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NUMERICAL AND THEORETICAL STUDY FOR SOLVING MULTI-TERM LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS USING A COLLOCATION METHOD BASED ON THE GENERALIZED LAGUERRE POLYNOMIALS

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Abstract. In this paper, a direct solution technique for solving multi-order linear fractional differential equations (LFDEs) with variable coefficients is developed using a collocation method based on the generalized Laguerre polynomials. Taking the advantages of the Laguerre polynomials, to introduce an approximate formula of the derivatives of any fractional order. The fractional derivatives are presented in terms of the Caputo sense. Special attention is given to study the convergence analysis and estimate an upper bound of the error of the proposed formula. The properties of Laguerre polynomials are utilized to reduce LFDEs to a system of algebraic equations which can be solved using an efficient numerical method. Several numerical examples are provided to confirm the theoretical results and the efficiency of the proposed method.

1. Introduction

The multi-term fractional differential equations with initial values are increasingly used to model problems in fluid flow, finance, engineering, and other areas of applications. This kind of problems is more complex than ordinary differential equations. In the field of numerical treatment of this kind, a great attention has been recently dedicated to the development efficient and accurate numerical methods ([10], [12], [19]). In the last three decades, spectral methods such as Tau method [2] and collocation methods ([3], [4], [7], [8], [16], [18]) have a considerable attention to seek with this kind of problems. Orthogonal polynomials are fundamental concepts in approximation theory and form the basis of spectral methods of the solution of differential equations such as shifted Chebyshev polynomials [6], shifted Legendre polynomials [22], sinc and rational Legendre functions [14] and the generalized Laguerre polynomials ([1], [9], [20]-[22]). The classical generalized Laguerre polynomials constitute a complete orthogonal set of functions on the semi-infinite interval $[0, \infty)$. Collocation methods are efficient and highly accurate techniques for numerical solution of linear differential equations [15]. The basic idea

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of the spectral collocation method is to assume that the unknown solution $u(x)$ can be approximated by a linear combination of some basis functions, called the trial functions.

The main aim of the present paper is to introduce an approximate formula of the Caputo fractional derivative and the application of this approach to obtain the numerical solution of multi-order linear fractional differential equations with variable coefficients of the form

$$D^{\nu}u(x)+\sum_{j=1}^{n-1}\gamma_j(x)\,D^{\beta_j}\,u(x)+\gamma_n(x)u(x)=g(x), \tag{1}$$

with the following initial conditions

$$u^{(j)}(0)=u_j, \qquad j=0,1,...,n-1, \tag{2}$$

where $n<\nu\le n+1$, $n\in\mathbb{N}$, $0<\beta_1<\beta_2<...<\beta_{n-1}<\nu$ and D^{ν} denotes Caputo fractional derivative of order ν and $g(x)$ is the source term, here the functions $\gamma_i(x)$, $i=1,2,...,n$ are given functions.

2. Preliminaries and notations

In this section, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

2.1: The Caputo fractional derivative

Definition 1: The Caputo fractional derivative operator D^{ν} of order ν is defined in the following form

$$D^{\nu}f(x)=\frac{1}{\Gamma(n-\nu)}\int_0^x\frac{f^{(n)}(t)}{(x-t)^{\nu-n+1}}dt, \qquad x>0,$$

where $n-1<\nu\le n,\ n\in\mathbb{N}$ and $\Gamma(.)$ is the Gamma function.
Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation

$$D^{\nu}\left(\lambda\,f(x)+\mu\,g(x)\right)=\lambda\,D^{\nu}\,f(x)+\mu\,D^{\nu}\,g(x), \tag{3}$$

where λ and μ are constants. For the Caputo's derivative we have

$$D^{\nu}\,C=0, \qquad \text{C is a constant}, \tag{4}$$

$$D^{\nu}\,x^n=\begin{cases}0, & \text{for}\quad n\in\mathbb{N}_0 \text{ and } n<\lceil\nu\rceil;\\ \frac{\Gamma(n+1)}{\Gamma(n+1-\nu)}x^{n-\nu}, & \text{for}\quad n\in\mathbb{N}_0 \text{ and } n\ge\lceil\nu\rceil.\end{cases} \tag{5}$$

We use the ceiling function $\lceil\nu\rceil$ to denote the smallest integer greater than or equal to ν and $\mathbb{N}_0=\{0,1,2,...\}$. Recall that for $\nu\in\mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.
For more details on fractional derivatives definitions and its properties see ([5], [17]).

2.2: The definition and properties of the generalized Laguerre polynomials

The generalized Laguerre polynomials $[L_n^{(\alpha)}(x)]_{n=0}^{\infty}$, $\alpha>-1$ are defined on the semi-infinite interval $[0,\infty)$ and can be determined with the aid of the following recurrence formula [4]

$$(n+1)L_{n+1}^{(\alpha)}(x)+(x-2n-\alpha-1)L_n^{(\alpha)}(x)+(n+\alpha)L_{n-1}^{(\alpha)}(x)=0, \quad n=1,2,..., \tag{6}$$

where, $L_0^{(\alpha)}(x)=1$ and $L_1^{(\alpha)}(x)=\alpha+1-x$.

The explicit formula of these polynomials of degree n is given by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} x^k = \binom{n+\alpha}{n} \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!}\,, \tag{7}$$

with $(a)_0 := 1$ and $(a)_k := a(a+1)(a+2)...(a+k-1),\ k = 1, 2, 3, ...$ and $L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}$. These polynomials are orthogonal on the interval $[0, \infty)$ with respect to the weight function $w(x) = \frac{1}{\Gamma(1+\alpha)} x^\alpha e^{-x}$ and the orthogonality relation is

$$\frac{1}{\Gamma(1+\alpha)} \int_0^\infty x^\alpha e^{-x} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) dx = \binom{n+\alpha}{n} \delta_{mn}. \tag{8}$$

Also, they satisfy the differentiation formula

$$D^k L_n^{(\alpha)}(x) = (-1)^k L_{n-k}^{(\alpha+k)}(x), \qquad k = 0, 1, ..., n. \tag{9}$$

Any function $u(x)$ belongs to the space $L_w^2[0, \infty)$ of all square integrable functions on $[0, \infty)$ with weight function $w(x)$, can be expanded in the following Laguerre series

$$u(x) = \sum_{i=0}^{\infty} c_i L_i^{(\alpha)}(x), \tag{10}$$

where the coefficients c_i are given by

$$c_i = \frac{\Gamma(i+1)}{\Gamma(i+\alpha+1)} \int_0^\infty x^\alpha e^{-x} L_i^{(\alpha)}(x) u(x) dx, \quad i = 0, 1, 2, \ldots. \tag{11}$$

Consider only the first $(m + 1)$ terms of generalized Laguerre polynomials, so we can write

$$u_m(x) \cong \sum_{i=0}^{m} c_i L_i^{(\alpha)}(x). \tag{12}$$

3. The approximated fractional derivatives of $L_n^{(\alpha)}(x)$ and its convergence analysis

The main goal of this section is to introduce the following four theorems to derive an approximate formula of the fractional derivatives of the generalized Laguerre polynomials and study the truncating error for the approximated formula.

The main approximate formula of the fractional derivative of $u(x)$ is given in the following theorem.

Theorem 1 [11]

Let $u(x)$ be approximated by the generalized Laguerre polynomials as (12) and also suppose $\nu > 0$ then, its approximated fractional derivative can be written in the following form

$$D^\nu(u_m(x)) = \sum_{i=\lceil\nu\rceil}^{m} \sum_{k=\lceil\nu\rceil}^{i} c_i\, w_{i,\,k}^{(\nu)}\, x^{k-\nu}, \tag{13}$$

where $w_{i,\,k}^{(\nu)}$ is given by

$$w_{i,\,k}^{(\nu)} = \frac{(-1)^k}{\Gamma(k+1-\nu)} \binom{i+\alpha}{i-k}. \tag{14}$$

Theorem 2 [11]

The Caputo fractional derivative of order ν for the generalized Laguerre polynomials can be expressed in terms of the generalized Laguerre polynomials themselves in the following form

$$D^{\nu} L_i^{(\alpha)}(x) = \sum_{k=\lceil\nu\rceil}^{i} \sum_{j=0}^{k-\lceil\nu\rceil} \Omega_{i\,j\,k}\, L_j^{(\alpha)}(x), \qquad i = \lceil\nu\rceil, \lceil\nu\rceil+1, ..., m\,, \tag{15}$$

where

$$\Omega_{i\,j\,k} = \frac{(-1)^{j+k}\,(\alpha+i)!\,(k-\nu+\alpha)!}{(i-k)!\,(\alpha+k)!\,(k-\nu-j)!\,(\alpha+j)!}\,.$$

Theorem 3

The error $|E_T(m)| = |D^{\nu}u(x) - D^{\nu}u_m(x)|$ in approximating $D^{\nu}u(x)$ by $D^{\nu}u_m(x)$ is bounded by [11]

$$|E_T(m)| \leq \sum_{i=m+1}^{\infty} c_i \Pi_{\nu}(i,j)\, \frac{(\alpha+1)_j}{j!}\, e^{x/2}, \qquad \alpha \geq 0,\;\; x \geq 0,\;\; j = 0,1,\dots\,, \tag{16}$$

$$|E_T(m)| \leq \sum_{i=m+1}^{\infty} c_i \Pi_{\nu}(i,j)\, \Big(2 - \frac{(\alpha+1)_j}{j!}\Big)\, e^{x/2}, \qquad -1 < \alpha \leq 0,\;\; x \geq 0,\;\; j = 0,1,\dots\,, \tag{17}$$

where, $\Pi_{\nu}(i,j) = \sum_{k=\lceil\nu\rceil}^{i} \sum_{j=0}^{k-\nu} \Omega_{i\,j\,k}$.

4. Procedure of solution for the multi-order LFDEs

Consider the multi-order linear fractional differential equation of type given in Eq.(1). Let $w(x) = \frac{1}{\alpha!} x^{\alpha} e^{-x}$ be a positive weight function on the interval $I = [0, \infty)$ and $L^2_w(I)$ is the weighted space L^2 with inner product

$$(u, v)_w = \int_0^{\infty} w(x) u(x) v(x) dx,$$

and the associated norm $\|u\|_w = (u,u)_w^{\frac{1}{2}}$. It is well known that $[L_n^{(\alpha)}(x) : n \geq 0]$ forms a complete orthogonal system in $L^2_w(I)$, so if we define

$$S_m(I) = \mathrm{Span}[L_0^{(\alpha)}(x), L_1^{(\alpha)}(x), ..., L_m^{(\alpha)}(x)], \tag{18}$$

then, the Laguerre spectral solution of Eq.(1) is to find $u_m \in S_m(I)$ such that Eq.(12) achieved. From Eq.(1) and (12) and Theorem 1 we have

$$\begin{aligned} \sum_{i=\lceil\nu\rceil}^{m} \sum_{k=\lceil\nu\rceil}^{i} c_i\, w_{i,\,k}^{(\nu)}\, x^{k-\nu} + \sum_{j=1}^{r-1} \gamma_j(x) \Big(\sum_{i=\lceil\beta_j\rceil}^{m} \sum_{k=\lceil\beta_j\rceil}^{i} c_i\, w_{i,\,k}^{(\beta_j)}\, x^{k-\beta_j} \Big) \\ + \gamma_r(x) \sum_{i=0}^{m} c_i\, L_i^{(\alpha)}(x) = g(x). \end{aligned} \tag{19}$$

We now collocate Eq.(19) at $(m+1-\lceil\nu\rceil)$ points x_p, $p=0,1,...,m-\lceil\nu\rceil$ as

$$\sum_{i=\lceil\nu\rceil}^{m}\sum_{k=\lceil\nu\rceil}^{i} c_i w_{i,\,k}^{(\nu)} x_p^{k-\nu} + \sum_{j=1}^{r-1}\gamma_j(x_p)\Big(\sum_{i=\lceil\beta_j\rceil}^{m}\sum_{k=\lceil\beta_j\rceil}^{i} c_i\, w_{i,\,k}^{(\beta_j)} x_p^{k-\beta_j}\Big) + \gamma_r(x_p)\sum_{i=0}^{m} c_i L_i^{(\alpha)}(x_p) = g(x_p). \tag{20}$$

For suitable collocation points we use roots of the generalized Laguerre polynomial $L_{m+1-\lceil\nu\rceil}^{(\alpha)}(x)$.
Also, by substituting Eq.(9) in the initial conditions (2) and using the property $L_i^{(\alpha)}(0) = \left(\begin{array}{c}\alpha+i\\ i\end{array}\right)$ we can obtain $\lceil\nu\rceil$ of equations

$$\sum_{i=0}^{m} c_i\,(-1)^j \left(\begin{array}{c}\alpha+i\\ i-j\end{array}\right) = u_j, \qquad j=0,1,2,...,m-1. \tag{21}$$

Eqs.(20) together with $\lceil\nu\rceil$ equations of the initial conditions (21), give $(m+1)$ of linear algebraic equations which can be solved, for the unknowns $c_i,\ i=0,1,...,m$, using a suitable numerical method, as described in the following section 6.

5. Error estimate of the collocation method for non-homogenous linear FDEs

In this section, we introduce some necessary notations and prove the theorems of error estimate. Consider the following form of non-homogenous linear FDEs

$$[D^{n\nu}+a_1D^{(n-1)\nu}+...+a_{n-1}D+a_n]u(x)=f(x),\ D^ju(0)=0,\ \ j=0,1,...,m-1, \tag{22}$$

where, m is the smallest integer greater than or equal to $n\nu$ and $x\in[0,\infty)$.

Theorem 4

Let $f(x)$ be a piecewise continuous on $(0,\infty)$, integrable and of exponential order on $[0,\infty)$. Let $P(t)=t^n+a_1t^{n-1}+...+a_n$ be the indicial polynomial of LFDE (22) and let $K(x)$ is the fractional Green function defined by

$$K(x)=\ell^{-1}[P(s^\nu)]^{-1}, \tag{23}$$

where ℓ^{-1} is the inverse Laplace transform. Then,

$$u(x)=\int_0^x K(x-\zeta)f(\zeta)d\zeta, \tag{24}$$

is the unique solution of (22).
For more details about this theorem, its proof and the fractional Green function see [13].

Theorem 5

The truncation error of the spectral Laguerre solution (12) of the non-homogenous linear FDEs (22) is estimated using the following integral form

$$e(x)=\int_0^x K(x-\zeta)\delta f(\zeta)d\zeta. \tag{25}$$

Proof.

We may analyze the error of the preceding Laguerre collocation method by the use of backward error analysis. Let $u(x)$ denotes the exact solution of the problem

(22). Let $u_m(x)$ which defined by (12) be the approximate solution obtained by collocation method and let $f_n(x)$ be the $n\nu$-th derivative of $u_m(x)$. Then $u_m(x)$ is itself the exact solution to the following similar problem

$$[D^{n\nu}+a_1 D^{(n-1)\nu}+...+a_{n-1}D+a_n]u_m(x) = f_m(x),\ \ D^j u_m(0) = 0,\ \ j = 0,1,...,m-1. \tag{26}$$

Since Eqs.(22) and (26) are both linear, and have the same initial conditions and the error $e(x) := u(x)-u_m(x)$ must be the solution of the non-homogeneous initial-value problem

$$[D^{n\nu} + a_1 D^{(n-1)\nu} + ... + a_n]e(x) = \delta f(x), \quad D^j e(0) = 0,\ \ j = 0,1,...,m-1, \tag{27}$$

where

$$\delta f(x) = f(x) - f_m(x). \tag{28}$$

The solution of Eq.(27) corresponds to Theorem 5 can be rewritten in the integral form (24) which leads to the desired result.

6. Numerical simulation and comparison

In order to illustrate the effectiveness of the proposed method, we implement it to solve the linear multi-term fractional orders differential equations with different four examples.

Example 1:

Consider the following linear fractional initial value problem

$$D^2u(x) + x^{\frac{1}{2}} D^{1.234}u(x) + x^{\frac{1}{3}} Du(x) + x^{\frac{1}{4}} D^{0.333}u(x) + x^{\frac{1}{5}} u(x) = g(x), \tag{29}$$

where, $g(x) = -1 - \frac{x^{1.266}}{\Gamma(1.766)} - x^{\frac{4}{3}} - \frac{x^{1.817}}{\Gamma(2.766)} + x^{\frac{1}{5}}(2 - \frac{x^2}{2})$,
with the following initial conditions

$$u(0) = 2, \quad u'(0) = 0. \tag{30}$$

The exact solution for this problem is $u(x) = 2 - \frac{1}{2}x^2$.
We apply the suggested method with $m = 3$, and approximate the solution $u(x)$ as follows

$$u_3(x) = \sum_{i=0}^{3} c_i\, L_i^{(\alpha)}(x). \tag{31}$$

Using Eq.(19), with $\nu = 2,\ \beta_1 = 1.234,\ \beta_2 = 1.0,\ \beta_3 = 0.333$, and $\alpha = -0.5,$ we have

$$\sum_{i=2}^{3}\sum_{k=2}^{i} c_i\, w_{i,k}^{(2)}\, x^{k-2} + x^{\frac{1}{2}} \sum_{i=2}^{3}\sum_{k=2}^{i} c_i\, w_{i,k}^{(1.234)}\, x^{k-1.234} + x^{\frac{1}{3}} \sum_{i=1}^{3}\sum_{k=1}^{i} c_i\, w_{i,k}^{(1)}\, x^{k-1}$$
$$+ x^{\frac{1}{4}} \sum_{i=1}^{3}\sum_{k=1}^{i} c_i\, w_{i,k}^{(0.333)}\, x^{k-0.333} + x^{\frac{1}{5}} \sum_{i=0}^{3} c_i\, L_i^{(\alpha)}(x) = g(x). \tag{32}$$

Now, we collocate Eq.(32) at the roots x_p as

$$\sum_{i=2}^{3}\sum_{k=2}^{i} c_i\, w_{i,\,k}^{(2)}\, x_p^{k-2} + x_p^{\frac{1}{2}} \sum_{i=2}^{3}\sum_{k=2}^{i} c_i\, w_{i,\,k}^{(1.234)}\, x_p^{k-1.234} + x_p^{\frac{1}{3}} \sum_{i=1}^{3}\sum_{k=1}^{i} c_i\, w_{i,\,k}^{(1)}\, x_p^{k-1} + x_p^{\frac{1}{4}} \sum_{i=1}^{3}\sum_{k=1}^{i} c_i\, w_{i,\,k}^{(0.333)}\, x_p^{k-0.333} + x_p^{\frac{1}{5}} \sum_{i=0}^{3} c_i\, L_i^{(\alpha)}(x_p) = g(x_p), \tag{33}$$

where x_p are the roots of the generalized Laguerre polynomial $L_2^{(\alpha)}(x)$, i.e.,

$$x_0 = 2+\alpha-\sqrt{2+\alpha} = 0.275255, \qquad x_1 = 2+\alpha+\sqrt{2+\alpha} = 2.72474\,.$$

By using Eqs.(21) and (33) we obtain the following linear system of algebraic equations

$$\sum_{i=2}^{3}\sum_{k=2}^{i} c_i\, w_{i,\,k}^{(2)}\, x_0^{k-2} + x_0^{\frac{1}{2}} \sum_{i=2}^{3}\sum_{k=2}^{i} c_i\, w_{i,\,k}^{(1.234)}\, x_0^{k-1.234} + x_0^{\frac{1}{3}} \sum_{i=1}^{3}\sum_{k=1}^{i} c_i\, w_{i,\,k}^{(1)}\, x_0^{k-1} + x_0^{\frac{1}{4}} \sum_{i=1}^{3}\sum_{k=1}^{i} c_i\, w_{i,\,k}^{(0.333)}\, x_0^{k-0.333} + x_0^{\frac{1}{5}} \sum_{i=0}^{3} c_i L_i^{(\alpha)}(x_0) = g(x_0), \tag{34}$$

$$\sum_{i=2}^{3}\sum_{k=2}^{i} c_i\, w_{i,\,k}^{(2)}\, x_1^{k-2} + x_1^{\frac{1}{2}} \sum_{i=2}^{3}\sum_{k=2}^{i} c_i\, w_{i,\,k}^{(1.234)}\, x_1^{k-1.234} + x_1^{\frac{1}{3}} \sum_{i=1}^{3}\sum_{k=1}^{i} c_i\, w_{i,\,k}^{(1)}\, x_1^{k-1} + x_1^{\frac{1}{4}} \sum_{i=1}^{3}\sum_{k=1}^{i} c_i\, w_{i,\,k}^{(0.333)}\, x_1^{k-0.333} + x_1^{\frac{1}{5}} \sum_{i=0}^{3} c_i\, L_i^{(\alpha)}(x_1) = g(x_1), \tag{35}$$

$$r_0c_0 + r_1c_1 + r_2c_2 + r_3c_3 = 2, \tag{36}$$

$$s_0c_0 + s_1c_1 + s_2c_2 + s_3c_3 = 0, \tag{37}$$

where, $r_i = \begin{pmatrix} \alpha+i \\ i \end{pmatrix}$, $s_i = \begin{pmatrix} \alpha+i \\ i-1 \end{pmatrix}$, $i = 0,1,2,3.$

By solving the system of Eqs.(34)-(37) using conjugate gradient method we obtain

$$c_0 = \frac{13}{8}, \qquad c_1 = \frac{3}{2}, \qquad c_2 = -1, \qquad c_3 = 0.$$

Therefore,

$$u(x) = \left(\tfrac{13}{8}\,, \quad \tfrac{3}{2}\,, \quad -1\,, \quad 0 \right) \begin{pmatrix} 1 \\ -x+0.5 \\ 0.5x^2-1.5x+0.375 \\ -0.1667x^3+1.25x^2-1.875x+0.313 \end{pmatrix} = 2-\frac{1}{2}x^2,$$

which is the exact solution of this problem.

It is clear that in this example the presented method can be considered as an efficient method.

Example 2:

In this example, we consider the following Cauchy initial value problem

$$D^{1.5}u(x)+2Du(x)+3\sqrt{x}D^{0.5}u(x)+(1-x)u(x)=g(x), \tag{38}$$

where $g(x)=\frac{2}{\Gamma(1.5)}x^{0.5}+4x+\frac{4}{\Gamma(1.5)}x^{2}+(1-x)x^{2}$ and subject to the initial conditions

$$u(0)=u'(0)=0. \tag{39}$$

The exact solution of this example is $u(x)=x^2$.
To solve this example, by applying the proposed technique described in section 4 with $m=3$, we approximate the solution as

$$u_3(x)=c_0L_0^{(\alpha)}(x)+c_1L_1^{(\alpha)}(x)+c_2L_2^{(\alpha)}(x)+c_3L_3^{(\alpha)}(x).$$

Using Eq.(19), with $\nu=1.5$, $\beta_1=1$, $\beta_2=0.5$ and $\alpha=-0.25$ we have

$$\begin{aligned}\sum_{i=2}^{3}\sum_{k=2}^{i}c_i\,w_{i,\,k}^{(1.5)}\,x^{k-1.5}+2\sum_{i=1}^{3}\sum_{k=1}^{i}c_i\,w_{i,\,k}^{(1)}\,x^{k-1}+3\sqrt{x}\sum_{i=1}^{3}\sum_{k=1}^{i}c_i\,w_{i,\,k}^{(0.5)}\,x^{k-0.5}\\+(1-x)\sum_{i=0}^{3}c_iL_i^{(\alpha)}(x)=g(x).\end{aligned} \tag{40}$$

Now, we collocate Eq.(40) at the roots x_p as

$$\begin{aligned}\sum_{i=2}^{3}\sum_{k=2}^{i}c_i\,w_{i,\,k}^{(1.5)}\,x_p^{k-1.5}+2\sum_{i=1}^{3}\sum_{k=1}^{i}c_i\,w_{i,\,k}^{(1)}\,x_p^{k-1}+3\sqrt{x_p}\sum_{i=1}^{3}\sum_{k=1}^{i}c_i\,w_{i,\,k}^{(0.5)}\,x_p^{k-0.5}\\+(1-x_p)\sum_{i=0}^{3}c_iL_i^{(\alpha)}(x_p)=g(x_p),\end{aligned} \tag{41}$$

where x_p are roots of the generalized Laguerre polynomial $L_2^{(\alpha)}(x)$.
By using Eqs.(21), (39) and (41) we obtain the following linear system of algebraic equations

$$\begin{aligned}\sum_{i=2}^{3}\sum_{k=2}^{i}c_i\,w_{i,\,k}^{(1.5)}\,x_0^{k-1.5}+2\sum_{i=1}^{3}\sum_{k=1}^{i}c_i\,w_{i,\,k}^{(1)}\,x_0^{k-1}+3\sqrt{x_0}\sum_{i=1}^{3}\sum_{k=1}^{i}c_i\,w_{i,\,k}^{(0.5)}\,x_0^{k-0.5}\\+(1-x_0)\sum_{i=0}^{3}c_iL_i^{(\alpha)}(x_0)=g(x_0),\end{aligned} \tag{42}$$

$$\begin{aligned}\sum_{i=2}^{3}\sum_{k=2}^{i}c_i\,w_{i,\,k}^{(1.5)}\,x_1^{k-1.5}+2\sum_{i=1}^{3}\sum_{k=1}^{i}c_i\,w_{i,\,k}^{(1)}\,x_1^{k-1}+3\sqrt{x_1}\sum_{i=1}^{3}\sum_{k=1}^{i}c_i\,w_{i,\,k}^{(0.5)}\,x_1^{k-0.5}\\+(1-x_1)\sum_{i=0}^{3}c_iL_i^{(\alpha)}(x_1)=g(x_1),\end{aligned} \tag{43}$$

$$r_0c_0+r_1c_1+r_2c_2+r_3c_3=0, \tag{44}$$

$$s_0c_0+s_1c_1+s_2c_2+s_3c_3=0, \tag{45}$$

where, $\quad r_i = \begin{pmatrix} \alpha+i \\ i \end{pmatrix}, \quad s_i = \begin{pmatrix} \alpha+i \\ i-1 \end{pmatrix}, \quad i = 0,1,2,3\,.$
By solving Eqs.(42)-(45) we obtain

$$c_0 = \frac{3}{4}, \qquad c_1 = -3, \qquad c_2 = 2, \qquad c_3 = 0.$$

Therefore,

$$u(x) = \begin{pmatrix} \frac{3}{4}, & -3, & 2, & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -x+0.5 \\ 0.5x^2-1.5x+0.375 \\ -0.1667x^3+1.25x^2-1.875x+0.313 \end{pmatrix} = x^2,$$

which is the exact solution of this problem.

Example 3:

In this example, we consider the following linear fractional differential equation

$$D^2u(x) + \sin(x)D^{\frac{1}{2}}u(x) + xu(x) = g(x), \tag{46}$$

where, $f(x) = x^9 - x^8 + 56x^6 - 42x^5 + \sin(x)\left(\frac{32768}{6435}x^{\frac{15}{2}} - \frac{2048}{429}x^{\frac{13}{2}}\right)$,
and subject to the initial conditions

$$u(0) = u'(0) = 0. \tag{47}$$

The unique analytical solution for this problem is $u(x) = x^8 - x^7$.
To solve this example, by applying the proposed technique with $m = 8$, we approximate the solution as follows

$$u_8(x) = \sum_{i=0}^{8} c_i\, L_i^{(\alpha)}(x).$$

Using Eq.(19), with $\nu = 2,\ \beta_1 = 0.5,\ \alpha = -0.75$ we have

$$\sum_{i=2}^{8}\sum_{k=2}^{i} c_i\, w_{i,\,k}^{(2)}\, x^{k-2} + \sin(x)\sum_{i=1}^{8}\sum_{k=1}^{i} c_i\, w_{i,\,k}^{(0.5)}\, x^{k-0.5} + x\sum_{i=0}^{8} c_i\, L_i^{(\alpha)}(x) = g(x). \tag{48}$$

Now, we collocate Eq.(48) at the roots x_p as

$$\sum_{i=2}^{8}\sum_{k=2}^{i} c_i\, w_{i,\,k}^{(2)}\, x_p^{k-2} + \sin(x_p)\sum_{i=1}^{8}\sum_{k=1}^{i} c_i\, w_{i,\,k}^{(0.5)}\, x_p^{k-0.5} + x_p\sum_{i=0}^{8} c_i\, L_i^{(\alpha)}(x_p) = g(x_p), \tag{49}$$

where x_p are roots of the generalized Laguerre polynomial $L_7^{(\alpha)}(x)$, i.e.,

$x_0 = 0.039$, $x_1 = 0.649$, $x_2 = 1.988$, $x_3 = 4.134$, $x_4 = 7.237$, $x_5 = 11.614$, $x_6 = 18.089$.

By using Eqs.(21), (47) and (49) we obtain a linear system of algebraic equations and by solving it we obtain

$$c_0 = 1991.7011,\ c_1 = -65009.1247,\ c_2 = 371189.3555,\ c_3 = -1.00888\times 10^6,$$

$$c_4 = 1.581398\times 10^6,\ c_5 = -1.51593\times 10^6,\ c_6 = 88200,\ c_7 = -28728,\ c_8 = 40320.$$

Therefore, $u(x)$ has the form

$$u(x) \simeq \sum_{i=0}^{8} c_i\, L_i^{(\alpha)}(x) = x^8 - x^7\,.$$

It is clear that in this example the presented method can be considered as an efficient method.

Example 4:

In this example, we consider the following linear fractional differential equation

$$D^{\frac{1}{2}}u(x) - aD^{\frac{1}{4}}u(x) = \sin(bx), \tag{50}$$

where a and b are arbitrary constants and subject to the following initial condition

$$u(0) = 0. \tag{51}$$

The exact solution for this problem is

$$u(x) = \frac{b}{a^8 + b^2} \sum_{j=2}^{5} a^{j-2}[a^4 E_x(j\nu, a^4) - a^4 C_x(j\nu, b) + bS_x(j\nu, b)].$$

To solve the above problem, by applying the proposed technique with $m = 5$, we approximate the solution as

$$u_5(x) \cong \sum_{i=0}^{5} c_i \, L_i^{(\alpha)}(x).$$

Using Eq.(19), with $\nu = 0.5,\ \beta_1 = 0.25,$ and $\alpha = -0.75$ we have

$$\sum_{i=1}^{5}\sum_{k=1}^{i} c_i \, w_{i,\,k}^{(0.5)} \, x^{k-0.5} - a \sum_{i=1}^{5}\sum_{k=1}^{i} c_i \, w_{i,\,k}^{(0.25)} \, x^{k-0.25} = \sin(bx). \tag{52}$$

Now, we collocate Eq.(52) at the roots x_p as

$$\sum_{i=1}^{5}\sum_{k=1}^{i} c_i \, w_{i,\,k}^{(0.5)} \, x_p^{k-0.5} - a \sum_{i=1}^{3}\sum_{k=1}^{i} c_i \, w_{i,\,k}^{(0.25)} \, x_p^{k-0.25} = \sin(bx_p), \tag{53}$$

where x_p are roots of the generalized Laguerre polynomial $L_5^{(\alpha)}(x)$, i.e.,

$$x_0 = 0.055, \quad x_1 = 0.908, \quad x_2 = 2.829, \quad x_3 = 6.075, \quad x_4 = 11.383.$$

By using Eqs.(21), (51) and (53) we obtain a linear system of algebraic equations and by solving it we obtain

$$c_0 = 0.1239,\ c_1 = 0.1688,\ c_2 = 0.0453,\ c_3 = -0.0005,\ c_4 = -0.0009,\ c_5 = 0.0001.$$

Therefore, $u(x)$ has the form

$$\begin{aligned} u(x) \cong\ & 0.3009x^{1.5} + 0.2487x^{1.75} + 0.2x^2 + 0.1569x^{2.25} + 0.6018x^{2.5} + 0.4522x^{2.75} \\ & + 0.3333x^3 + 0.2414x^{3.25} - 0.5158x^{3.5} - 0.3617x^{3.75} + 0.01667x^4 - 0.2667x^4 \\ & - 0.1704x^{4.25} - 0.1146x^{4.5} - 0.0762x^{4.75} - 0.0533x^5 - 0.0324x^{5.25}\,. \end{aligned}$$

The approximate solutions of this example are presented in the figure 1, with different values of m, $(m = 5, 8, 10)$. From this figure it is clear that the presented method can be considered as an efficient method and more applicable to solve numerically for such problems.

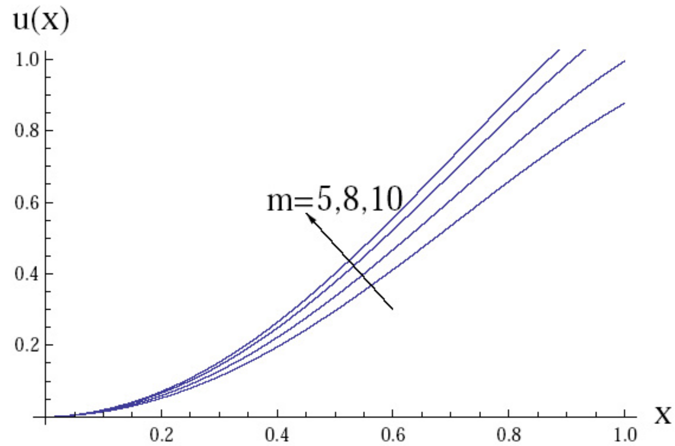

Figure 1. The behavior of the exact solution and the approximate solutions at $m = 5, 8$ and 10.

7. Conclusion and remarks

We have presented a complete analysis for the collocation method based on the generalized Laguerre polynomials to solve LFDEs. A special family of the generalized Laguerre polynomials was used as an approximation basis. Some error estimates are derived to demonstrate the spectral accuracy for the proposed method. The numerical solutions obtained from this approach show that LFDEs can be solved effectively. In addition, a small number of Laguerre polynomials is needed to achieve the satisfactory result. All numerical results are obtained by building fast algorithms using Matlab 7.1.

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