - p) = 0 \end{aligned} \tag{18}$$

and $-1 \leq B < A \leq 1$, $\gamma \in \mathbb{C}\backslash\{0\}$, $\mu < p+1$, $-\infty < \lambda < \eta+p+1$, $p \in \mathbb{N}$, $\mu, \eta \in \mathbb{R}$.

Proof. Since $g(z) \in \mathcal{M}_{0,z,q}^{p,\lambda,\mu,\eta}(\gamma, A, B)$, we readily obtain from (13) that, if

$$1 - \frac{1}{\gamma}\left[\frac{z(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} g(z))^{(q+1)}}{(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} g(z))^{(q)}} + p + q\right] = \frac{1 + Aw(z)}{1 + Bw(z)} \tag{19}$$

where w denotes the well known class of bounded analytic functions in Δ and

$$w(0) = 0 \quad \text{and} \quad |w(z)| \leq |z|, \quad (z \in \Delta). \tag{20}$$

From(19) we get

$$\frac{z(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} g(z))^{(q+1)}}{(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} g(z))^{(q)}} = -\frac{(p+q) + [(p+q)B + \gamma(A-B)]w(z)}{1 + Bw(z)}.$$

Using (12) in the above equation, we get,

$$(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} g(z))^{(q)} = \frac{(\lambda+\eta-p)[1 + Bw(z)]}{(\lambda+\eta-p) + [(\lambda+\eta-p)B - \gamma(A-B)]\ w(z)} (\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} g(z))^{(q)}. \tag{21}$$

Hence, by making use of (20), we get,

$$|(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} g(z))^{(q)}| \leq \frac{(\lambda+\eta-p)[1 + |B|\ |z|]}{(\lambda+\eta-p) - |(\lambda+\eta-p)B - \gamma(A-B)|\ |z|} |(\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} g(z))^{(q)}|. \tag{22}$$

Since $(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q)}$ is majorized by $(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} g(z))^{(q)}$ in Δ^* from (3), we have

$$(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q)} = \phi(z)(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} g(z))^{(q)}.$$

Differentiating the above equation w.r.t z and multiplying by z, we have,

$$z(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q+1)} = z\phi'(z)(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} g(z))^{(q)} + z\phi(z)(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} g(z))^{(q+1)}.$$

By using (12), we get,

$$(\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} f(z))^{(q)} = \frac{z}{(\lambda+\eta-p)} \phi'(z)(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} g(z))^{(q)} + \phi(z)(\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} g(z))^{(q)}. \tag{23}$$

Noting that the Schwarz function $\phi(z)$ satisfies

$$|\phi^{'}(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2} \tag{24}$$

and using (22) and (24) in (23) we have

$$|(\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} f(z))^{(q)}| \leq \left(|\phi(z)| + \frac{(1-|\phi(z)|^2)}{(1-|z|^2)} \cdot \frac{|z|\ (1+|B|\ |z|)}{(\lambda+\eta-p) - |(\lambda+\eta-p)B - \gamma(A-B)|\ |z|}\right) |(\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} g(z))^{(q)}|$$

which upon setting $|z| = r$ and $|\phi(z)| = \rho$, $(0 \leq \rho \leq 1)$ leads us to the inequality

$$|(\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} f(z))^{(q)}| \leq \frac{\theta(\rho)}{(1-r^2)\{(\lambda+\eta-p) - |(\lambda+\eta-p)B - \gamma(A-B)|r\}} |(\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} g(z))^{(q)}|, \tag{25}$$

where

$$\theta(\rho) = \rho(1-r^2)\{\ (\lambda+\eta-p) - |(\lambda+\eta-p)B - \gamma(A-B)|\ r\} + (1-\rho^2)(1+|B|\ r)r$$

takes its maximum value at $\rho = 1$. Furthermore, if $0 \leq \sigma \leq r_1$, the function $\varphi(\rho)$ defined by

$$\varphi(\rho) = \rho(1-\sigma^2)\{\ (\lambda+\eta-p) - |(\lambda+\eta-p)B - \gamma(A-B)|\ \sigma\} + (1-\rho^2)(1+|B|\ \sigma)\sigma$$

is an increasing function on $(0 \leq \rho \leq 1)$ so that

$$\varphi(\rho) \leq \varphi(1) = (1-\sigma^2)\{\ (\lambda+\eta-p) - |(\lambda+\eta-p)B - \gamma(A-B)|\ \sigma\}.$$

Therefore, from this fact, (25) gives the inequality (17). □

3. Corollaries and their Consequences

By taking $A = 1$ and $B = -1$ and $\rho = 1$ in Theorem 1 we state the following corollary without proof.

Corollary 1. *Let the function $f \in \Sigma_p$ and $g(z) \in \mathcal{M}_{0,z}^{p,\lambda,\mu,\eta}(\gamma)$ if $(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q)}$ is majorized by $(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} g(z))^{(q)}$ in Δ^* then*

$$|(\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} f(z))^{(q)}| \leq |(\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} g(z))^{(q)}|,\ \ |z| \leq r_1,$$

where $r_1 = r_1(\lambda,\eta,\mu)$ is the smallest positive root of the equation

$$|\lambda+\eta-p+2\gamma|r^3 - (\lambda+\eta-p+2)r^2 - (|\lambda+\eta-p+2\gamma|+2)r + (\lambda+\eta-p) = 0,$$

$$r_1 = \frac{L_1 - \sqrt{L_1^2 - 4|\lambda+\eta-p+2\gamma|(\lambda+\eta-p)}}{2|\lambda+\eta-p+2\gamma|} \text{ and } L_1 = \lambda+\eta-p+2+|\lambda+\eta-p+2\gamma|.$$

By setting $p = 1$ in Corollary 1, we state the following Corollary.

Corollary 2. *Let the function $f \in \Sigma_1$ and $g(z) \in \mathcal{M}_{0,z}^{\lambda,\mu,\eta}(\gamma)$ if $(\mathfrak{J}_{0,z}^{\lambda,\mu,\eta} f(z))^{(q)}$ is majorized by $(\mathfrak{J}_{0,z}^{\lambda,\mu,\eta} g(z))^{(q)}$ in Δ^* then*

$$|(\mathfrak{J}_{0,z}^{\lambda-1,\mu,\eta} f(z))^{(q)}| \leq |(\mathfrak{J}_{0,z}^{\lambda-1,\mu,\eta} g(z))^{(q)}|,\ \ |z| \leq r_2,$$

where $r_2 = r_2(\eta,\mu)$ is the smallest positive root of the equation

$$|\lambda+\eta-1+2\gamma|r^3 - (\lambda+\eta+1)r^2 - (|\lambda+\eta-1+2\gamma|+2)\ r + (\lambda+\eta-1) = 0,$$

$$r_2 = \frac{L_2 - \sqrt{L_2^2 - 4|\lambda+\eta-1+2\gamma|(\lambda+\eta-1)}}{2|\lambda+\eta-1+2\gamma|} \text{ and } L_2 = \lambda+\eta+1+|\lambda+\eta-1+2\gamma|.$$

By taking $\gamma = (p-\alpha)cos\ \beta e^{-i\beta}$ $((|\beta| < \frac{\pi}{2}, \delta(0 \leq \beta < p),)$ in Corollaries 1, we state the following Corollaries without proof.

Corollary 3. *Let the function $f \in \Sigma_p$ and $g(z) \in \mathcal{M}_{0,z}^{p,\lambda,\mu,\eta}(\alpha,\beta)$ if $(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q)}$ is majorized by $(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} g(z))^{(q)}$ in Δ^* then*

$$|(\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} f(z))^{(q)}| \leq |(\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} g(z))^{(q)}|,\ \ |z| \leq r_3,$$

where $r_3 = r_3(\alpha,\eta,\mu)$ is the smallest positive root of the equation

$$|\lambda+\eta-p+2(p-\alpha)cos\ \beta e^{-i\beta}|r^3 - (\lambda+\eta-p+2)r^2$$
$$-(|\lambda+\eta-p+2(p-\alpha)cos\ \beta e^{-i\beta}|+2)r + (\lambda+\eta-p) = 0,$$
$$r_3 = \frac{L_3 - \sqrt{L_3^2 - 4|\lambda+\eta-p+2(p-\alpha)cos\ \beta e^{-i\beta}|(\lambda+\eta-p)}}{2|\lambda+\eta-p+2(p-\alpha)cos\ \beta e^{-i\beta}|}$$

and $L_3 = \lambda+\eta-p+2+|\lambda+\eta-p+2(p-\alpha)cos\ \beta e^{-i\beta}|$.

Corollary 4. *Let the function* $f \in \Sigma_p$ *and* $g(z) \in \mathcal{M}_{0,z}^{p,\lambda,\mu,\eta}(\alpha)$ *if* $(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} f(z))^{(q)}$ *is majorized by* $(\mathfrak{J}_{0,z}^{p,\lambda,\mu,\eta} g(z))^{(q)}$ *in* Δ^* *then*

$$|(\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} f(z))^{(q)}| \leq |(\mathfrak{J}_{0,z}^{p,\lambda-1,\mu,\eta} g(z))^{(q)}|, \ \ |z| \leq r_4,$$

where $r_4 = r_4(\alpha, \eta, \mu)$ *is the smallest positive root of the equation*

$$|\lambda + \eta + p - 2\alpha| r^3 - (\lambda + \eta - p + 2) r^2 - (|\lambda + \eta + p - 2\alpha| + 2) r + (\lambda + \eta - p) = 0,$$

$$r_4 = \frac{L_4 - \sqrt{L_4^2 - 4|\lambda+\eta+p-2\alpha|(\lambda+\eta-p)}}{2|\lambda+\eta+p-2\alpha|} \ \text{and} \ L_4 = \lambda + \eta - p + 2 + |\lambda + \eta + p - 2\alpha|.$$

Corollary 5. *Let the function* $f \in \Sigma_1$ *and* $g(z) \in \mathcal{M}_{0,z}^{\lambda,\mu,\eta}(\alpha)$ *if* $(\mathfrak{J}_{0,z}^{\lambda,\mu,\eta} f(z))^{(q)}$ *is majorized by* $(\mathfrak{J}_{0,z}^{\lambda,\mu,\eta} g(z))^{(q)}$ *in* Δ^* *then*

$$|(\mathfrak{J}_{0,z}^{\lambda-1,\mu,\eta} f(z))^{(q)}| \leq |(\mathfrak{J}_{0,z}^{\lambda-1,\mu,\eta} g(z))^{(q)}|, \ \ |z| \leq r_5,$$

where $r_5 = r_5(\alpha, \eta, \mu)$ *is the smallest positive root of the equation*

$$|\lambda + \eta + 1 - 2\alpha| r^3 - (\lambda + \eta + 1) r^2 - (|\lambda + \eta + 1 - 2\alpha| + 2) r + (\lambda + \eta - 1) = 0,$$

$$r_5 = \frac{L_5 - \sqrt{L_5^2 - 4|\lambda+\eta+1-2\alpha|(\lambda+\eta-1)}}{2|\lambda+\eta+1-2\alpha|} \ \text{and} \ L_5 = \lambda + \eta + 1 + |\lambda + \eta + 1 - 2\alpha|$$

Concluding Remarks: Further specializing the parameters λ, η one can define the various other interesting subclasses of Σ_p involving the various integral operators and the corresponding Corollarylaries as mentioned above can be derived easily. The details involved may be left as an exercise for the interested reader.

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References

[1] O. Altintaş, Ö.Ozkan and H. M. Srivastava, Majorization by starlike functions ofcomplex order, Complex Variables Theory Appl., 46, no. 3, 207–218, 2001.

[2] J.K. Prajapat, R.K. Raina and H.M. Srivastava, Some inclusion properties for certain subclasses of strongly starlike and strongly convex functions involving a family of fractional integral operators, Integral Transform. Spec. Funct., 18, 639 – 651, 2007.

[3] S. P.Goyal and P. Goswami, Majorization for certain classes of meromorphic functions defined by integral operator, Annales. Univ. Marie-Curie-Sklod. Math. Lett., 66, no. 2, 57–62, 2012.

[4] S.P. Goyal and J.K. Prajapat, A new class of analytic $p-$valent functions with negative coefficients and fractional calculus operators, Tamsui Oxf. J Math. Sci., 20, 175 – 186, 2004.

[5] R. J. Libera, Univalent α spiral-like functions, Canadian. J. Math. Soc., 19, 449–456, 1967.

[6] A.Y. Lashin, On certain subclasses of meromorphic functions associated with certain integral operators, Comput. Math Appl., 59 (1), 524–531, 2010.

[7] T. H.MacGregor,: Majorization by univalent functions,Duke Math. J., 34, 95–102, 1967.

[8] S. S. Miller,and P.T.Mocanu, Differential subordinations, Monographs and Textbooks in Pure and Applied Mathematics, 225, Dekker, New York, 2000.

[9] S. Owa, On the distortion theorems, I. Kyungpook Math. J., 18, 53 - 59, 1978.

[10] L. *Špaček,Přı́spěvek* k teorii funkci *prostýčh,Časopis Pěst* Math., 63, 12–19, 1933.

[11] H.M.Srivastava and S.Owa, A note on certain class of spiral-like functions, Rend.Sem.Mat.Univ.Padavo., 80, 17–24, 1988.

[12] H.M. Srivastava, M. Saigo and S. Owa, A class of distortion theorems involving certain operators of fractional calculus, J. Math. Anal. Appl.,131, 412 – 420, 1988.

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