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EXISTENCE OF SOLUTIONS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS WITH INTEGRAL BOUNDARY VALUE CONDITIONS

DANDAN YANG

Abstract. In this paper, we investigate the existence of solutions for fractional boundary value problems with integral boundary value conditions in the autonomous case:

$$\begin{cases} {}^cD^{\alpha}_{0+}y(t) \in F(y(t)), & t \in (0,1), \\ y(0) + y'(0) = g(y), & \displaystyle\int_0^1 y(t)dt = m, \\ y''(0) = y'''(0) = ... = y^{(n-1)}(0) = 0, \end{cases}$$

where ${}^cD^{\alpha}_{0+}$ is the Caputo fractional derivatives, $F : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map, and $g : C([0,1], \mathbb{R}) \to \mathbb{R}$ is a continuous function and $m \in \mathbb{R}$, $n - 1 < \alpha < n$, $n \geq 2$. By means of some standard fixed point theorems, sufficient conditions for the existence of solutions for the fractional differential inclusions with integral boundary value problems are presented. An example is presented to illustrate our main result. Our result generalizes the single known results to the multi-valued ones.

1. Introduction

Fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non-integer order, which is a wonderful technique to understand of memory and hereditary properties of materials and processes. Some recent contributions to fractional differential equations have been carried out, see the monographs ([3]-[5],[11],[13]-[15]), and the references cited therein. On the other hand, much attention has been focused on the study of integral boundary conditions, which are applied in different fields, such as blood flow problems, chemical engineering, underground water flow, populations dynamics and so on. Problems can be expressed as nonlocal problems with integral boundary conditions, for details, please see([1],[3],[11],[13]-[14]) and the references therein.

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In 2014, Yan, Sun and Lu etc. in [13] considered the following fractional differential boundary value problem with integral boundary conditions:

$$\begin{aligned} &{}^cD^{\alpha}_{0+}x(t) = f(t, x(t), {}^cD^{\beta}_{0+}x(t)), \quad t \in (0,1), \\ &x(0) + x'(0) = y(x), \quad \int_0^1 x(t)dt = m, \\ &x''(0) = x'''(0) = ... = x^{(n-1)}(0) = 0, \end{aligned} \tag{1}$$

where ${}^cD^{\alpha}_{0+}, {}^cD^{\beta}_{0+}$ are the Caputo fractional derivatives, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. $y : C([0,1], \mathbb{R}) \to \mathbb{R}$ is a continuous function and $m \in \mathbb{R}$, $n-1 < \alpha < n$, $n \geq 2$, $0 < \beta < 1$ is a real number. By using the Banach fixed-point theorem and the Schauder fixed-point theorem, some existence of solutions are obtained.

In 2013, Ahmad, Ntouyas and Alsaedi in [1] investigated the flowing fractional differential inclusions with anti-periodic type integral boundary conditions given by

$$\begin{aligned} &{}^cD^q x(t) \in F(t, x(t)), \quad t \in (0,T), \quad 2 < q \leq 3, \\ &x^{(j)}(0) + \lambda_j x^{(j)}(T) = \mu_j \int_0^T g_j(s, x(s))ds, \quad j = 0,1,2, \end{aligned} \tag{2}$$

where ${}^cD^q$ denotes the Caputo derivative of fractional order q, x^j denotes jth derivative of x, $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, $g_j : [0,T] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $\lambda_j, \mu_j \in \mathbb{R}(\lambda_j \neq 1)$. By beans of some standard fixed point theorems for inclusions, the authors established the existence of solutions for fractional differential inclusions of order $q \in (2,3]$ with anti-periodic type integral boundary conditions.

This paper is motivated by [13], in which the authors considered (1) with F as a single-valued map. We study the existence of solutions to the following fractional inclusions with integral boundary value conditions in the autonomous case:

$$\begin{aligned} &{}^cD^{\alpha}_{0+}y(t) \in F(y(t)), \quad t \in (0,1), \\ &y(0) + y'(0) = g(y), \quad \int_0^1 y(t)dt = m, \\ &y''(0) = y'''(0) = ... = y^{(n-1)}(0) = 0, \end{aligned} \tag{3}$$

where ${}^cD^{\alpha}_{0+}$ is the Caputo fractional derivatives, $F : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, and $g : C([0,1], \mathbb{R}) \to \mathbb{R}$ is a continuous function and $m \in \mathbb{R}$, $n-1 < \alpha < n$, $n \geq 2$. Sufficient conditions for the existence of solutions are given by means of the fixed point theorem for multi-valued mapping. The methods used are well known, but the exposition in the framework of problem is new. The rest of this paper is organized as follows. We first present some basic definitions of fractional calculus and multi-valued maps. In section 3, the main result on the existence of solutions for integral boundary value problem (3) is presented. An example is given to illustrate our main result in last section.

2. Preliminaries

In this section, we recall some notations, definitions and preliminaries about fractional calculus ([9],[12])and multi-valued maps ([2],[6]-[8],[10]) that will be used in the remainder.

Definition 1. The αth fractional order integral of the function $u : (0, \infty) \mapsto R$ is defined by

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

where $\alpha > 0$, Γ is the gamma function, provided the right side is pointwise defined on $(0, \infty)$.

Definition 2. The αth fractional order derivative of a continuous function $u : (0, \infty) \mapsto R$ is defined by

$$D_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-\alpha-1} u(s) ds,$$

where $\alpha > 0$, $n = [\alpha]+1$, provided that the right side is pointwise defined on $(0, \infty)$.

Definition 3. Caputo fractional derivative of order $\alpha > 0$ for a function u defined on $[0, \infty)$ is given by

$${}^c D_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, \infty)$. For normed space $(X, \| \cdot \|)$, let

$$P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{is closed}\},$$

$$P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{is bounded}\}.$$

$$P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{is compact}\},$$

$$P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{is convex and compact}\}.$$

For each $y \in C([0,1], \mathbb{R})$, denote the selection set of F as

$$S_{F,y} := \{v \in L^1([0,1], \mathbb{R}) : v(t) \in F(y(t)) \quad a.e. t \in [0,1]\}.$$

To set the frame for our main results, we introduce the following lemmas.

Lemma 1 (Nonlinear alternative for Kakutani maps , [7]). Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \bar{U} \to \mathcal{P}_{c,cv}(C)$ is a upper semicontinuous compact map; here $\mathcal{P}_{c,cv}(C)$ denotes the family of nonempty, compact convex subsets of C. Then either

(i) F has a fixed point in $\bar{U}$, or

(ii) there is a $u \in \partial U$ and $\lambda \in (0,1)$ such that $u \in \lambda F(u)$.

Lemma 2 ([10]). Let X be a Banach space. Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathedory multivalued map and let H be a linear continuous mapping from $L^1([0,1], X) \to C([0,1], X)$. Then the operator

$$\Theta \circ S_F : C([0,1], X) \to \mathcal{P}_{cp,c}(C([0,1], X)), x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,1], X) \times C([0,1], X)$.

In order to study the problem (3), we need the following lemma,which is presented in [13], so we omit the proof here.

Lemma 3. For a given $2 < \alpha < 3, \lambda \neq 2$, $h \in AC([0,1],\mathbb{R}), g \in C([0,1],\mathbb{R})$, the unique solution of the boundary value problem:

$$\begin{aligned} &{}^cD_{0+}^{\alpha} y(t) = h(t), \quad t \in (0,1),\\ &y(0) + y'(0) = g(y), \quad \int_0^1 y(t)dt = m, \\ &y''(0) = y'''(0) = ... = y^{(n-1)}(0) = 0, \end{aligned} \tag{4}$$

has a unique solution

$$y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + 2(1-t)m + (2t-1)g(y) + \frac{2(t-1)}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} h(s)ds.$$

For convenience, denote

$$G = \max_{t\in[0,1]} |g(y(t))|.$$

3. Main results

Let us list the following assumptions:

(A_1) $F : \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is Carathéodory and has nonempty compact and convex values.

(A_2) there exists a continuous nondecreasing function each $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,1],\mathbb{R}^+)$, such that

$$\|F(y)\| = \sup\{|f| : f \in F(y)\} \leq p(t)\psi(\|y\|), \text{for} \quad y \in \mathbb{R}.$$

(A_3) There exists a constant $l_1 > 0$ such that

$$|g(y_1) - g(y_2)| \leq l_1\|y_1 - y_2\|, \quad \text{for} \quad \text{each} \quad y_1, y_2 \in C([0,1],\mathbb{R}).$$

Theorem 1. Suppose that $(A_1)-(A_3)$. If there exists a positive constant $M > 0$ such that

$$\frac{M}{\psi(M)[\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}p(s)ds + \frac{2}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha-1}p(s)ds] + 2|m| + G} > 1, \tag{5}$$

then problem (3) has at least one solution on $[0,1]$.

Proof. Define the operator $T : C([0,1],\mathbb{R}) \to \mathcal{P}(C[0,1],\mathbb{R})$ as follows:

$$\begin{aligned} T(y) = \{h(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + 2(1-t)m + (2t-1)g(y) \\ &+ \frac{2(t-1)}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha} f(s)ds, \quad h \in C([0,1],\mathbb{R})\}, \end{aligned} \tag{6}$$

for $f \in S_{F,y}$. We shall prove that the operator T satisfies all the conditions in Lemma 1. We shall divide the proof to several steps.

Step 1, for each $y \in C([0,1],\mathbb{R})$ the operator T is convex. For $S_{F,y}$ is convex, it is easy to check it.

Step 2, T maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.

For a positive number r, let $B_r = \{y \in C([0,1],\mathbb{R}) : \|y\| \leq r\}$ be a bounded ball in $C([0,1],\mathbb{R})$, then for $h \in T(y)$, $x \in B_r$, there exists $f \in S_{F,y}$ such that

$$h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + 2(1-t)m + (2t-1)g(y) + \frac{2(t-1)}{\Gamma(\alpha+1)} \int_0^1 (1-s)^\alpha f(s)ds,$$

and we have

$$|h(t)| \leq \psi(\|y\|)[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)ds + \frac{2}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha-1} p(s)ds] + 2|m| + G.$$

Thus, we obtain

$$\|h\| \leq \psi(\|r\|)[\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} p(s)ds + \frac{2}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha-1} p(s)ds] + 2|m| + G.$$

Step 3, T maps the bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t', t'' \in [0,1]$, and $t' < t''$, $y \in B_r$, where B_r is a bounded set in$C([0,1],\mathbb{R})$, for $h \in T(y)$, we have

$$\begin{aligned} |h(t'') - h(t')| \leq & \left| \int_0^{t'} \frac{(t''-s)^{\alpha-1} - (t'-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + \int_{t'}^{t''} \frac{(t''-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds \right| \\ & + 2(1 - |t'' - t'|)m + (2|t'' - t'| - 1)|g(y)| \\ & + \frac{2|t'' - t'| - 1}{\Gamma(\alpha+1)} \int_0^1 (1-s)^\alpha f(s)ds. \end{aligned}$$

the right side hand of above inequality tends to 0 independent of $y \in B_r$ as $t'' \to t'$. By means of Ascoli-Arzelá Theorem, T is completely continuous.

Step 4, T has a closed graph. Set $y_n \to y_*$, $h_n \in T(y_n)$ and $h_n \to h_*$. Then, We shall show that $h_* \in T(y_*)$. for $h_n \in T(y_n)$, there exist $f_n \in S_{F,y_n}$ such that

$$h_n(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_n(s)ds + 2(1-t)m + (2t-1)g(y) + \frac{2(t-1)}{\Gamma(\alpha+1)} \int_0^1 (1-s)^\alpha f_n(s)ds.$$

Thus, it suffices to show that there exists $f_* \in S_{F,y}$, such that for each $t \in [0,1]$,

$$h_*(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_*(s)ds + 2(1-t)m + (2t-1)g(y) + \frac{2(t-1)}{\Gamma(\alpha+1)} \int_0^1 (1-s)^\alpha f_*(s)ds.$$

Consider the continuous linear the operator $\Phi : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ as follows:

$$\begin{aligned} f \mapsto \Phi(f)(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + 2(1-t)m \\ & + (2t-1)g(y) + \frac{2(t-1)}{\Gamma(\alpha+1)} \int_0^1 (1-s)^\alpha f(s)ds. \end{aligned}$$

Notice that

$$\begin{aligned} \|h_n(t) - h_*(t)\| = & \| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (f_n(s) - f_*(s))ds \\ & + \frac{2(t-1)}{\Gamma(\alpha+1)} \int_0^1 (1-s)^\alpha (f_n(s) - f_*(s))ds\| \to 0, \end{aligned}$$

as $n \to \infty$. Thus, by Lemma 2, $\Phi \circ S_F$ is a closed graph operator. Moreover, we have $h_n(t) \in \Phi(S_{F,y_n})$. By $y_n \to y_*$, we get

$$h_*(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_*(s)ds + 2(1-t)m + (2t-1)g(y) + \frac{2(t-1)}{\Gamma(\alpha+1)} \int_0^1 (1-s)^\alpha f_*(s)ds,$$

for some $f_* \in S_{F,y_*}$.

Step 5, there exists a open set $U \subset C([0,1],\mathbb{R})$, $y \in T(y)$ for $\lambda \in (0,1)$, $y \in \partial U$. Let $\eta \in (0,1)$, $y \in \eta T(y)$. Then for $t \in [0,1]$, there exists $f \in L^1([0,1],\mathbb{R})$, with $f \in S_{F,y}$ such that for $t \in (0,1)$, we have

$$h(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds + 2(1-t)m + (2t-1)g(y) + \frac{2(t-1)}{\Gamma(\alpha+1)} \int_0^1 (1-s)^\alpha f(s)ds.$$

Similar to the discussion of step 2, we have

$$\|h\| \le \psi(\|y\|)[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)ds + \frac{2}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha-1} p(s)ds] + 2|m| + G.$$

Thus,

$$\frac{\|y\|}{\psi(\|y\|)[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)ds + \frac{2}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha-1} p(s)ds] + 2|m| + G} \le 1.$$

By (5), there exist M such that $\|y\| \ne M$. Let

$$U = \{y \in C([0,1],\mathbb{R}) : \|y\| < M+1\}.$$

Note that the operator $T : \bar{U} \to \mathcal{P}(C([0,1],\mathbb{R}))$ is upper semicontinuous and completely continuous. By the choice of U, there is no $y \in \partial U$ such that $y \in \eta T(s)$ for some $\eta \in (0,1)$. Thus, by means of Lemma 1, we can get the conclusion that there exists a fixed point $y \in \bar{U}$, that is, it is a solution of problem (3). We complete the proof.

4. Application

In this section, we present an example to illustrate our main result.

Consider the fractional differential inclusion with integral boundary value conditions

$$\begin{gathered} {}^cD_{0+}^{\frac{5}{2}} y(t) \in F(y(t)), \quad t \in (0,1), \\ y(0) + y'(0) = \sum_{i=1}^n c_i, \quad \int_0^1 y(t)dt = 1, \\ y''(0) = y'''(0) = ... = y^{(n-1)}(0) = 0, \end{gathered} \tag{7}$$

where $\Sigma_{i=1}^n c_i < \frac{1}{5}$. Set $\alpha = \frac{5}{2}$. Obviously, condition (A_3) is satisfied.

$$y \to F(y(t)) := [\frac{|y|^5}{|y|^5+3} + 4, \frac{|y|}{|y|+1} + 2], \quad y \in \mathbb{R},$$

$$|f| \le \max(\frac{|y|^5}{|y|^5+3} + 4, \frac{|y|}{|y|+1} + 1 + 2) \le 5, \quad y \in \mathbb{R},$$

and

$$\|F(y)\| := \sup\{|v| : v \in F(y)\} \le 5 := p(t)\psi(|y|), \quad y \in \mathbb{R},$$

where $p(t) = 1$, $\psi(|y|) = 5$, we can find a positive constant M such that

$$\frac{M}{5[\frac{1}{\Gamma(\frac{5}{2})}\int_0^1 (1-s)^{\frac{3}{2}}ds + \frac{2}{\Gamma(\frac{7}{2})}\int_0^1 (1-s)^{\frac{3}{2}}ds] + 2 + \frac{1}{5}} > 1,$$

that is, $M > 4.90811$. All the conditions in Theorem 1 are satisfied. Therefore, fractional differential inclusion with integral boundary value conditions (7) has at least one solution.

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Dandan Yang
1.School of Mathematical Science, Huaiyin Normal University, Huaian 223300, China
2.Department of Mathematics, The University of Iowa, Iowa city, Iowa State 52240, USA
E-mail address: ydd423@sohu.com