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ON FRACTIONAL ORDER HIGH TEMPERATURE SUPERCONDUCTIVITY

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ABSTRACT. It is argued that fractional order formulation is suitable for strongly interacting systems. Also that high-temperature superconductivity is still an open problem.that needs at least two mechanisms to explain.

1. Superconductivity and Quantum Entanglement

1.1. **Quantum Entanglement** [1]**.** Quantum entanglement: is a physical phenomenon used in quantum theory to describe the way that two particles (or group) of matter can become correlated and interact with each other regardless of the the distance between each other.

Entanglement indicates that nonlocality should be included in formulating quantum system.

1.2. **Definition of Superconductivity**[2]**.** Superconductivity: is a quantum mechanical phenomenon which occurs when the electrical resistance (R) completely disappear in solids when they are cooled below a characteristic temperature. (since the resistivity of a superconductor goes to zero). This temperature is called transition temperature or critical temperature, conductivity ($\to \infty$)

Since $R = \frac{\rho L}{A}$ and $\rho = 0$ then $R = 0$

$G = \sigma \frac{A}{L}$, $\sigma = \frac{1}{\rho}$ then $G = \frac{A}{\rho L}$ so $G \to \infty$

Tc: is the critical temperature at which the resistivity of a superconductor goes to zero. Above this temperature the material is non-superconducting, while below it, the material becomes superconducting.

Superconductivity [2] is a phenomenon of exactly zero electrical resistance and expulsion of magnetic fields occurring in certain materials when cooled below a characteristic critical temperature.

1.3. **Types Superconductor :** 1- Low-temperature superconductors, or LTS.
2- High-temperature superconductors, or HTS.

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1.3.1. *Low-Temperature Superconductors*[3]. The critical temperature of the low-temperature superconductors is below 77K and their properties are:

No electric resistance ($R = 0$).

No magnetic field ($B = 0$).

Mechanism of Low-Temperature superconductivity

Isotope effect, T_c depends on the mass of atoms

$$T_c \propto \frac{1}{\sqrt{mass\ of\ atoms\ constituting\ the\ crystal\ lattice}}$$

Interaction between electrons and lattice atoms is critical for the existence of superconductive state.

In LHS $T_c \propto M^{-\alpha}$, $\alpha = 0.5$ so the the electron − phonon is exist

Meissner Effect[4] : is a phenomenon of exactly zero electrical resistance and expulsion of magnetic fields occurring in certain materials when cooled below a characteristic critical temperature.

The superconducting state is characterized by two properties: no electric resistance ($R = 0$) and no magnetic field ($B = 0$).

Meissner Effect from London Equation: The London equation relates the curl of the current density J to the magnetic field:

$$\overrightarrow{\nabla} \times \overrightarrow{J} = -\frac{1}{\mu_0 \lambda_L^2} \overrightarrow{B} \tag{1}$$

S.T $\overrightarrow{J}$: is the current density

By relating the London equation to Maxwell's equations, it can be shown that the Meissner effect arises from the London equation. One of Maxwell's equations is

$$\overrightarrow{\nabla} \times \overrightarrow{B} = \mu_0 \overrightarrow{J} \tag{2}$$

Using the vector calculus identity:

$$\overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} \times \overrightarrow{B}\right) = \overrightarrow{\nabla} \times \mu_0 \overrightarrow{J} \to \nabla^2 \overrightarrow{B} = \mu_0 \left(-\frac{1}{\mu_0 \lambda_L^2} \overrightarrow{B}\right) \tag{3}$$

$$\therefore \overrightarrow{\nabla^2} B = \frac{1}{\lambda_L^2} \overrightarrow{B} \tag{4}$$

by substitution

Then:

$$\nabla^2 \overrightarrow{B} = \frac{B}{\lambda_L^2} \tag{5}$$

Since $\nabla^2 B = 0$ by Maxwell's equations, the value for B inside the superconductor must be identically zero unless the penetration depth is infinite (i.e., not a superconductor). This is one of the theoretical approaches to explaining the Meissner effect.

Alternatively Meissner Effect[3] is a phenomenon of expulsion of magnetic fields occurring in certain materials when cooled below a characteristic critical temperature

2. The Fractional Order Formalism of Superconductivity[4]

In this section we will review the linear London equations modified by fractional derivatives for non differintial equations.

2.1. **Fractional Derivative**[12]**:** Fractional calculus is three centuries old as the conventional calculus, but not very popular amongst science and or engineering community. The beauty of this subject is that fractional derivatives (and integrals) are not a local (or point) property (or quantity). Thereby this considers the history and non-local distributed effects. In other words perhaps this subject translates the reality of nature better. Fractional calculus is one of the generalizations of the classical calculus and it has been used successfully in various fields of science and engineering. Really there are New possibilities in mathematics and theoretical physics appear, when the order of the differential operator or the integral operator becomes an arbitrary parameter.In this paper we are concerned with some notes on Riemann-Liouville fractional integral, Riemann-Liouville fractional derivative, Caputo fractional derivative and Caputo via Riemann-Liouville fractional derivative which are the most famous definitions in fractional calcuclus

2.1.1. *Basic definition.* We will use the notations:

$$D^{\alpha} f(x) = f^{(\alpha)}(x) = \frac{d^{\alpha} f(x)}{dx^{\alpha}}, I^{1-\alpha} = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ds \tag{6}$$

for the fractional derivative.

2.1.2. *Fractional Calculus operators:*[12]. The main Fractional Calculus operators are the following :

1) Riemann–Liouville fractional order integral
2) The Fractional-order derivative (Caputo sense)
3) The Riemann-Liouville Fractional order derivative
4) The Caputo derivative via Riemann-Liouville derivative

Riemann–Liouville fractional order integral. Let $\beta \in (0,1)$. The fractional order integral (R-L) operator is given by the singular integral operator of convolution type

$$I^{\beta} f(t) =_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds \ , \qquad t \ \in [0,T] \tag{7}$$

or

$$I^{\beta} f(t) =_0^t \frac{s^{\beta-1}}{\Gamma(\beta)} f(t-s) ds \ , \qquad t \ \in [0,T] \tag{8}$$

The Fractional-order derivative (Caputo sense). Let $\alpha \in (0,1)$ and $f \in AC \ [0,T]$. Then the fractional order derivative is defined by the singular integro differential operator of convolution type.

$$\begin{aligned} D^{\alpha} f(t) &= I^{1-\alpha} \frac{d}{dt} f(t) \\ &= {}_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{ds} f(s) ds9 \end{aligned} \tag{1}$$

Let $\alpha \in (n-1,n)$ (one step only) and $\frac{d^n}{dt^n} f \ \in AC \ [0,T]$. Then

$$D^{\alpha} f(t) = I^{n-\alpha} \frac{d^n}{dt^n} f(t) \tag{10}$$

Some properties;

$$\lim_{\alpha \to 1} D^{\alpha} f(t) = \frac{d}{dt} f(t) \tag{11}$$

$$\lim_{\alpha \to 0} D^{\alpha} f(t) = f(t) - f(0) \tag{12}$$

$$D^{\alpha} k = 0 \ , \ k \text{ is constant} \tag{13}$$

Let $\alpha, \ \beta \in (0,1)$. If $f^{/}$ is bounded. Then

$$D^{\alpha} D^{\beta} f(t) = D^{\alpha+\beta} f(t) \ , \ \alpha + \beta \in (0,1] \tag{14}$$

Riemann–Liouville fractional derivative. The Riemann–Liouville fractional derivative of order $\alpha \in (n-1, n)$ where n=1,2,3,.... is defined by the singular differential integral operator of convolution type.

is defined by :

$$D^{\alpha}_{R-L} (t) = \boldsymbol{D}^n \ \boldsymbol{I}^{n-\alpha} f(t) \tag{15}$$

and the derivative of order $\alpha \in (0,1)$, is given by :

$$D^{\alpha}_{R-L} (t) = \frac{d}{dt} \ \boldsymbol{I}^{1-\alpha} f(t) \tag{16}$$

which exists provided that $I^{1-\alpha} f(t)$ is differentiable.
Remarks:

$$\lim_{\alpha \to 1} D^{\alpha}_{R-L} \ f(t) \neq \frac{d}{dt} f(t) \tag{17}$$

If $f(t) = k \neq 0$, k is a constant, then $D^{\alpha}_{R-L} k \neq 0$, but equal $\frac{k+\alpha}{\Gamma(1-\alpha)}$
Now, want to prove that Caputo $\cong$ Riemann–Liouville fractional order.

$${}^{c}D^{\alpha}_{a} x(t) = D^{\alpha}_{R} x(t) \tag{18}$$

where

$$I^{1-\alpha} \frac{d}{dt} x(t) = \frac{d}{dt} I^{1-\alpha} x(t)$$

if x is differentiable and $x(0) = 0$ then

$${}^{c}D^{\alpha} x(t) = D^{\alpha}_{R} x(t)$$

if $x(0) \neq 0 = x_0$ then

$${}^{c}D^{\alpha} x(t) = \frac{-t-\alpha}{\Gamma(1-\alpha)} + D^{\alpha}_{R} x(t)$$

by integrating by parts

$$\begin{aligned} {}^{c}D^{\alpha}x(t) &= I^{1-\alpha}\frac{d}{dt}x(t) \\ &= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\frac{d}{ds}x(s)ds \end{aligned}$$

let

$$u = \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}$$

$$du = \frac{-(t-s)^{-\alpha-1}}{\Gamma(-\alpha)}ds$$

and

$$dv = \frac{d}{ds}x(s)ds$$

$$v = x(s)$$

$$\begin{aligned} {}^{c}D^{\alpha}x(t) &= \left.\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}x(s)\right|_0^t + \int_0^t \frac{(t-s)^{-\alpha-1}}{\Gamma(-\alpha)}x(s)ds \\ &= \frac{-t^{-\alpha}}{\Gamma(1-\alpha)}x(0) + \frac{d}{dt}\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}x(s)ds \\ &= \frac{-t^{-\alpha}}{\Gamma(1-\alpha)}x(0) + \frac{d}{dt}I^{1-\alpha}x(t) \\ &= \frac{-t^{-\alpha}}{\Gamma(1-\alpha)}x(0) + \boldsymbol{D}_R^{\alpha}x(t) \end{aligned}$$

if $x(0) = 0$

$${}^{c}D^{\alpha}x(t) = D_R^{\alpha}x(t)$$

ThenCaputo $\cong$ Riemann–Liouville fractional order.

The Caputo fractional order derivative via Riemann-Liouville derivative. Let $x \in AC[0,T]$, the relation between the Riemann-Liouville and Caputo derivative is given by

$${}^{C-R-L}D^{\alpha}x(t) = {}^{R-L}\ D^{\alpha}[x(t) - x(0)] \tag{19}$$

In the case when $x \notin AC[0,T]$, we simulate (or can define) the Caputo derivative via Riemann-Liouville one by the relation

In many recent papers the Caputo via Riemann-Liouville fractional derivative is called modified Riemann-Liouville fractional derivative

Some advantages can be cited, first of all, by using the CVR difinition it is found that derivative of constant is zero, and second, it is can be used so much for differentiable as non differentiable functions.

Simple rules:

1-

$$D^{\alpha}K = 0 \tag{20}$$

Since:

$$D^{\alpha}x(t)=\frac{1}{\Gamma\left(1-\alpha\right)}\frac{d}{dt}\int_{0}^{t}\left(t-s\right)^{-\alpha}\left(x\left(s\right)-x\left(0\right)\right)ds;\qquad 0\prec\alpha\prec 1. \tag{21}$$

If $x(t)=K$,
then

$$x\left(s\right)=K \qquad \text{and} \qquad x\left(0\right)=K$$

$$D^{\alpha}x(t)=\frac{1}{\Gamma\left(1-\alpha\right)}\frac{d}{dt}\int_{0}^{t}\left(t-s\right)^{-\alpha}\left(K-K\right)dt=0\ ;\qquad 0\prec\alpha\prec 1.$$

Then

$$D^{\alpha}K=0$$

By using The Caputo fractional order derivative via Riemann-Liouville derivativewe we want to proof that :

$$D^{\alpha}t^{\alpha}=\frac{\Gamma\left(\gamma+1\right)}{\Gamma\left(\gamma-\alpha+1\right)}t^{\gamma-\alpha} \tag{22}$$

since $x\left(t\right)=t^{\gamma}$ so $x\left(s\right)=s^{\gamma}$ and $x\left(0\right)=0$ then eq. (21) can be written as

$$D^{\alpha}t^{\gamma}\ =\frac{1}{\Gamma\left(1-\alpha\right)}\frac{d}{dt}\int_{0}^{t}\left(t-s\right)^{-\alpha}s^{\gamma}\ ds\ ;$$

Let, $s=tx$

$$\begin{aligned}
D^{\alpha}x^{\gamma} &= \frac{1}{\Gamma\left(1-\alpha\right)}\frac{d}{dt}\int_{0}^{1}\left(1-x\right)^{-\alpha}t^{-\alpha}\left(tx\right)^{\gamma}xdt;\\
&= \frac{1}{\Gamma\left(1-\alpha\right)}\frac{d}{dt}\left[t^{\gamma-\alpha+1}\right]\int_{0}^{1}\left(1-x\right)^{-\alpha}x^{\gamma}dx;\\
&= \frac{1}{\Gamma\left(1-\alpha\right)}\left[\left(\gamma-\alpha+1\right)t^{\gamma-\alpha}\right]\int_{0}^{1}\left(1-x\right)^{(1-\alpha)-1}x^{(\gamma+1)-1}dx; \qquad 23(2)
\end{aligned}$$

Since the definition of Beta function is

$$\begin{aligned}
B\left(\alpha,\gamma\right) &= \int_{0}^{1}\left(1-x\right)^{\alpha-1}x^{\gamma-1}dx\\
&= \frac{\Gamma\left(\alpha\right)\Gamma\left(\gamma\right)}{\Gamma\left(\alpha+\gamma\right)}24 \qquad (3)
\end{aligned}$$

So eq. (23) becomes

$$\begin{aligned}
D^{\alpha}t^{\gamma} &= t^{\gamma-\alpha}\ \left(\gamma-\alpha+1\right)\frac{1}{\Gamma\left(1-\alpha\right)}\ \frac{\Gamma\left(1-\alpha\right)\Gamma\left(1+\gamma\right)}{\Gamma\left(\gamma-\alpha+1+1\right)}\\
&= t^{\gamma-\alpha}\ \frac{\Gamma\left(1+\gamma\right)}{\Gamma\left(\gamma-\alpha+1\right)}25 \qquad (4)
\end{aligned}$$

2.2. **Linear Fractional London Equation**[6]**.** To investigating the magnetic-field distribution feature in the fractional formalism, we need the modified London equation. for comparison with that in the integer case we will first review the derivation of the modified london equation which is usually written as[7]:

$$B + \lambda^2 \nabla \times (\nabla \times B) = 0 \tag{27}$$

where λ is the London penetration depth and B is the magnetic induction vector field. With the standard form[8],

$$\lambda^2 \nabla^2 B(r) - B(r) = -\Phi_0 \delta^{(2)}(r)\, \acute{z}, \tag{28}$$

here $\Phi_0 = 2\pi\hbar/e^*$ is the quantum flux and ∇^2 is the vectorial Laplacian. The magnetic field is in $\acute{z}$ direction and depends only on radial coordinates r.

This equation have a well known exact solution given by:

$$B = \frac{\Phi_0}{2\pi\lambda} K_0\left(\frac{r}{\lambda}\right), \tag{29}$$

where K_0 is the zero order Hankel function.

With this 2-D geometry we can write the operator ∇^2 as

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) \tag{30}$$

letting to rewrite eq. (28) as

$$-\frac{\lambda^2}{r}\frac{d}{dr}\left(r\frac{dB}{dr}\right) + B(r) = \Phi_0 \delta^{(2)}(r - r_0). \tag{31}$$

In order to rewrite the equation above with fractional derivatives we use a generalized fractional Laplacian, in the MRL's sense, of a form given by[9], [10]

$$\frac{1}{r^\alpha} D_r^\alpha \left(r^\alpha D_r^\alpha B(r)\right).$$

With this fractional operator we can write the fractional London equations as

$$\frac{\lambda^{2\alpha}}{r^\alpha} D_r^\alpha \left(r^\alpha D_r^\alpha B(r)\right) + B(r) = \Phi_0 \delta^{(2)}(r - r_0). \tag{32}$$

3. Fractional quantum field theory, path integral, and stochastic differential equation for strongly interacting many-particle systems[11]

We Explain Low-temperature Superconductors, and weak ineractions ,by using Equation (1-15)(Ginzburg – Landau theory [3]).

$$F = F_0 + a(T - T_c)|\psi|^2 + \frac{b}{2}|\psi|^2 \quad , \quad a\ , b \succ 0,\ T \succ T_c$$

but in high- T_c superconductivity can use The fractional Schrodinger equation ,

The fractional Schrodinger equation It was discovered by Nick Laskin (1999) ,which is[12]:

$$ih\frac{\partial \Psi(r,t)}{\partial t} = D_\alpha(-h^2\frac{\partial^2}{\partial r^2})^{\frac{\alpha}{2}}\Psi(r,t) + V(r,t)\Psi(r,t) \tag{33}$$

where : r is the 3-dimensional position vector, ħ is the reduced Planck constant, Ψ(r, t) is the wavefunction, which is the quantum mechanical probability amplitude for the particle to have a given position r at any given time t, $V(r,t)$ is a potential energy, $\frac{\partial^2}{\partial r^2}$ is the Laplace operator. $D_\alpha$is a scale constant with physical dimension $[D\alpha] = [energy]^{1-\alpha} \cdot [length]^{\alpha}[time]^{-\alpha}$, at $\alpha = 2$, $D_2 = 1/2m$, where m is a particle mass and the operator (- $h^2 \frac{\partial^2}{\partial r^2})^{\frac{\alpha}{2}}$ is the 3-dimensional fractional quantum Riesz derivative. The fractional Schrodinger equation is a fundamental equation of fractional quantum mechanics. . The fractional Schrodinger equation has many problems, such as the nonvalidity of the quantum superposition law, the violation of unitarity of the time evolution, and the violation of probability conservation which can produce nonsensical probabilities $\succ$ 1. However, these problems exist only if we restrict ourselves only to the free effective action, and this is meaningless, since the entire theory is only defined by the effective action in the strong-coupling limit and this contains necessarily additional nonquadratic terms. Hence it does not possess free quasiparticles as in the time-honored Landau theory of Fermi liquids. There is always an interaction that invalidates the standard discussion of Schrödinger equations. In fact, the theory of high- T_c superconductivity must probably be build as a true strong-coupling theory of this type with electrons being non-fermi liquids.

4. High-temperature superconductors:

4.1. Weak Nonzero Isotopic Effect. The critical temperature of the high-temperature superconductors is above 77K, The first high-Tc superconductor was discovered in 1986 by IBM researchers Karl Müller and Johannes Bednorz , [3],and their properties are:

1- planar (Exist in two dimensions and the third dimension is an insulator
2- No magnetic field (B = 0).
3- At $T \succ T_c \rightarrow \rho = 0$
4- T_c ¿ 90 K

Mechanism of high-temperature superconductivity
Isotope effect, T_c depends on the mass of atoms

$$T_c \propto \frac{1}{\sqrt{mass\ of\ atoms\ constituting\ the\ crystal\ lattice}} \tag{34}$$

In HTS

$$T_c \propto M^{-\alpha}, \alpha \cong 0.05 \tag{35}$$

so the electron-phonon exists.

There has to be a further strong interaction since

$$T_c > 90K \tag{36}$$

We think that, we need two interactions to explain high temperature superconductivity.

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