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EXISTENCE OF SOLUTION FOR A NONLOCAL BOUNDARY VALUE PROBLEM WITH FRACTIONAL Q-DERIVATIVES

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Abstract. The authors investigate the existence of solutions to the following boundary value problem fractional q-derivative

$$(D_q^{\alpha}u)(t) + f(t, u(t)) = 0,\ t \in (0,1),\ 1 < \alpha \leq 2,$$
$$u(0) = 0,\ (D_q^{\alpha}u)(1) = \beta u(\xi),$$

where $0 < \beta\xi^{\alpha-1} < 1$, $0 < \xi < 1$, D_q^{α} denotes the q-derivative of Riemmane-Liouville type of order α. By applying generalized Banach contraction principle and Schauder fixed point theorem some new existence and uniqueness results of solutions are obtained. We give an example to illustrate our results.

1. Introduction

In the recent years, fractional calculus is one of the interest issues that attracts many scientists, specially mathematics and engineering sciences. Many natural phenomena can be present by boundary value problems of fractional differential equations. Many authors in different fields such as chemical physics, fluid flows, electrical networks, visco-elasticity, try to modeling of these phenomena by boundary value problems of fractional differential equations [[1]-[4]]. To achieve extra information in fractional calculus, specially boundary value problems, reader can refer to valuable papers or books that are written by authors [[5]-[20]].
The fractional q-calculus is the q-extension of ordinary fractional calculus. Recently, there seems to be a significant increase in study in the topic of the q-calculus due to application of the q-calculus in mathematics statistics and physics [[21]-[24]]. Early developments for q-fractional calculus can be seen in the papers that presented by Al-Salam [[25]] and Agarwal [[26]] on the existence theory of fractional q-difference. We can mention to the attentions of several researches [[27]-[30]]. Furthermore, some new existence study in the existence of solutions for boundary value problems with fractional q-derivative [[31]-[34]].
In this paper, we investigate the existence and uniqueness of a positive and nondecreasing solution for a nonlocal boundary value problem for fractional q-derivatives

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equation of the form

$$(D_q^\alpha u)(t) + f(t, u(t)) = 0,\ t \in (0,1),\ 1 < \alpha \leq 2, \tag{1}$$

$$u(0) = 0,\ (D_q^\alpha u)(1) = \beta u(\xi), \tag{2}$$

where D_q^α is the q-derivative of Riemann-Liouville type of order α, $0 < \xi < 1$ and $0 < \beta\xi^{\alpha-1} < 1$, and $f \in C([0,1] \times [0,+\infty), [0,+\infty))$.

2. Preliminaries q-calculus and lemmas

We now given preliminaries q-calculus, definitions and lemmas that will be used in the remainder of this paper. The presentation here can be found in, for example, [[22],[24],[35], [36]].

Let $q \in (0,1)$ and define

$$[a]_q = \frac{1-q^a}{1-q},\quad a \in \mathbf{R}.$$

The q-analogue of the power function $(a-b)^n$ with $n \in \mathbf{N_0} := \{0,1,2,...\}$ is defined by

$$(a-b)^{(0)} = 1,\ (a-b)^{(n)} = \prod_{k=0}^{n-1}(a - bq^k),\ n \in \mathbf{N}, a, b \in \mathbf{R}.$$

More generally, if $\alpha \in \mathbf{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty} \frac{a - bq^k}{a - bq^{\alpha+k}},\ a \neq 0.$$

Clearly, if $b = 0$, then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}},\quad x \in \mathbf{R} - \{0, -1, -2, ...\},$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function f is defined by

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x},\quad (D_q f)(0) = \lim_{x \to 0}(D_q f)(x),$$

and q-derivatives of higher order by

$$(D_q^0 f)(x) = f(x),\qquad (D_q^n f)(x) = D_q(D_q^{n-1} f)(x),\ n \in \mathbf{N}.$$

The q-integral of a function f defined in the interval $[0,b]$ is given by

$$(I_q f)(x) = \int_0^x f(s) d_q s = x(1-q)\sum_{k=0}^{\infty} f(xq^k)q^k,\quad x \in [0,b].$$

If $a \in [0,b]$ and f is defined in the interval $[0,b]$, then its integral from a to b is defined by

$$\int_a^b f(s) d_q s = \int_0^b f(s) d_q s - \int_0^a f(s) d_q s.$$

Similar to that for derivatives, an operator I_q^n is given by

$$(I_q^0 f)(x) = f(x),\quad (I_q^n f)(x) = I_q(I_q^{n-1} f)(x),\qquad n \in \mathbf{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

The following formulas will be used later, namely, the integration by parts formula:

$$\int_0^x f(s)(D_q g)(s) d_q s = [f(s)g(s)]_{s=0}^{s=x} - \int_0^x (D_q f)(s) g(qs) d_q s,$$

and

$$[a(t-s)]^{(\alpha)} = a^{\alpha}(t-s)^{(\alpha)}, \tag{3}$$

$${}_tD_q(t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)}, \tag{4}$$

$${}_sD_q(t-s)^{(\alpha)} = -[\alpha]_q (t-qs)^{(\alpha-1)}, \tag{5}$$

$$({}_xD_q \int_0^x f(x,s) d_q s)(x) = \int_0^x {}_xD_q f(x,s) d_q s + f(qx, x). \tag{6}$$

where ${}_tD_q$ denotes the derivative with respect to the variable t.

Definition 1. Let $\alpha \geq 0$ and f be a function defined on $[0,1]$. The fractional q-integral of Riemann-Liouville type is $(I_q^0 f)(x) = f(x)$ and

$$(I_q^{\alpha} f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qs)^{(\alpha-1)} f(s) d_q s, \qquad \alpha > 0, \qquad x \in [0,1].$$

Definition 2. The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by $(D_q^0 f)(x) = f(x)$ and

$$(D_q^{\alpha} f)(x) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f)(x), \qquad \alpha > 0,$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

Lemma 1. [see [22]] Assume that $\alpha \geq 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$.

Lemma 2. Let $\alpha, \beta \geq 0$ and f be a function defined on $[0,1]$. Then the following formulas hold:

(1) $(I_q^{\beta} I_q^{\alpha} f)(x) = (I_q^{\alpha+\beta} f)(x)$,

(2) $(D_q^{\alpha} I_q^{\alpha} f)(x) = f(x)$.

Lemma 3. [see [22]] Let $\alpha > 0$ and n be a positive integer. Then the following equality holds:

$$(I_q^{\alpha} D_q^n f)(x) = (D_q^n I_q^{\alpha} f)(x) - \sum_{k=0}^{n-1} \frac{x^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)} (D_q^k f)(0).$$

Lemma 4. [see [24]] Let $\alpha \in \mathbf{R}^+$, $\lambda \in (-1, +\infty)$, the following is valid:

$$I_q^{\alpha}((t-a)^{(\lambda)}) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)} (t-a)^{(\alpha+\lambda)}, \ 0 < a < t < b.$$

Particularly, for $\lambda = 0$, $a = 0$, using q-integration by parts, we have

$$\begin{aligned} (I_q^{\alpha} 1)(t) &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} d_q s = \frac{1}{\Gamma_q(\alpha)} \int_0^t \frac{{}_sD_q((t-s)^{(\alpha)})}{-[\alpha]_q} d_q s \\ &= -\frac{1}{\Gamma_q(\alpha+1)} \int_0^t {}_sD_q((t-s)^{(\alpha)}) d_q s = \frac{1}{\Gamma_q(\alpha+1)} t^{(\alpha)}. \end{aligned}$$

Obviously, we have $\int_0^t (t-qs)^{(\alpha-1)} d_q s = \frac{1}{[\alpha]_q} t^{(\alpha)}$, and

$$\begin{aligned}\int_0^t (1-qs)^{(\alpha-1)} d_q s &= \int_0^t \frac{{}_sD_q((1-s)^{(\alpha)})}{-[\alpha]_q} d_q s \\ &= -\frac{1}{[\alpha]_q}\int_0^t {}_sD_q((1-s)^{(\alpha)}) d_q s = \frac{1}{[\alpha]_q}[1-(1-t)^{(\alpha)}].\end{aligned}$$

In order to define the solution for the our problem, we need the following lemma.

Lemma 5. For given $y \in C[0,1]$, the unique solution of the boundary value problem

$$(D_q^\alpha u)(t) + y(t) = 0,\ t \in (0,1),\ 1 < \alpha \le 2, \tag{7}$$

$$u(0) = 0,\ (D_q^\alpha u)(1) = \beta u(\xi), \tag{8}$$

is given by

$$u(t) = \int_0^1 G(t,qs) y(s) d_q s, \tag{9}$$

where $G(t,qs)$, is

$$\begin{cases} \frac{[\alpha-1]_q(1-qs)^{(\alpha-2)}t^{\alpha-1} - \beta(\xi-qs)^{(\alpha-1)}t^{\alpha-1} - ([\alpha-1]_q - \beta\xi^{\alpha-1})(t-qs)^{(\alpha-1)}}{([\alpha-1]_q - \beta\xi^{\alpha-1})\Gamma_q(\alpha)}, & 0 \le s \le t \le 1, s \le \xi, \\ \frac{[\alpha-1]_q(1-qs)^{(\alpha-2)}t^{\alpha-1} - ([\alpha-1]_q - \beta\xi^{\alpha-1})(t-qs)^{(\alpha-1)}}{([\alpha-1]_q - \beta\xi^{\alpha-1})\Gamma_q(\alpha)}, & 0 < \xi \le s \le t \le 1, \\ \frac{[\alpha-1]_q(1-qs)^{(\alpha-2)}t^{\alpha-1} - \beta(\xi-qs)^{(\alpha-1)}t^{\alpha-1}}{([\alpha-1]_q - \beta\xi^{\alpha-1})\Gamma_q(\alpha)}, & 0 \le t \le s \le \xi < 1, \\ \frac{[\alpha-1]_q(1-qs)^{(\alpha-2)}t^{\alpha-1}}{([\alpha-1]_q - \beta\xi^{\alpha-1})\Gamma_q(\alpha)}, & 0 \le t \le s \le 1, \xi \le s. \end{cases}$$

Proof. Since $1 < \alpha \le 2$, we put $n = 2$. In view of Definition 1 and Lemma 2, we see that

$$(D_q^\alpha u)(t) = -y(t) \Leftrightarrow (I_q^\alpha D_q^2 D_q^{2-\alpha}) = -(I_q^\alpha y)(t).$$

Then it follows from Lemma 3 that the solution $u(t)$ of (7) and (8) is given by

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_q s, \tag{10}$$

for some constants $c_1, c_2 \in \mathbf{R}$. Since $u(0) = 0$, we have $c_2 = 0$.
Differentiating both sides of (10) and with the help of (4) and (6), we obtain

$$(D_q u)(t) = [\alpha-1]_q c_1 t^{\alpha-2} + [\alpha-2]_q c_2 t^{\alpha-3} - \int_0^t \frac{[\alpha-1]_q (t-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)} y(s) d_q s.$$

Using the boundary conditions $D_q u(1) = \beta u(\xi)$, we get

$$\begin{aligned} c_1 &= \frac{1}{[\alpha-1]_q - \beta\xi^{\alpha-1}} \int_0^1 \frac{[\alpha-1]_q (1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)} y(s) d_q s \\ &\quad - \frac{\beta}{[\alpha-1]_q - \beta\xi^{\alpha-1}} \int_0^\xi \frac{(\xi-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_q s. \end{aligned}$$

Now, substitution of c_1 into (10) gives

$$\begin{aligned} u(t) &= \frac{t^{\alpha-1}}{[\alpha-1]_q - \beta\xi^{\alpha-1}} \int_0^1 \frac{[\alpha-1]_q (1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)} y(s) d_q s \\ &\quad - \frac{\beta t^{\alpha-1}}{[\alpha-1]_q - \beta\xi^{\alpha-1}} \int_0^\xi \frac{(\xi-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_q s \\ &\quad - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) d_q s \\ &= \int_0^1 G(t,qs) y(s) d_q s. \end{aligned}$$

This completes the proof of the lemma.

3. Main results

Let $E = C[0,1]$ be a Banach space endowed with the norm $\|u\| = \max_{0\leq t\leq 1} |u(t)|$. Assume that $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. Define the operator $T : E \to E$ as follows:

$$(Tu)(t) = \int_0^1 G(t,qs) f(s,u(s)) d_q s = c_1 t^{\alpha-1} - I_q^\alpha f(t,u(t)). \tag{11}$$

where c_1 is the same in Lemma 5. Clearly, the fixed points of the operator T are solutions of problem (1) and (2).

Lemma 6. [see [35]] Let $\alpha \in \mathbf{R}^+$ and $f : (0,a] \to \mathcal{C}$ be a function. If $f \in L_q^1[0,a]$ then $I_q^\alpha f \in L_q^1[0,a]$ and

$$\|I_q^\alpha f\|_1 \leq \frac{a^\alpha}{\Gamma_q(\alpha+1)} \|f\|_1.$$

Theorem 1. Suppose that $f(t,u)$ satisfies the following condition

$$\|f\| \leq \Gamma_q(\alpha+1)(\frac{r}{2} - c_1), \tag{12}$$

where r is a positive number and $r > 2c_1$. Then problem (1) and (2) has at least one positive solution.

Proof. Define the operator $T : E \to E$ by

$$(Tu)(t) = \int_0^1 G(t,qs) f(s,u(s)) d_q s = c_1 t^{\alpha-1} - I_q^\alpha f(t,u(t)).$$

From continuity of f and $G(t,s)$, the operator T is continuous.

Let $B_r = \{u \in E \mid \|u - I_q^\alpha f(t,u(t))\| \leq r\}$, be a convex, bounded, and closed subset of the Banach space E.

We prove that $T : B_r \to B_r$. By (12) and Lemma 6, for $u \in B_r$, we have

$$\begin{aligned} |Tu(t) - I_q^\alpha f(t,u(t))| &= |c_1 t^{\alpha-1} - I_q^\alpha f(t,u(t)) - I_q^\alpha f(t,u(t))| \\ &= |c_1 t^{\alpha-1} - 2I_q^\alpha f(t,u(t))| \\ &\leq 2|I_q^\alpha f(t,u(t))| + c_1 |t^{\alpha-1}| \\ &\leq \frac{2\|f\|}{\Gamma_q(\alpha+1)} + c_1 \\ &\leq r. \end{aligned}$$

This show that T maps B_r into B_r. Next, we shall prove that T is completely continuous. Let $M = \max_{t\in[0,1],u\in B_r} |f(t, u(t))| + 1$, then, for $u \in B_r$, we have

$$\begin{aligned} |Tu(t)| = |c_1 t^{\alpha-1} - I_q^\alpha f(t, u(t))| &\leq |I_q^\alpha f(t, u(t))| + |c_1 t^{\alpha-1}| \\ &\leq \frac{|f(t, u(t))|}{\Gamma_q(\alpha+1)} + c_1 \\ &\leq \frac{M}{\Gamma_q(\alpha+1)} + c_1, \end{aligned}$$

hence, $T(B_r)$ is bounded. Now, we show that $T(B_r)$ is equicontinuous. For $u \in B_r$ and $t_1, t_2 \in [0, 1]$ such that $t_1 < t_2$ we get,

$$\begin{aligned} |Tu(t_2) - Tu(t_1)| &\leq \left| \int_0^{t_2} \frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_q s - \int_0^{t_1} \frac{(t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_q s \right| \\ &\quad + \left| \frac{(t_2^{\alpha-1} - t_1^{\alpha-1})}{[\alpha-1]_q - \beta\xi^{\alpha-1}} \int_0^1 \frac{[\alpha-1]_q (1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)} f(s, u(s)) d_q s \right| \\ &\quad + \left| \frac{\beta(t_2^{\alpha-1} - t_1^{\alpha-1})}{[\alpha-1]_q - \beta\xi^{\alpha-1}} \int_0^\xi \frac{(\xi - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s)) d_q s \right| \\ &\leq \frac{M}{\Gamma_q(\alpha)} \left| \int_0^{t_1} [(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)}] d_q s + \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha-1)} d_q s \right| \\ &\quad + \frac{M|t_2^{\alpha-1} - t_1^{\alpha-1}|}{[\alpha-1]_q - \beta\xi^{\alpha-1}} \int_0^1 \frac{[\alpha-1]_q (1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha)} d_q s \\ &\quad + \frac{M\beta|t_2^{\alpha-1} - t_1^{\alpha-1}|}{[\alpha-1]_q - \beta\xi^{\alpha-1}} \int_0^\xi \frac{(\xi - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \\ &\leq \frac{M}{\Gamma_q(\alpha+1)} \left| t_2^{(\alpha)} - t_1^{(\alpha)} \right| + \frac{M|t_2^{\alpha-1} - t_1^{\alpha-1}|[\alpha-1]_q}{([\alpha-1]_q - \beta\xi^{\alpha-1})\Gamma_q(\alpha+1)} \\ &\quad + \frac{M\beta\xi^{(\alpha)}|t_2^{\alpha-1} - t_1^{\alpha-1}|}{([\alpha-1]_q - \beta\xi^{\alpha-1})\Gamma_q(\alpha+1)}. \end{aligned}$$

It is easy to see that functions t^α and $t^{\alpha-1}$ are uniformly continuous on [0,1]. Then, $T(B_r)$ is equicontinuous. By the Arzela-Ascoli theorem $\overline{T(B_r)}$ is compact and so $T : B_r \to B_r$, is completely continuous. The Schauder fixed point theorem now implies that the BVP (1) and (2) has a solution.

Theorem 2. There exists a nonnegative function $h \in C[0, 1]$ such that $f(t, u)$ satisfies

$$|f(t, u) - f(t, v)| \leq h(t)|u - v|, \qquad t \in [0, 1], u, v \in [0, \infty). \tag{13}$$

Then, the problem (1) and (2) has a unique solution provided

$$A = \int_0^1 (1 - qs)^{(\alpha-2)} s^{\alpha-1} h(s) d_q s < \frac{([\alpha-1]_q - \beta\xi^{\alpha-1})\Gamma_q(\alpha)}{2}. \tag{14}$$

Proof. We shall prove that under the assumptions (13) and (14), T^n is a contraction operator for n sufficiently large. By the definition of $G(t, qs)$, we have

$$G(t, qs) = \frac{[\alpha-1]_q (1 - qs)^{(\alpha-2)} t^{\alpha-1}}{([\alpha-1]_q - \beta\xi^{\alpha-1})\Gamma_q(\alpha)},$$

For $u, v \in E$, we have the estimate

$$\begin{aligned}|Tu(t)-Tv(t)| &= \left|\int_0^1 G(t,qs)[f(s,u(s))-f(s,v(s))]d_qs\right| \\ &\leq \int_0^1 G(t,qs)|f(s,u(s))-f(s,v(s))|d_qs \\ &\leq \|u-v\|\int_0^1 G(t,qs)h(s)d_qs \\ &= \frac{[\alpha-1]_q\|u-v\|t^{\alpha-1}}{([\alpha-1]_q-\beta\xi^{\alpha-1})\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)}h(s)d_qs \\ &= \frac{B[\alpha-1]_q\|u-v\|t^{\alpha-1}}{([\alpha-1]_q-\beta\xi^{\alpha-1})\Gamma_q(\alpha)},\end{aligned}$$

where $B=\int_0^1(1-qs)^{(\alpha-2)}h(s)d_qs$.
Consequently,

$$\begin{aligned}|(T^2)u(t)-(T^2)v(t)| &= |T(Tu(t))-T(Tv(t))| \\ &\leq \int_0^1 G(t,qs)|f(s,Tu(s))-f(s,Tv(s))|d_qs \\ &\leq \int_0^1 G(t,qs)h(s)|Tu(s)-Tv(s)|d_qs \\ &\leq \frac{B[\alpha-1]_q\|u-v\|}{([\alpha-1]_q-\beta\xi^{\alpha-1})\Gamma_q(\alpha)}\int_0^1 G(t,qs)s^{\alpha-1}h(s)d_qs \\ &= \frac{B[\alpha-1]_q\|u-v\|}{([\alpha-1]_q-\beta\xi^{\alpha-1})\Gamma_q(\alpha)}\int_0^1 \frac{[\alpha-1]_q(1-qs)^{(\alpha-2)}t^{\alpha-1}}{([\alpha-1]_q-\beta\xi^{\alpha-1})\Gamma_q(\alpha)}s^{\alpha-1}h(s)d_qs \\ &= \frac{B([\alpha-1]_q)^2\|u-v\|t^{\alpha-1}}{[([\alpha-1]_q-\beta\xi^{\alpha-1})\Gamma_q(\alpha)]^2}\int_0^1 (1-qs)^{(\alpha-2)}s^{\alpha-1}h(s)d_qs \\ &= \frac{AB([\alpha-1]_q)^2t^{\alpha-1}}{[([\alpha-1]_q-\beta\xi^{\alpha-1})\Gamma_q(\alpha)]^2}\|u-v\|,\end{aligned}$$

where $A=\int_0^1(1-qs)^{(\alpha-2)}s^{\alpha-1}h(s)d_qs$. By introduction, we obtain

$$|(T^nu)(t)-(T^nv)(t)| \leq \frac{BA^{n-1}([\alpha-1]_q)^n t^{\alpha-1}}{[([\alpha-1]_q-\beta\xi^{\alpha-1})\Gamma_q(\alpha)]^n}\|u-v\|.$$

From the condition (14), we get

$$\begin{aligned}\frac{BA^{n-1}}{[([\alpha-1]_q-\beta\xi^{\alpha-1})\Gamma_q(\alpha)]^n} &= \frac{B}{A}\left[\frac{A}{([\alpha-1]_q-\beta\xi^{\alpha-1})\Gamma_q(\alpha)}\right]^n \\ &\leq \frac{B}{A}(\frac{1}{2})^n \\ &< \frac{1}{4},\end{aligned}$$

since $0<[\alpha-1]_q<1$, thus for n sufficiently large, we have

$$\|T^nu-T^nv\| \leq \frac{1}{4}\|u-v\|.$$

Hence, it follows from the generalized Banach contraction principle that the BVP (1) and (2) has a unique solution.

4. An Example

Example 1. Consider the following fractional q-difference boundary value problem

$$\begin{cases} D_{0.5}^{1.5}u(t) + \frac{t^2}{3(1+t^2)}(tan^{-1}u + t) = 0, \quad 0 < t < 1, \\ u(0) = 0, \ D_{0.5}u(1) = \frac{1}{5}u(\frac{1}{2}), \end{cases} \tag{15}$$

where, $\alpha = 1.5$, $q = 0.5$, $\beta = \frac{1}{5}$, $\xi = \frac{1}{2}$ and

$$f(t,u) = \frac{t^2}{3(1+t^2)}(tan^{-1}u + t), \quad (t,u) \in [0,1] \times (0,\infty),$$

is a continuous function, and $h(t) = \frac{t^2}{3(1+t^2)}$. It is easy for $(t,u),(t,v) \in [0,1]\times(0,\infty)$ to prove that

$$\begin{aligned} |f(t,u) - f(t,v)| &= \frac{t^2}{3(1+t^2)}|tan^{-1}u - tan^{-1}v| \\ &\leq h(t)|u-v|. \end{aligned}$$

By simple calculation, we get

$$([\alpha-1]_q - \beta\xi^{\alpha-1})\Gamma_q(\alpha) \approx 0.46,$$

and

$$\begin{aligned} A = \int_0^1 (1-qs)^{(\alpha-2)}s^{\alpha-1}h(s)d_qs &= \int_0^1 (1-qs)^{(\alpha-2)}s^{\alpha-1}\frac{s^2}{3(1+s^2)}d_qs \\ &\leq \frac{1}{3}\int_0^1 (1-qs)^{(\alpha-2)}s^{\alpha-1}d_qs \\ &\leq \frac{1}{3}\int_0^1 (1-qs)^{(\alpha-2)}d_qs \approx 0.166, \end{aligned}$$

which implies that

$$A = \int_0^1 (1-qs)^{(\alpha-2)}s^{\alpha-1}h(s)d_qs < \frac{([\alpha-1]_q - \beta\xi^{\alpha-1})\Gamma_q(\alpha)}{2} \approx 0.23.$$

Obviously, for any $n \geq 2$, we have

$$\frac{BA^{n-1}}{[([\alpha-1]_q - \beta\xi^{\alpha-1})\Gamma_q(\alpha)]^n} \leq \frac{0.166}{0.46 \times 2^{n-1}} < 0.1804 < \frac{1}{4}.$$

Thus, Theorem 2 implies that the boundary value problem (15) has a unique solution.

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