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TOWARDS FRACTIONAL ORDER SYSTEMS ON GRAPHS AND EXTREME EVENTS

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Abstract. It is argued that modeling epidemics requires graphs. Some argue that adaptive graphs are needed. Fractional order dynamics is presented on graphs. Also it is argued that fractals are expected in extreme events.

1. Introduction

In modeling epidemics using differential equations it is standard to write infection term as si where s (i) is the fraction of susceptible (infectives and infected). In reality this is true only after the disease has spread such that uniform mixing occurs between susceptible and infectives. Therefore modeling epidemics on networks has an increasing importance ([8]). Typically static graphs has been used ([1]). But it has been pointed out ([4]-[10]) that as the epidemic spread the behavior of susceptible change the network topology. As an example a rewinding is allowed to replace an si link by an ss link with probability w. Hence the threshold infection rate becomes

$$P = w/\{< z > [1 - e^{(-w/r)}]\} \tag{1}$$

Where $< z >$ is the average number of nearest neighbors for each site and r is the recovery probability. If re-winding is neglected one regains $p = r/ < z >$. The equations modeling this process are not easy to derive. Choosing special graphs may simplify this problem ([5]) but it is unrealistic. Therefore a present problem is to model epidemics on realistic time dependent adaptive graphs.

2. Equilibrium and stability of fractional order systems on graphs:

On a general graph the fractional order dynamical system takes the form

$$D^{\alpha} x_i = f_i(x_i) + \sum_{j \neq i} A_{ij}(t) g_{ij}(x_i, x_j) \tag{2}$$

Averaging on the graph time dependence one gets

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$$D^{\alpha}x_i = f_i(x_i) + \sum_{j\neq i}\overline{A}_{ij}g_{ij}(x_i, x_j) \tag{3}$$

The equilibrium solution is given by

$$f_i(x_i) + \sum_{j\neq i}\overline{A}_{ij}g_{ij}(x_i, x_j) = 0$$

and it is stable if all the eigenvalues of the following matrix satisfy $|\arg(\lambda)| > \alpha\pi/2$

$$B_{ij} = [\partial f_i/\partial x_j + \sum_{j\neq i}\overline{A_{ij}}\partial g_{ij}/\partial x_i]\delta_{ik} + \overline{A_{ik}}\partial g_{ik}/\partial x_k$$

where B_{ij} is calculated at equilibrium.

As an example we present a coupled map lattice model ([2]) modified to fractional order

$$D^{\alpha}x_i = f(x_i) + g(x_{i+1}) + g(x_{i-1}) \tag{4}$$

The equilibrium is given by

$$f(x_i) + g(x_{i+1}) + g(x_{i-1}) = 0$$

and its stability is given by the condition that $|\arg(\lambda)| > \alpha\pi/2$ where for the homogeneous equilibria case $x_i = x_{eq}\forall i$

$$\lambda = f'(x_{eq}) + 2g'(x_{eq})\cos 2\pi r/n, r = 0, 1, ..., n-1$$

3. Fractals in risk management:

In ecosystems, markets and climates, three major warning signals were identified as signs of pre-crisis periods ([9]):

(i) Slower recovery from perturbations.
(ii) Increased self similarity.
(iii) Increased variance of fluctuations.

Definition 1 The real valued stochastic process $(X_{t1}, X_{t2}, ...X_{tn})$, n is a positive integer, is self similar with index H (H_{ss}) if for every $a > 0$, and for every $\{t_1, t_2, ..., t_n\}$

$$(X_{at_1}, X_{at_2}, ..., X_{at_n}) = a^H(X_{t_1}, X_{t_2}, ..., X_{t_n})$$

For Brownian motion H=1/2.

Proposition 1 ([3]) **.** Let $(X_{t_1}, X_{t_2}, ..., X_{t_n})$ be H_{ss} with $0 < H < 1$ then for all t_0 fixed

$$\lim_{t\to t_0^+}\sup\frac{|X_t - X_{t_0}|}{(t-t_0)}$$

is unbounded i.e. the sample paths of H_{ss} processes are nowhere differentiable.

Proof. Without loss of generality choose $t_0 = 0$. Let (t_n) be a sequence such that $t_n \to 0$. Then by H-self similar

$$\lim_{t\to\infty} \sup_{0\le s\le t_n} \frac{|X_s|}{s} \ge \lim_{t\to\infty} \sup \frac{|X_{t_n}|}{t_n} = \lim_{t\to\infty} (\sup(t_n^{H-1}) \mid X_1 \mid = \infty,$$

since $0 < H < 1$. Thus fractals are expected to occur in many extreme events. □

We conclude by giving an example of continuous nondifferentiable (CND) function namely Riemann function ([7])

$$R(t) = \sum_{k=1}^{\infty} \sin \frac{2\pi t k^2}{k^2}$$

It is known ([6]) that this map appears as a part of a solution of Schrodinger equation hence the trajectory of a quantum mechanical particle can be continuous nondifferentiable.

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