*Journal of Fractional Calculus and Applications*,
Vol. 6(1) Jan. 2015, pp. 160-174.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

# EXISTENCE SOLUTIONS FOR THREE POINT BOUNDARY VALUE PROBLEM FOR DIFFERENTIAL EQUATIONS

MOHAMED HOUAS, MAAMAR BENBACHIR

Abstract. In this paper, under weak assumptions, we study the existence and uniqueness of solutions for a nonlinear fractional boundary value problem. New existence and uniqueness results are established using Banach contraction principle. Other existence results are obtained using scheafer and krasnoselskii's fixed point theorem. At the end, some illustrative examples are presented.

## 1. Introduction

Differential equations of fractional order is rapidly growing area of differential equations both theoretically and in practical point of view to real world problems. The theory of existence of solutions to nonlinear boundary value problems corresponding to fractional differential equations have recently been attracted the attention of many researchers, for more details, we refer the reader to [6, 8, 9, 10, 11, 14, 15, 16, 17] and references therein. Recently, boundary value problems for fractional differential equations have been studied in many papers, (see [6, 7, 20, 22, 26]). More recently, some basic theory for the initial boundary value problems of fractional differential equations has been discussed in [1, 2, 3, 5, 23, 24]. Moreover, existence and uniqueness of solutions to boundary value problems for fractional differential equations had attracted the attention of many authors, see for example,[4, 13, 20, 24] and the references therein.

In [1, 2, 3, 4, 23, 24], the existence and uniqueness of solutions was investigated for a nonlinear fractional differential equations with fractional nonlocal integral boundary conditions by using Schauder and Krasnoselskii's fixed point theorem. In this paper we give an improvement of the results in [15, 16, 17], we investigate the existence and uniqueness of solutions for the following problem:

$$\begin{cases} D^{\alpha}x\left(t\right)+f\left(x\left(t\right),D^{\beta}x\left(t\right)\right)=0, t\in J, \\ x\left(0\right)=x_{0}, x^{'}\left(0\right)=0, x^{'}\left(1\right)=\lambda J^{\sigma}x\left(\eta\right). \end{cases} \tag{1}$$

where $2<\alpha\leq 3$ and $1<\beta\leq 2$, $0<\eta<1$, and $D^{\alpha}$ and $D^{\beta}$ are the Caputo fractional derivatives, $J=[0,1]$, $\lambda$ is real constant and $f$ continuous function on

2010 *Mathematics Subject Classification.* 26A33, 34B25, 34B15.
*Key words and phrases.* Caputo derivative, Boundary Value Problem, fixed point theorem, local conditions.
Submitted Aug. 10, 2014.

$\mathbb{R}^2$. The paper is organized as follows. In section 2, we present some preliminaries and lemmas. Section 3 is devoted to the existence of solution of (1). In section 4, we will give examples to illustrate our main results.

## 2. Preliminaries

The following notations, definitions, and preliminary facts will be used throughout this paper.

**Definition 1** The fractional (arbitrary) order integral of the function $f \in \mathbb{L}^1([a,b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

where $\Gamma$ is the gamma function. When $a = 0$, we write $I^\alpha f(t) = (f * \varphi_\alpha)(t)$, where $\varphi_\alpha(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha \longrightarrow \delta(t)$ as $\alpha \longrightarrow 0$, where $\delta$ is the delta function.

**Definition 2** For a function $f$ given on the interval $[a,b]$, the $\alpha^{th}$ Riemann-Liouville fractional-order derivative of $f$, is defined by

$$\left(D_{a^+}^\alpha f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \quad n = [\alpha] + 1.$$

For more details, we refer the reader to [21, 25].

The following lemmas give some properties of Riemann-Liouville fractional integral and Caputo fractional derivative [18, 19].

**Lemma 1** Let $r, s > 0, f \in L_1([a,b])$. Then $J^r J^s f(t) = J^{r+s} f(t)$, $D^s J^s f(t) = f(t)$, $t \in [a,b]$.

**Lemma 2** Let $s > r > 0, f \in L_1([a,b])$. Then $D^r J^s f(t) = J^{s-r} f(t)$, $t \in [a,b]$.

Let us now introduce the following Banach space $X = \{x : x \in C([0,1]), D^\beta x \in C([0,1])\}$, endowed with the norm $\| x \|_X = \| x \| + \| D^\beta x \|$; $\| x \| = \sup_{t \in J} |x(t)|$ and $\| D^\beta x \| = \sup_{t \in J} \left|D^\beta x(t)\right|$.

We give the following lemmas [21]:

**Lemma 3** Let $\alpha > 0$. If $x \in C(0,1) \cap L(0,1)$, then the fractional differential equation

$$D_{0^+}^\alpha x(t) = 0$$

has solution $x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$, $n = [\alpha] + 1$.

**Lemma 4** Assume that $x \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$. Then

$$J_{0^+}^\alpha D_{0^+}^\alpha x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n},$$

We give also the following result

**Lemma 5** Let $g$ be function absolutely continuous on $J$, the solution of the boundary value problem

$$\begin{cases} D^\alpha x(t) + g(t) = 0, t \in J, 2 < \alpha \leq 3, \\ x(0) = x_0, x'(0) = 0, x'(1) = \lambda J^\sigma x(\eta) \end{cases} \quad (2)$$

is given by:

$$x(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)\,ds + x_0 \tag{3}$$
$$- \frac{\Gamma(\sigma+3)\,t^2}{2(\lambda\eta^{\sigma+2} - \Gamma(\sigma+3))\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} g(s)\,ds$$
$$+ \frac{\lambda\Gamma(\sigma+3)\,t^2}{2(\lambda\eta^{\sigma+2} - \Gamma(\sigma+3))\Gamma(\sigma+\alpha)}\int_0^\eta (\eta-s)^{\sigma+\alpha-1} g(s)\,ds$$
$$- \frac{x_0\lambda\Gamma(\sigma+3)\,\eta^\sigma t^2}{2(\lambda\eta^{\sigma+2} - \Gamma(\sigma+3))\Gamma(\sigma+1)}.$$

**Proof** For $c_i \in \mathbb{R}, i = 0, 1, 2$, and by Lemmas 3 and 4 the general solution of (2) is given by

$$x(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)\,ds - c_0 - c_1 t - c_2 t^2. \tag{4}$$

Thanks to Lemma 3, we get

$$J^\sigma x(t) = -\frac{1}{\Gamma(\sigma+\alpha)}\int_0^t (t-s)^{\sigma+\alpha-1}\,ds + \frac{x_0\eta^\sigma}{\Gamma(\sigma+1)} - c_2\frac{\Gamma(3)\,\eta^{\sigma+2}}{\Gamma(\sigma+3)}.$$

Using the boundary conditions for (2), we find that $c_0 = -x_0$ and $c_1 = 0$.
For $c_2$, we have

$$c_2 = -\frac{\lambda\Gamma(\sigma+3)}{2(\lambda\eta^{\sigma+2} - \Gamma(\sigma+3))\Gamma(\sigma+\alpha)}\int_0^\eta (\eta-s)^{\sigma+\alpha-1} g(s)\,ds$$
$$+ \frac{\Gamma(\sigma+3)}{2(\lambda\eta^{\sigma+2} - \Gamma(\sigma+3))\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} g(s)\,ds$$
$$+ \frac{x_0\lambda\Gamma(\sigma+3)\,\eta^\sigma}{2(\lambda\eta^{\sigma+2} - \Gamma(\sigma+3))\Gamma(\sigma+1)}.$$

Substituting the value of $c_{0,}$ $c_1$ and $c_2$ in (4), we get (3).

## 3. Main Results

Let us introduce the following notations

$$M_1 = \frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(\sigma+3)}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)} + \frac{|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)},$$
$$M_2 = \frac{1}{\Gamma(\alpha-\beta+1)} + \frac{\Gamma(\sigma+3)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)\Gamma(3-\beta)} + \frac{|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)\Gamma(3-\beta)},$$
$$N = \frac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)} + \frac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}$$
$$+ \frac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\delta)},$$
$$\theta_1 = \frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(\sigma+3)}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)},$$
$$\theta_2 = \frac{1}{\Gamma(\alpha-\beta+1)} + \frac{\Gamma(\sigma+3)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)\Gamma(3-\beta)}.$$

we need the following hypotheses
$(H1)$ : The function $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous.
$(H2)$ : We suppose that the partial derivative of $f$ with respect to $x, y$ exist and

are bounded. This condition implies in particular, the existence non negative real numbers $\omega, \varpi$ such that for all $(x,y), (x_1,y_1) \in \mathbb{R}^2$, we have

$$|f(t,x,y) - f(t,x_1,y_1)| \leq \omega |x - x_1| + \varpi |y - y_1|,$$

Our first result is obtained by use of the Banach's contraction principle.
**Theorem 1** Suppose that $\lambda\eta^{\sigma+2} \neq \Gamma(\sigma+3)$ and assume that the hypothesis (H2) holds.

If

$$(M_1 + M_2)(\omega + \varpi) < 1, \tag{5}$$

then the problem (1) has a unique solution on $J$.

**Proof** Consider the operator $\phi : X \to X$ defined by

$$\begin{aligned}
\phi x(t) :=& -\tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(x(s), D^\beta x(s)\right) ds + x_0 \\
&- \tfrac{\Gamma(\sigma+3)t^2}{2(\lambda\eta^{\sigma+2} - \Gamma(\sigma+3))\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} f\left(x(s), D^\beta x(s)\right) ds \\
&+ \tfrac{\lambda\Gamma(\sigma+3)t^2}{2(\lambda\eta^{\sigma+2} - \Gamma(\sigma+3))\Gamma(\sigma+\alpha)} \int_0^\eta (\eta-s)^{\sigma+\alpha-1} f\left(x(s), D^\beta x(s)\right) ds \\
&- \tfrac{x_0\lambda\Gamma(\sigma+3)\eta^\sigma t^2}{2(\lambda\eta^{\sigma+2} - \Gamma(\sigma+3))\Gamma(\sigma+1)}.
\end{aligned}$$

We shall prove that $\phi$ is a contraction
For $x, y \in X$ and for each $t \in J$, we obtain

$$\begin{aligned}
|\phi x(t) - \phi y(t)| \leq& \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|f\left(x(s), D^\beta x(s)\right) - f\left(y(s), D^\beta y(s)\right)\right| ds \\
&+ \tfrac{\Gamma(\sigma+3)t^2}{2|\lambda\eta^{\sigma+2} - \Gamma(\sigma+3)|\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \left|f\left(x(s), D^\beta x(s)\right) - f\left(y(s), D^\beta y(s)\right)\right| ds \\
&+ \tfrac{|\lambda|\Gamma(\sigma+3)t^2}{2|\lambda\eta^{\sigma+2} - \Gamma(\sigma+3)|\Gamma(\sigma+\alpha)} \int_0^\eta (\eta-s)^{\sigma+\alpha-1} \left|f\left(y(s), D^\beta y(s)\right) - f\left(y(s), D^\beta y(s)\right)\right| ds.
\end{aligned}$$

Using the $(H2)$, we can write

$$\begin{aligned}
|\phi x(t) - \phi y(t)| \leq& \tfrac{\omega\|x-y\| + \varpi\left\|D^\beta x - D^\beta y\right\|}{\Gamma(\alpha+1)} \\
&+ \tfrac{\Gamma(\sigma+3)\left(\omega\|x-y\| + \varpi\left\|D^\beta x - D^\beta y\right\|\right)}{2|\lambda\eta^{\sigma+2} - \Gamma(\sigma+3)|\Gamma(\alpha)} \\
&+ \tfrac{|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}\left(\omega\|x-y\| + \varpi\left\|D^\beta x - D^\beta y\right\|\right)}{2|\lambda\eta^{\sigma+2} - \Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)}.
\end{aligned}$$

Consequently we obtain,

$$\|\phi(x) - \phi(y)\| \leq M_1(\omega + \varpi)\left(\|x-y\| + \left\|D^\beta x - D^\beta y\right\|\right), \tag{6}$$

and

$$\begin{aligned}
\left|D^\beta \phi x(t) - D^\beta \phi y(t)\right| \leq& \tfrac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} \left|f\left(x(s), D^\beta x(s)\right) - f\left(y(s), D^\beta y(s)\right)\right| ds \\
&+ \tfrac{\Gamma(\sigma+3)t^{2-\beta}}{|\lambda\eta^{\sigma+2} - \Gamma(\sigma+3)|\Gamma(\alpha-1)\Gamma(3-\beta)} \int_0^1 (1-s)^{\alpha-2} \left|f\left(x(s), D^\beta x(s)\right) - f\left(y(s), D^\beta y(s)\right)\right| ds \\
&+ \tfrac{|\lambda|\Gamma(\sigma+3)t^{2-\beta}}{|\lambda\eta^{\sigma+2} - \Gamma(\sigma+3)|\Gamma(\sigma+\alpha)\Gamma(3-\beta)} \int_0^\eta (\eta-s)^{\sigma+\alpha-1} \left|f\left(x(s), D^\beta x(s)\right) - f\left(y(s), D^\beta y(s)\right)\right| ds.
\end{aligned}$$

By $(H2)$, we have

$$\left|D^{\beta}\phi x\left(t\right)-D^{\beta}\phi y\left(t\right)\right|\leq\frac{\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|}{\Gamma(\alpha-\beta+1)}+\frac{\Gamma(\sigma+3)\left(\omega+\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)\Gamma(3-\beta)}$$
$$+\frac{|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}\left(\omega\|x-y\|+\varpi\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)\Gamma(3-\beta)}.$$

Thus,

$$\left\|D^{\beta}\phi\left(x\right)-D^{\beta}\phi\left(y\right)\right\|\leq M_{2}\left(\omega+\varpi\right)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right),\tag{7}$$

It follows from (6) and (7) that

$$\|\phi\left(x\right)-\phi\left(y\right)\|_{X}\leq\left(M_{1}+M_{2}\right)\left(\omega+\varpi\right)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right).$$

Thanks to (5), we deduce that $\phi$ is a contraction. As a consequence of Banach contraction principle, the problem (1) has a unique solution on $J$.

The second result is based on the Scheafer's fixed point theorem.

**Ttheorem 2** Suppose that $\lambda\eta^{\sigma+2}\neq\Gamma\left(\sigma+3\right)$ and assume that the (H1), (H2) hold.

Then, the problem (1) has at least one solution on $[0,1]$.

**Proof** We use Scheafer's fixed point theorem to prove that $\phi$ has at least a fixed point on $X$. The proof will be given in several steps:

**Step 1** $\phi$ is continuous on $X$ : In view of the continuity of $f$, we conclude that the operator $\phi$ is continuous.

**Step 2** The operator $\phi$ maps bounded sets into bounded sets in $X$ : For $\mu>0$, we take $x\in B_{\mu}=\{x\in X;\|x\|_{X}\leq\mu\}$.

For $x\in B_{\mu}$, and for each $t\in J$, we have:

$$\left|\phi x\left(t\right)\right|\leq\frac{1}{\Gamma(\alpha)}\int_{0}^{t}\left(t-s\right)^{\alpha-1}\left|f\left(x\left(s\right),D^{\beta}x\left(s\right)\right)\right|ds+|x_{0}|$$
$$+\frac{\Gamma(\sigma+3)t^{2}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha-1)}\int_{0}^{1}\left(1-s\right)^{\alpha-2}\left|f\left(x\left(s\right),D^{\beta}x\left(s\right)\right)\right|ds$$
$$+\frac{|\lambda|\Gamma(\sigma+3)t^{2}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha)}\int_{0}^{\eta}\left(\eta-s\right)^{\sigma+\alpha-1}\left|f\left(x\left(s\right),D^{\beta}x\left(s\right)\right)\right|ds$$
$$+\frac{|\lambda x_{0}|\Gamma(\sigma+3)\eta^{\sigma}t^{2}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}$$

Thus,

$$\left|\phi x\left(t\right)\right|\leq\frac{1}{\Gamma(\alpha+1)}\left(\omega+\varpi\right)\|x\|_{X}+|x_{0}|$$
$$+\frac{\Gamma(\sigma+3)}{|\lambda\Gamma(3)\eta^{\sigma+2}-2\Gamma(\sigma+3)|\Gamma(\alpha)}\left(\omega+\varpi\right)\|x\|_{X}$$
$$+\frac{|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{|\lambda\Gamma(3)\eta^{\sigma+2}-2\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)}\left(\omega+\varpi\right)\|x\|_{X}$$
$$+\frac{|\lambda x_{0}|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\Gamma(3)\eta^{\sigma+2}-2\Gamma(\sigma+3)|\Gamma(\sigma+1)}.$$

We can write

$$\left|\phi x\left(t\right)\right|\leq\frac{(\omega+\varpi)\mu}{\Gamma(\alpha+1)}+|x_{0}|+\frac{(\omega+\varpi)\mu\Gamma(\sigma+3)}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)}$$
$$+\frac{(\omega+\varpi)\mu|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)}+\frac{|\lambda x_{0}|\Gamma(\sigma+3)\eta^{\sigma}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}.$$

Thus,

$$|\phi x(t)| \leq (\omega+\varpi)\,\mu\left[\tfrac{1}{\Gamma(\alpha+1)}+\tfrac{\Gamma(\sigma+3)}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)}+\tfrac{|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)}\right] + |x_0| + \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)},$$

which implies that

$$\|\phi(x)\| \leq (\omega+\varpi)\,\mu M_1 + |x_0| + \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}, \tag{8}$$

and

$$\begin{aligned}\left|D^{\beta}\phi x(t)\right| \leq{}& \tfrac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\left|f\left(x(s),D^{\beta}x(s)\right)\right|ds\\ &+\tfrac{\Gamma(\sigma+3)t^{2-\beta}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha-1)\Gamma(3-\beta)}\int_0^1 (1-s)^{\alpha-2}\left|f\left(x(s),D^{\beta}x(s)\right)\right|ds\\ &+\tfrac{|\lambda|\Gamma(\sigma+3)t^{2-\beta}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha)\Gamma(3-\beta)}\int_0^{\eta} (\eta-s)^{\sigma+\alpha-1}\left|f\left(x(s),D^{\beta}x(s)\right)\right|ds\\ &+\tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}t^{2-\beta}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}\end{aligned}$$

We obtain

$$\begin{aligned}\left|D^{\beta}\phi x(t)\right| \leq{}& \tfrac{(\omega+\varpi)\|x\|_X}{\Gamma(\alpha-\delta+1)}+\tfrac{(\omega+\varpi)\|x\|_X\Gamma(\sigma+3)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)\Gamma(3-\beta)}\\ &+\tfrac{(\omega+\varpi)\|x\|_X\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)\Gamma(3-\beta)}+\tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}.\end{aligned}$$

Consequently,

$$\left\|D^{\beta}\phi(x)\right\| \leq (\omega+\varpi)\,\mu M_2 + \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}. \tag{9}$$

Thanks to (8) and (9), yields

$$\begin{aligned}\|\phi(x)\|_X \leq{}& (\omega+\varpi)\,\mu(M_1+M_2)+|x_0|+\tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}\\ &+\tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\delta)}+\tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}.\end{aligned}$$

Therefore,

$$\|\phi(x)\|_X < \infty.$$

**Step 3** $\phi$ is equicontinuous on $J$:

Let us take $x \in B_{\mu}, t_1, t_2 \in J, t_1 < t_2$, we have:

$$\begin{aligned}&|\phi x(t_2)-\phi x(t_1)|\\ &\leq \tfrac{1}{\Gamma(\alpha)}\int_0^{t_1}\left((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right)\left|f\left(x(s),D^{\beta}x(s)\right)\right|ds\\ &+\tfrac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left|f\left(x(s),D^{\beta}x(s)\right)\right|ds\\ &+\tfrac{\Gamma(\sigma+3)\left(t_1^2-t_2^2\right)}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}\left|f\left(x(s),D^{\beta}x(s)\right)\right|ds\\ &+\tfrac{|\lambda|\Gamma(\sigma+3)\left(t_2^2-t_1^2\right)}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha)}\int_0^{\eta}(\eta-s)^{\sigma+\alpha-1}\left|f\left(x(s),D^{\beta}x(s)\right)\right|ds\\ &+\tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}\left(t_1^2-t_2^2\right)}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}.\end{aligned}$$

Thus,

$$\begin{aligned}&\left|\phi x\left(t_2\right)-\phi x\left(t_1\right)\right|\\&\leq \tfrac{(\omega+\varpi)\mu}{\Gamma(\alpha+1)}\left(t_1^{\alpha}-t_2^{\alpha}\right)+\tfrac{(\omega+\varpi)\mu}{\Gamma(\alpha+1)}\left(t_2-t_1\right)^{\alpha}+\tfrac{(\omega+\varpi)\mu\Gamma(\sigma+3)}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)}\left(t_1^2-t_2^2\right)\\&+\tfrac{(\omega+\varpi)\mu|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha)}\left(t_2^2-t_1^2\right)+\tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}\left(t_1^2-t_2^2\right).\end{aligned}$$

and

$$\begin{aligned}&\left|D^{\beta}\phi x\left(t_2\right)-D^{\beta}\phi x\left(t_1\right)\right|\\&\leq \tfrac{1}{\Gamma(\alpha-\beta)}\int_0^{t_1}\left(\left(t_1-s\right)^{\alpha-\beta-1}-\left(t_2-s\right)^{\alpha-\delta-1}\right)\left|f\left(x\left(s\right),D^{\beta}x\left(s\right)\right)\right|ds\\&+\tfrac{1}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_2}\left(t_2-s\right)^{\alpha-\beta-1}\left|f\left(x\left(s\right),D^{\beta}x\left(s\right)\right)\right|ds\\&+\tfrac{\Gamma(\sigma+3)\left(t_1^{2-\beta}-t_2^{2-\beta}\right)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha-1)\Gamma(3-\beta)}\int_0^1\left(1-s\right)^{\alpha-2}\left|f\left(x\left(s\right),D^{\beta}x\left(s\right)\right)\right|ds\\&+\tfrac{|\lambda|\Gamma(\sigma+3)\left(t_2^{2-\beta}-t_1^{2-\beta}\right)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha)\Gamma(3-\beta)}\int_0^{\eta}\left(\eta-s\right)^{\sigma+\alpha-1}\left|f\left(x\left(s\right),D^{\beta}x\left(s\right)\right)\right|ds\\&+\tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}\left(t_1^{2-\beta}-t_2^{2-\beta}\right)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\left|D^{\beta}\phi x\left(t_2\right)-D^{\beta}\phi x\left(t_1\right)\right|\\&\leq \tfrac{(\omega+\varpi)\mu}{\Gamma(\alpha-\beta+1)}\left(t_1^{\alpha-\beta}-t_2^{\alpha-\beta}\right)+\tfrac{(\omega+\varpi)\mu}{\Gamma(\alpha-\beta+1)}\left(t_2-t_1\right)^{\alpha-\beta}\\&+\tfrac{(\omega+\varpi)\mu\Gamma(\sigma+3)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)\Gamma(2-\beta)}\left(t_1^{2-\beta}-t_2^{2-\beta}\right)\\&+\tfrac{(\omega+\varpi)\mu|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)\Gamma(2-\beta)}\left(t_2^{2-\beta}-t_1^{2-\beta}\right)\\&+\tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(2-\beta)}\left(t_1^{2-\beta}-t_2^{2-\beta}\right).\end{aligned}$$

Hence,

$$\begin{aligned}&\left\|\phi x\left(t_2\right)-\phi x\left(t_1\right)\right\|_X\\&\leq \tfrac{(\omega+\varpi)\mu}{\Gamma(\alpha+1)}\left(t_1^{\alpha}-t_2^{\alpha}\right)+\tfrac{(\omega+\varpi)\mu}{\Gamma(\alpha+1)}\left(t_2-t_1\right)^{\alpha}+\tfrac{(\omega+\varpi)\mu}{\Gamma(\alpha-\beta+1)}\left(t_1^{\alpha-\beta}-t_2^{\alpha-\beta}\right)\\&+\tfrac{((\omega+\varpi)\mu+L)}{\Gamma(\alpha-\beta+1)}\left(t_2-t_1\right)^{\alpha-\beta}+\tfrac{(\omega+\varpi)\mu\Gamma(\sigma+3)}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)}\left(t_1^2-t_2^2\right)\\&+\tfrac{(\omega+\varpi)\mu|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha)}\left(t_2^2-t_1^2\right)+\tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}\left(t_1^2-t_2^2\right)\\&+\tfrac{((\omega+\varpi)\mu+L)|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)\Gamma(3-\beta)}\left(t_2^{2-\beta}-t_1^{2-\beta}\right)+\tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}\left(t_1^{2-\beta}-t_2^{2-\beta}\right).\end{aligned}$$

which implies $\left\|\phi x\left(t_2\right)-\phi x\left(t_1\right)\right\|_X\rightarrow 0$ as $t_2\rightarrow t_1$. By Arzela-Ascoli theorem, we conclude that $\phi$ is completely continuous operator.

**Step 4** We show that the set $\Omega$ defined by:

$$\Omega=\{x\in X,x=\rho\phi\left(x\right),0<\rho<1\},$$

is bounded:

Let $x \in \Omega$, then $x = \rho\phi(x)$, for some $0 < \rho < 1$. Thus, for each $t \in J$, we have:

$$\begin{aligned}\frac{1}{\rho}|x(t)| &\leq \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(x(s), D^\beta x(s)\right)\right| ds + |x_0| \\ &+ \tfrac{\Gamma(\sigma+3)t^2}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\left|f\left(x(s), D^\beta x(s)\right)\right| ds \\ &+ \tfrac{|\lambda|\Gamma(\sigma+3)t^2}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha)}\int_0^\eta (\eta-s)^{\sigma+\alpha-1}\left|f\left(x(s), D^\beta x(s)\right)\right| ds \\ &+ \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^\sigma t^2}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}.\end{aligned}$$

So, we can write

$$\frac{1}{\rho}|x(t)| \leq (\omega+\varpi)\mu M_1 + |x_0| + \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^\sigma}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}.$$

Therefore,

$$\|x\| \leq \rho\left[(\omega+\varpi)\mu M_1 + |x_0| + \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^\sigma}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}\right], \tag{10}$$

and

$$\begin{aligned}&\frac{1}{\rho}\left|D^\beta x(t)\right| \\ &\leq \tfrac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\left|f\left(x(s), D^\beta x(s)\right)\right| ds \\ &+ \tfrac{\Gamma(\sigma+3)t^{2-\beta}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha-1)\Gamma(3-\beta)}\int_0^1 (1-s)^{\alpha-2}\left|f\left(x(s), D^\beta x(s)\right)\right| ds \\ &+ \tfrac{|\lambda|\Gamma(\sigma+3)t^{2-\beta}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha)\Gamma(3-\beta)}\int_0^\eta (\eta-s)^{\sigma+\alpha-1}\left|f\left(x(s), D^\beta x(s)\right)\right| ds \\ &+ \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^\sigma t^{2-\beta}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}.\end{aligned}$$

As a consequence, we get

$$\begin{aligned}\frac{1}{\rho}\left|D^\beta x(t)\right| &\leq (\omega+\varpi)\mu\left[\tfrac{1}{\Gamma(\alpha-\beta+1)} + \tfrac{\Gamma(\sigma+3)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)\Gamma(3-\beta)}\right] \\ &+ (\omega+\varpi)\mu\tfrac{|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)\Gamma(3-\beta)} \\ &+ \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^\sigma}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}.\end{aligned}$$

Therefore,

$$\left|D^\beta x(t)\right| \leq \rho\left[(\omega+\varpi)\mu M_2 + \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^\sigma}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}\right]. \tag{11}$$

Thus, from (10) and (11), we obtain

$$\begin{aligned}\|x\|_X &\leq \rho\left[(\omega+\varpi)\mu(M_1+M_2) + |x_0|\right] \\ &+ \rho\left[\tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^\sigma}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)} + \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^\sigma}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}\right]\end{aligned}$$

Hence,

$$\|\phi(x)\|_X < \infty.$$

This shows that $\Omega$ is bounded.

As consequence of Schaefer's fixed point theorem, the problem (1) has at least one solution on $[0,1]$.

Now, we state Krasnoselskii's fixed point theorem [21] which is needed to prove our next existence result.

**Theorem 3** Suppose that $\lambda\eta^{\sigma+2} \neq \Gamma(\sigma+3)$ and assume that the hypotheses (H1)-(H2) are satisfied, such that

$$(\theta_1+\theta_2)(\omega+\varpi) < 1. \tag{12}$$

If there exist $\upsilon \in \mathbb{R}$ such that

$$\upsilon \geq \upsilon(\omega+\varpi)(M_1+M_2)+|x_0|+N, \tag{13}$$

then there exists at least one solution of the boundary value problem (1) on $[0,1]$.

**Proof** Suppose that (13) holds and let us take

$$\phi x(t) := Tx(t) + Rx(t),$$

where

$$\begin{aligned} Tx(t) &:= -\tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(x(s), D^\beta x(s)\right) ds + x_0 \\ &\quad - \tfrac{\Gamma(\sigma+3)t^2}{2(\lambda\eta^{\sigma+2}-\Gamma(\sigma+3))\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2} f\left(x(s), D^\beta x(s)\right) ds, \end{aligned}$$

and

$$\begin{aligned} Rx(t) &:= \tfrac{\lambda\Gamma(\sigma+3)t^2}{2(\lambda\eta^{\sigma+2}-\Gamma(\sigma+3))\Gamma(\sigma+\alpha)}\int_0^\eta (\eta-s)^{\sigma+\alpha-1} f\left(x(s), D^\beta x(s)\right) ds \\ &\quad - \tfrac{x_0\lambda\Gamma(\sigma+3)\eta^\sigma t^2}{2(\lambda\eta^{\sigma+2}-\Gamma(\sigma+3))\Gamma(\sigma+1)}. \end{aligned}$$

$(1:*)$:We shall prove that for any $x, y \in B_\upsilon$, then $T(x)+R(y) \in B_\upsilon$. Such that $B_\upsilon = \{x \in X; \|x\|_X \leq \upsilon\}$.

For any $x, y \in B_\upsilon$ and for each $t \in J$ we have:

$$\begin{aligned} |Tx(t)+Ry(t)| &\leq \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(x(s), D^\beta x(s)\right)\right| ds + |x_0| \\ &\quad + \tfrac{\Gamma(\sigma+3)t^2}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\left|f\left(x(s), D^\beta x(s)\right)\right| ds \\ &\quad + \tfrac{|\lambda|\Gamma(\sigma+3)t^2}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha)}\int_0^\eta (\eta-s)^{\sigma+\alpha-1}\left|f\left(x(s), D^\beta x(s)\right)\right| ds \\ &\quad + \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^\sigma t^2}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}. \end{aligned}$$

We obtain:

$$\begin{aligned} |Tx(t)+Ry(t)| &\leq \tfrac{(\omega+\varpi)\upsilon}{\Gamma(\alpha+1)} + |x_0| + \tfrac{(\omega+\varpi)\upsilon\Gamma(\sigma+3)}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)} \\ &\quad + \tfrac{(\omega+\varpi)\upsilon|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)} + \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^\sigma}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}. \end{aligned}$$

Consequently,

$$\begin{aligned} &|Tx(t)+Ry(t)| \\ &\leq (\omega+\varpi)\upsilon\left[\tfrac{1}{\Gamma(\alpha+1)} + \tfrac{\Gamma(\sigma+3)}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)} + \tfrac{|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)}\right] \\ &+ |x_0| + \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^\sigma}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}, \end{aligned}$$

which implies that

$$\left\|T(x)+R(y)\right\| \leq (\omega+\varpi)\,\upsilon M_1 + |x_0| + \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}. \tag{14}$$

On the other hand,

$$\begin{aligned}
&\left|D^{\beta}Tx(t)+D^{\beta}Ry(t)\right| \\
&\leq \tfrac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\left|f\left(x(s),D^{\beta}x(s)\right)\right|ds \\
&+ \tfrac{\Gamma(\sigma+3)t^{2-\beta}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha-1)\Gamma(3-\beta)}\int_0^1 (1-s)^{\alpha-2}\left|f\left(x(s),D^{\beta}x(s)\right)\right|ds \\
&+ \tfrac{|\lambda|\Gamma(\sigma+3)t^{2-\beta}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha)\Gamma(3-\beta)}\int_0^{\eta} (\eta-s)^{\sigma+\alpha-1}\left|f\left(x(s),D^{\beta}x(s)\right)\right|ds \\
&+ \tfrac{2|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}t^{2-\beta}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)},
\end{aligned}$$

we have

$$\begin{aligned}
&\left|D^{\beta}Tx(t)+D^{\beta}Ry(t)\right| \\
&\leq \tfrac{(\omega+\varpi)\upsilon}{\Gamma(\alpha-\beta+1)} + \tfrac{(\omega+\varpi)\upsilon\Gamma(\sigma+3)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)\Gamma(3-\beta)} \\
&+ \tfrac{(\omega+\varpi)\upsilon|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)\Gamma(3-\beta)} + \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&\left|D^{\beta}Tx(t)+D^{\beta}Ry(t)\right| \\
&\leq (\omega+\varpi)\,\upsilon\left[\tfrac{1}{\Gamma(\alpha-\beta+1)} + \tfrac{\Gamma(\sigma+3)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)\Gamma(3-\beta)} + \tfrac{|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)\Gamma(3-\beta)}\right] \\
&+ \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}.
\end{aligned}$$

Hence,

$$\left\|D^{\beta}T(x)+D^{\beta}R(y)\right\| \leq (\omega+\varpi)\,\upsilon M_2 + \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}, \tag{15}$$

It follows (14) and (15) that

$$\begin{aligned}
&\left\|T(x)+R(y)\right\|_X \\
&\leq \left[(\omega+\varpi)\,\upsilon(M_1+M_2) + |x_0| + \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)} + \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}\right].
\end{aligned}$$

And consequently,

$$\left\|T(x)+R(y)\right\|_X \leq (\omega+\varpi)\,\upsilon(M_1+M_2) + |x_0| + N \leq \upsilon.$$

Using the condition (13) we conclude that $T(x)+R(y)\in B_{\upsilon}$.

$(2:*)$ : We shall prove that $R$ is continuous and compact.
$(2.1:*)$ : The continuity of $f$ implies that the operator $R$ is continuous.
$(2.2:*)$ : Now, we prove that $R$ maps bounded sets into bounded sets of $X$.
For $x\in B_{\upsilon}$ and for each $t\in J$, we have:

$$\begin{aligned}
|Rx(t)| \leq{}& \tfrac{|\lambda|\Gamma(\sigma+3)t^2}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha)}\int_0^{\eta}(\eta-s)^{\sigma+\alpha-1}\left|f\left(x(s),D^{\beta}x(s)\right)\right|ds \\
&+ \tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}t^2}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)},
\end{aligned}$$

using $(H2)$, we obtain

$$|R(x)| \leq \frac{(\omega+\varpi)\upsilon|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)} + \frac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}, \tag{16}$$

and

$$\begin{aligned}\left|D^{\beta}Rx(t)\right| &\leq \frac{|\lambda|\Gamma(\sigma+3)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha)\Gamma(3-\beta)}\int_0^1 (1-s)^{\alpha-2}\left|f\left(x(s),D^{\beta}x(s)\right)\right|ds \\ &+ \frac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}t^{2-\beta}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}.\end{aligned}$$

Indeed, we have

$$\left|D^{\beta}Rx(t)\right| \leq \frac{(\omega+\varpi)\upsilon|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)\Gamma(3-\beta)} + \frac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}. \tag{17}$$

Combining (16) and (17) yields

$$\|R(x)\|_X \leq (\omega+\varpi)\upsilon\left[\frac{|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)} + \frac{|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)\Gamma(3-\beta)}\right]+N.$$

And consequently,

$$\|R(x)\|_X < \infty.$$

Thus, it follows from the above inequalities that the operator $R$ is uniformly bounded.

$(2.3:*)$ : The operator $R$ maps bounded sets into equicontinuous sets of $X$.
Let $t_1, t_2 \in J; t_2 < t_1, x \in B_{\upsilon}$. Then, we have:

$$\begin{aligned}&|Rx(t_1) - Rx(t_2)| \\ &\leq \frac{|\lambda|\Gamma(\sigma+3)\left(t_1^2-t_2^2\right)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha)}\int_0^{\eta}(\eta-s)^{\sigma+\alpha-1}\left|f\left(x(s),D^{\beta}x(s)\right)\right|ds \\ &+ \frac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}\left(t_2^2-t_1^2\right)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)},\end{aligned}$$

we obtain

$$\begin{aligned}&|Rx(t_1) - Rx(t_2)| \\ &\leq \frac{(\omega+\varpi)\upsilon|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)}\left(t_1^2-t_2^2\right) + \frac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}\left(t_2^2-t_1^2\right).\end{aligned} \tag{18}$$

On the other hand,

$$\begin{aligned}&\left|D^{\beta}Rx(t_1) - D^{\beta}Rx(t_2)\right| \\ &\leq \frac{|\lambda|\Gamma(\sigma+3)\left(t_1^{2-\beta}-t_2^{2-\beta}\right)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha)\Gamma(3-\beta)}\int_0^{\eta}(\eta-s)^{\sigma+\alpha-1}\left|f\left(x(s),D^{\beta}x(s)\right)\right|ds \\ &+ \frac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}\left(t_2^{2-\beta}-t_1^{2-\beta}\right)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)},\end{aligned}$$

we can write:

$$\begin{aligned}\left|D^{\beta}Rx(t_1) - D^{\beta}Rx(t_2)\right| &\leq \frac{((\omega+\varpi)\upsilon+L)|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)\Gamma(3-\beta)}\left(t_1^{2-\beta}-t_2^{2-\beta}\right) \\ &+ \frac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}\left(t_2^{2-\beta}-t_1^{2-\beta}\right),\end{aligned} \tag{19}$$

It follows from (18) and (19) that

$$\begin{aligned}\|Rx(t_1)-Rx(t_2)\|_X &\le \tfrac{(\omega+\varpi)\upsilon|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)}\left(t_1^2-t_2^2\right)\\ &+\tfrac{((\omega+\varpi)\upsilon)|\lambda|\Gamma(\sigma+3)\eta^{\sigma+\alpha}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+\alpha+1)\Gamma(3-\beta)}\left(t_1^{2-\beta}-t_2^{2-\beta}\right)\\ &+\tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)}\left(t_2^2-t_1^2\right)\\ &+\tfrac{|\lambda x_0|\Gamma(\sigma+3)\eta^{\sigma}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\sigma+1)\Gamma(3-\beta)}\left(t_2^{2-\beta}-t_1^{2-\beta}\right).\end{aligned}$$

As $t_1\to t_2$ the right-hand side of this inequality tends to zero. Then, as a consequence of steps $[(2.1:*),(2.2:*),(2.3:*)]$; we can conclude that $R$ is continuous and compact.
$(3:*)$: Now, we prove that $T$ is contraction mapping.

Let $x,y\in X$. Then, for each $t\in J$, we have

$$|Tx(t)-Ty(t)|\le \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f\left(x(s),D^{\beta}x(s)\right)ds-f\left(y(s),D^{\beta}y(s)\right)\right|ds$$
$$+\tfrac{\Gamma(\sigma+3)t^2}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\left|f\left(x(s),D^{\beta}x(s)\right)ds-f\left(y(s),D^{\beta}y(s)\right)\right|ds.$$

By $(H1)$ ,we obtain

$$|Tx(t)-Ty(t)|\le \tfrac{(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{\Gamma(\alpha+1)}+\tfrac{\Gamma(\sigma+3)(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{2|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)}.$$

Consequently,

$$\|T(x)-T(y)\|\le\theta_1(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right), \tag{20}$$

and

$$\left|D^{\beta}Tx(t)-D^{\beta}Ty(t)\right|$$
$$\le\tfrac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\left|f\left(x(s),D^{\beta}x(s)\right)ds-f\left(y(s),D^{\beta}y(s)\right)\right|ds$$
$$+\tfrac{\Gamma(\sigma+3)t^{2-\beta}}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha-1)\Gamma(3-\beta)}\int_0^1 (1-s)^{\alpha-2}\left|f\left(x(s),D^{\beta}x(s)\right)ds-f\left(y(s),D^{\beta}y(s)\right)\right|ds.$$

Using the $(H1)$ ,we have

$$\left|D^{\beta}Tx(t)-D^{\beta}Ty(t)\right|\le\tfrac{(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{\Gamma(\alpha-\beta+1)}+\tfrac{\Gamma(\sigma+3)(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)\Gamma(3-\beta)}.$$

Consequently,

$$\begin{aligned}\left\|D^{\beta}T(x)-D^{\beta}T(y)\right\| &\le\tfrac{(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{\Gamma(\alpha-\beta+1)}\\ &+\tfrac{\Gamma(\sigma+3)(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right)}{|\lambda\eta^{\sigma+2}-\Gamma(\sigma+3)|\Gamma(\alpha)\Gamma(3-\beta)}\\ &\le\theta_2(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right).\end{aligned} \tag{21}$$

It follows from (20) and (21) that

$$\|T(x)-T(y)\|_X\le(\theta_1+\theta_2)(\omega+\varpi)\left(\|x-y\|+\left\|D^{\beta}x-D^{\beta}y\right\|\right).$$

Using the condition (12) we conclude that $T$ is a contraction mapping.
As a consequence of Krasnoselskii's fixed point theorem we deduce that $\phi$ has a fixed point which is a solution of $(1)$.

## 4. Examples

**Example 1**
Consider the following fractional problem.

$$\begin{cases} D^{\frac{5}{2}}x(t)+\frac{e^{-2\pi}|x(t)|}{20\left(\sqrt{\pi}+2\right)}+\frac{\left|D^{\frac{3}{2}}x(t)\right|}{(7\pi+4e^{-\pi})^2\left(1+\left|D^{\frac{3}{2}}x(t)\right|\right)}=0, t\in[0,1], \\ x(0)=\sqrt{2}, x'(0)=0, x'(1)=\frac{3}{4}J^{\frac{2}{3}}\left(\frac{1}{5}\right)=0. \end{cases} \tag{22}$$

So, we have:

$$f\left(x, D^{\frac{3}{2}}x(t)\right)=\frac{e^{-2\pi}|x(t)|}{20\left(\sqrt{\pi}+2\right)}+\frac{\left|D^{\frac{3}{2}}x(t)\right|}{(7\pi+4e^{-\pi})^2\left(1+\left|D^{\frac{3}{2}}x(t)\right|\right)}, t\in[0,1], x, y\in\mathbb{R}.$$

Let $x, y, x_1, y_1\in\mathbb{R}$ and $t\in J$. Then we have:

$$|f(x,y)-f(x_1,y_1)|\leq\frac{e^{-2\pi}}{20\left(\sqrt{\pi}+2\right)}|x-x_1|+\frac{1}{(7\pi+4e^{-\pi})^2}|y-y_1|.$$

So we can take

$$\omega=\frac{e^{-2\pi}}{20\left(\sqrt{\pi}+2\right)}, \varpi=\frac{1}{(7\pi+4e^{-\pi})^2}.$$

Then,

$$(M_1+M_2)(\omega+\varpi)<1.$$

Hence by Theorem $(6)$, the boundary value problem (22) has a unique solution on $[0,1]$.

**Example 2**
Let us consider the following boundary value problem.

$$\begin{cases} D^{\frac{9}{4}}x(t)+\frac{\sqrt{\pi}|x(t)|}{(20\pi+e)(1+|x(t)|)}+\frac{1}{8(\pi+1)^2}\sin\left|D^{\frac{1}{3}}x(t)\right|=0, t\in[0,1], \\ x(0)=\sqrt{3}, x'(0)=0, x'(1)=\frac{2}{5}J^{\frac{6}{7}}\left(\frac{2}{3}\right)=0. \end{cases} \tag{23}$$

Set,

$$f(t,x,y)=\frac{\sqrt{\pi}|x|}{(20\pi+e)(1+|x|)}+\frac{1}{8(\pi+1)^2}\sin|y|, t\in J, x, y\in\mathbb{R}.$$

For $t\in[0,1]$ and $x, y, x_1, y_1\in\mathbb{R}$, we have

$$|f(x,y)-f(x_1,y_1)|\leq\frac{\sqrt{\pi}}{(20\pi+e)}|x-x_1|+\frac{1}{8(\pi+1)^2}|y-y_1|.$$

So, we have

$$\omega=\frac{\sqrt{\pi}}{(20\pi+e)}, \varpi=\frac{1}{8(\pi+1)^2}.$$

It follows then that

$$(\theta_1+\theta_2)(\omega+\varpi)<1$$

Hence by Theorem $(8)$, the boundary value problem (23) has a solution.

**Acknowledgement** The authors thank the referee for his insightful comments and suggestions.

## References

[1] B. Ahmad, S.K. Ntouyas and A. Assolami: Caputo type fractional differential equations with nonlocal Riemann-Liouville integral boundary conditions. J Appl Math Comput. 41, pp. 339–350, 2013.
[2] B. Ahmad, J.J. Nieto: Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions. Bound. Value Probl. 36, 2011.
[3] B. Ahmad: On nonlocal boundary value problems for nonlinear integro-differential equations of arbitrary fractional order. Results. Math. 63, pp. 183-194, 2013.
[4] B. Ahmad, A. Alsaedi and A. Assolami: Existence results for Caputo type fractional differential equations with four-point nonlocal fractional integral boundary conditions. Electronic Journal of Qualitative Theory of Differential Equations. No. 93, pp.1-11, 2012.
[5] R. Ali khan: Three-point boundary value problems for higher order nonlinear fractional differential equations. J. Appl. Math. & Informatics. Vol. 31, No. 1-2, pp. 221-228, 2013.
[6] M. Benchohra, B. Hedia: Positive solutions for boundary value problems with fractional order. International Journal of Advanced Mathematical Sciences.Vol. 1, No. 1, pp. 12-22, 2013.
[7] G. Chai: Existence results of positive solutions for boundary value problems of fractional differential equations. Boundary Value Problems. 109, 2013.
[8] A. Cabada, G. Wang: Positive solutions of nonlinear fractional differential equations with integral boundary value conditions. J. Math. Anal. Appl. 389, pp. 403-411, 2012.
[9] D. Delbosco, L. Rodino: Existence and uniqueness for a nonlinear fractional differential equation, J. Math. Anal. Appl. 204. No. 3-4, pp. 429-440, 1996.
[10] K. Diethelm, N.J. Ford: Analysis of fractional differential equations. J. Math. Anal. Appl. 265(2), pp. 229-248, 2002.
[11] K. Diethelm, G. Walz: Numerical solution of fraction order differential equations by extrapolation, Numer. Algorithms. 16(3), pp. 231-253, 1998.
[12] A.M.A. El-Sayed: Nonlinear functional differential equations of arbitrary orders. Nonlinear Anal. 33(2), pp. 181-186, 1998.
[13] Guezane-Lakoud, A, Khaldi, R: Solvability of fractional boundary value problem with fractional integral condition. Nonlinear Anal. 75, pp. 2692-2700, 2012.
[14] R. Hilfer. Applications of fractional calculus in physics, World Scientific, Publishing Co., Inc., River Edge, NJ. 2000.
[15] M. Houas, Z. Dahmani: New Results for a Caputo Boundary Value Problem. American Journal of Computational and Applied Mathematics. 3(3), pp. 143-161, 2013.
[16] M. Houas, Z. Dahmani, M. Benbachir: New results for a boundary value problem for differential equations of arbitrary order. International Journal of Modern Mathematical Sciences. 7(2), pp.195-211, 2013.
[17] M. Houas, Z. Dahmani: New results for a coupled system of fractional differential equations . Facta A Universitatis (NIS) Ser. Math. Inform. Vol. 28, No 2, pp. 133-150, 2013.
[18] A.A. Kilbas, S.A. Marzan: Nonlinear differential equation with the Caputo fractional derivative in the space of continuously differentiable functions, Differ. Equ. 41(1), pp. 84-89, 2005.
[19] V. Lakshmikantham, A.S. Vatsala: Basic theory of fractional differential equations. Nonlinear Anal. 69(8), pp. 2677-2682, 2008.
[20] Z. Liu, X. Li: Existence and uniqueness of solutions for the nonlinear impulsive fractional differential equations. Communications in Nonlinear Science and Numerical Simulation. Vol. 18, pp. 1362-1373, 2013.
[21] F. Mainardi, Fractional calculus: Some basic problem in continuum and statistical mechanics. Fractals and fractional calculus in continuum mechanics, Springer, Vienna. (1997).
[22] S.M. Momani, S.B. Hadid: Some comparison results for integro- fractional differential inequalities. J. Fract. Calc. 24, pp. 37-44, 2003.
[23] S.K. Ntouyas: Existence results for first order boundary value problems for fractional differential equations and inclusions with fractional integral boundary conditions. Journal of Fractional Calculus and Applications. 3(9), pp. 1-14, 2012.

[24] S.K. Ntouyas: Boundary value problems for nonlinear fractional differential equations and inclusions with nonlocal and fractional integral boundary conditions. Opuscula Math. 33, No. 1, pp. 117–138, 2013.

[25] I. Podlubny, I. Petras, B.M. Vinagre, P. O'leary, L. Dorcak: Analogue realizations of fractional-order controllers. Fractional order calculus and its applications. Nonlinear Dynam. 29(4), pp. 281-296, 2002.

[26] M.U. Rehman ,R.A Khan, N.A. Asif :Three point boundary value problems for nonlinear fractional differential equations. Acta Mathematica Scientia. 31B(4). pp. 1337–1346, 2011.

Mohamed Houas, Faculty of Sciences and Technology, Khemis-Miliana University, Ain Defla, Algeria

*E-mail address*: houasmed@yahoo.fr

Maamar Benbachir, Faculty of Sciences and Technology, Khemis-Miliana University, Ain Defla, Algeria

*E-mail address*: mbenbachir201@gmail.com