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ON THE PROPERTIES OF SOLUTION OPERATORS OF FRACTIONAL EVOLUTION EQUATIONS

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Abstract. In this paper, we study solution operators for a fractional evolution equation involving an (almost) sectorial operator A. By employing fractional powers of operators notion, for an initial data lying in a domain of the fractional power of A, we obtain a solution for the Cauchy problem of the fractional evolution equation. Moreover, we also find a new semigroup-like property.

1. Introduction

Let H be a Banach space. We consider the fractional Cauchy problem

$$\begin{aligned} D_t^\alpha u &= Au, \ t > 0, \\ u(0) &= u_0, \end{aligned} \tag{1.1}$$

where $A : D(A) \subset H \to H$ is a linear operator and D_t^α is the Caputo fractional time derivative of order α with $0 < \alpha < 1$. There were some researches studying this problem, for intances, see [1, 2, 6]. Bajlekova [1] introduced a solution operator for (1.1) as follows. Let $B(H)$ be the set of all bounded linear operators on H.

Definition 1.1. *A family $\{S_\alpha(t)\}_{t\geq 0} \subset B(H)$ is called a solution operator for* (1.1) *if the following conditions are satisfied :*

(i) *$S_\alpha(t)$ is strongly continuous for $t \geq 0$ and $S_\alpha(0) = I$,*
(ii) *$S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$, $t \geq 0$,*
(iii) *$S_\alpha(t)x$ is a solution of*

$$u(t) = x + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Au(s)ds$$

for all $x \in D(A)$, $t \geq 0$.

Chen et al. [2] also introduced what they called as fractional resolvent operator functions defined by purely algebraic condition.

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Definition 1.2. *Let $\alpha > 0$. A function $S_\alpha : \mathbb{R}_+ \to B(H)$ is called an α-resovent operator function if the following conditions are satisfied :*

(i) *$S_\alpha(\cdot)$ is strongly continuous on $\mathbb{R}_+$ and $S_\alpha(0) = I$,*
(ii) *$S_\alpha(s)S_\alpha(t) = S_\alpha(t)S_\alpha(s)$ for all $s, t \geq 0$,*
(iii) *The functional equation*

$$\begin{aligned} S_\alpha(s)\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)d\tau - \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)d\tau S_\alpha(t) \\ = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)d\tau - \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)d\tau. \end{aligned} \tag{1.2}$$

holds for all $s, t \geq 0$.

In [6], Peng et al. introduced what they named as strongly continuous fractional semigroup of order α.

Definition 1.3. *Let $0 < \alpha < 1$. A one-parameter family $\{S_\alpha(t)\}_{t\geq 0}$ of bounded linear operators of H is called strongly continuous fractional semigroup of order α if it possesses the following two properties :*

(i) *For every $x \in H$, the mapping $t \mapsto S_\alpha(t)x$ is continuous over $[0, \infty)$,*
(ii) *$S_\alpha(0) = I$ and, for all $s, t \geq 0$,*

$$\begin{aligned} \int_0^{t+s} \frac{S_\alpha(\tau)}{(t+s-\tau)^\alpha} d\tau - \int_0^t \frac{S_\alpha(\tau)}{(t+s-\tau)^\alpha} d\tau - \int_0^s \frac{S_\alpha(\tau)}{(t+s-\tau)^\alpha} d\tau \\ = \alpha \int_0^t \int_0^s \frac{S_\alpha(\tau_1)S_\alpha(\tau_2)}{(t+s-\tau_1-\tau_2)^{1+\alpha}} d\tau_1 d\tau_2, \end{aligned} \tag{1.3}$$

where the integrals are defined in the strong operator topology.

In [1, 2, 6], the authors showed that the operator that each of them introduced in Definition 1.1, 1.2, and 1.3 is the solution operator for the problem (1.1) with each certain conditions. All of them found that, for $u_0 \in D(A)$ and $t \geq 0$, $u(t) = S_\alpha(t)u_0$ is the solution to the problem (1.1).

Wang et al., in [10], studied the Cauchy problem for the linear evolution equation

$$\begin{aligned} D_t^\alpha u(t) + Au(t) = f(t),\ t > 0, \\ u(0) = u_0, \end{aligned} \tag{1.4}$$

where D_t^α is the Caputo fractional time derivative of order α $(0 < \alpha < 1)$, $f : (0, \infty) \to H$, and $A : D(A) \subset H \to H$ is a linear operator satisfying the properties that there are constants $0 < \gamma < 1$ and $0 < \omega < \pi/2$ such that

$$\sigma(A) \subset \overline{\Sigma}_\omega \tag{1.5}$$

and, for every $\omega < \mu < \pi$, there exists a constant $C_\mu > 0$ such that

$$\|R(\lambda; A)\| \leq \frac{C_\mu}{|\lambda|^\gamma},\ \lambda \in \mathbb{C} \setminus \overline{\Sigma}_\mu, \tag{1.6}$$

where $\sigma(A)$ is the spectrum set of A, $R(\lambda; A) = (\lambda - A)^{-1}$ is the resolvent operator of A, and $\Sigma_\omega = \{\lambda \in \mathbb{C} : \lambda \neq 0, |\arg(\lambda)| < \omega\}$. They defined a pair of operators

$$S_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_\theta} E_{\alpha,1}(-\lambda t^\alpha)R(\lambda; A)d\lambda, \quad t \in \Sigma_{\pi/2-\omega},$$

$$P_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_\theta} E_{\alpha,\alpha}(-\lambda t^\alpha)R(\lambda; A)d\lambda, \quad t \in \Sigma_{\pi/2-\omega},$$

where the integral contour $\Gamma_\theta = \{\mathbb{R}_+ e^{i\theta}\} \cup \{\mathbb{R}_+ e^{-i\theta}\}$ is oriented counterclockwise with $\omega < \theta < \mu < \pi/2 - |\arg(t)|$ and $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function. They showed that this pair of the operators is the solution operator for the problem (1.4) and, for $u_0 \in D(A)$ and $t > 0$, $u(t) = S_\alpha(t)u_0$ is the solution to the homogeneous case of the problem (1.4).

In this paper, we study the fractional Cauchy problem

$$\begin{aligned} D_t^\alpha u(t) &= Au(t) + f(t),\ t > 0, \\ u(0) &= u_0, \end{aligned} \tag{1.7}$$

where D_t^α is the Caputo fractional time derivative of order α $(0 < \alpha < 1)$, $f : (0, \infty) \to H$, and $A : D(A) \subset H \to H$ is a linear operator satisfying the properties that there are constants $\theta \in (\pi/2, \pi)$, $M > 0$, and $0 < \gamma \le 1$ such that

$$\rho(A) \supset \Sigma_\theta, \tag{1.8}$$

$$\|R(\lambda; A)\| \le \frac{M}{|\lambda|^\gamma},\ \lambda \in \Sigma_\theta, \tag{1.9}$$

where $\rho(A)$ is the resolvent set of A. If $\gamma = 1$, we call A as a sectorial operator, otherwise, A is called as an almost sectorial operator. Here, we define a pair of operators

$$\begin{aligned} S_\alpha(t) &= \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha; A) d\lambda, \quad t > 0, \\ P_\alpha(t) &= \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} e^{\lambda t} R(\lambda^\alpha; A) d\lambda, \quad t > 0, \end{aligned}$$

where $r > 0$, $\pi/2 < \omega < \theta$, and the integral contour

$$\Gamma_{r,\omega} = \{\lambda \in \mathbb{C} : |\arg(\lambda)| = \omega, |\lambda| \ge r\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda)| \le \omega, |\lambda| = r\}$$

is oriented counterclockwise. We also show that this pair of the operators is the solution operator for the problem (1.7). By employing some estimates for $A^\beta S_\alpha(t)$ and $A^\beta P_\alpha(t)$, where A^β denotes the fractional power of A, we obtain that, not only for $u_0 \in D(A)$ but also for $u_0 \in D(A^\beta)$ with $1-\gamma < \beta < \gamma$ $(1/2 < \gamma \le 1)$ and $t > 0$, $u(t) = S_\alpha(t)u_0$ is the solution to the homogeneous case of the problem (1.7) and moreover, differently from (1.2) and (1.3), we get a new semigroup-like property, that is

$$S_\alpha(t+s)u_0 = S_\alpha(t)S_\alpha(s)u_0 - A \int_0^t \int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} P_\alpha(\tau)P_\alpha(r)u_0 dr d\tau,\ s, t > 0.$$

This paper is composed of four sections. In section 2, we introduce briefly the fractional integration and differentiation of Riemann-Liouville and Caputo operators. Some special function related to fractional differential equations and its properties are also introduced in this section. In section 3, we study analytic solution operators of a fractional evolution equation both in homogeneous and inhomogeneous case. In the last section, the fractional power of an (almost) sectorial operator is discussed and our main results are showed.

2. Fractional Time Derivative

Let $0 < \alpha < 1$, $a \geq 0$ and $I = (a, T)$ for some $T > 0$. The *Riemann-Liouville fractional integral* of order α is defined by

$$J^{\alpha}_{a,t} f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \quad f \in L^1(I),\ t > a. \tag{2.1}$$

We set $J^0_{a,t} f(t) = f(t)$. The fractional integral operator (2.1) obeys the semigroup property

$$J^{\alpha}_{a,t} J^{\beta}_{a,t} = J^{\alpha+\beta}_{a,t}, \quad 0 \leq \alpha,\ \beta < 1. \tag{2.2}$$

The *Riemann-Liouville fractional derivative* of order α is defined by

$$\mathcal{D}^{\alpha}_{a,t} f(t) = D_t \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s)ds,\ f \in L^1(I),\ t^{-\alpha} * f \in W^{1,1}(I),\ t > a, \tag{2.3}$$

where $*$ denotes the convolution of functions

$$(f * g)(t) = \int_a^t f(t-\tau) g(\tau) d\tau, \quad t > a,$$

and $W^{1,1}(I)$ is the set of all functions $u \in L^1(I)$ such that the distributional derivative of u exists and belongs to $L^1(I)$. The operator $\mathcal{D}^{\alpha}_{a,t}$ is a left inverse of $J^{\alpha}_{a,t}$, that is

$$\mathcal{D}^{\alpha}_{a,t} J^{\alpha}_{a,t} f(t) = f(t), \quad t > a,$$

but it is not a right inverse, that is

$$J^{\alpha}_{a,t} \mathcal{D}^{\alpha}_{a,t} f(t) = f(t) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} J^{1-\alpha}_{a,t} f(a), \quad t > a.$$

The *Caputo fractional derivative* of order α is defined by

$$D^{\alpha}_{a,t} f(t) = D_t \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} (f(s) - f(0))ds,\ t > a, \tag{2.4}$$

if $f \in L^1(I),\ t^{-\alpha} * f \in W^{1,1}(I)$, or

$$D^{\alpha}_{a,t} f(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} D_s f(s) ds,\ t > a, \tag{2.5}$$

if $f \in W^{1,1}(I)$. The operator $D^{\alpha}_{a,t}$ is also a left inverse of $J^{\alpha}_{a,t}$, that is

$$D^{\alpha}_{a,t} J^{\alpha}_{a,t} f(t) = f(t), \quad t > a, \tag{2.6}$$

but it is not also a right inverse, that is

$$J^{\alpha}_{a,t} D^{\alpha}_{a,t} f(t) = f(t) - f(a), \quad t > a. \tag{2.7}$$

The relation between the Riemann-Liouville and Caputo fractional derivative is

$$D^{\alpha}_{a,t} f(t) = \mathcal{D}^{\alpha}_{a,t} f(t) - \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} f(a), \quad t > a. \tag{2.8}$$

For $a = 0$, we set $J^{\alpha}_{a,t} = J^{\alpha}_t$, $\mathcal{D}^{\alpha}_{a,t} = \mathcal{D}^{\alpha}_t$, and $D^{\alpha}_{a,t} = D^{\alpha}_t$.

We also have the Laplace transform of the Riemann-Liouville and Caputo fractional derivative for $a = 0$. Those are

$$\mathcal{L}(\mathcal{D}^{\alpha}_t f)(\lambda) = \lambda^{\alpha} \mathcal{L}(f)(\lambda) - (J^{1-\alpha}_t) f(0), \tag{2.9}$$

$$\mathcal{L}\left(D^{\alpha}_t f\right)(\lambda) = \lambda^{\alpha} \mathcal{L}(f)(\lambda) - \lambda^{\alpha-1} f(0), \tag{2.10}$$

where the Laplace tranform is defined by

$$\mathcal{L}(f)(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt.$$

Next, we introduce the Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta > 0, \ z \in \mathbb{C}. \tag{2.11}$$

This function is entire and can be also represented in an integral form, that is, for $0 < \alpha < 2$ and an arbitrary complex number β,

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi\alpha i} \int_{\Gamma_{\epsilon,\mu}} \frac{e^{\zeta^{1/\alpha}} \zeta^{(1-\beta)/\alpha}}{\zeta - z} d\zeta, \quad z \in \Gamma^-_{\epsilon,\mu} \tag{2.12}$$

and

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} + \frac{1}{2\pi\alpha i} \int_{\Gamma_{\epsilon,\mu}} \frac{e^{\zeta^{1/\alpha}} \zeta^{(1-\beta)/\alpha}}{\zeta - z} d\zeta, \quad z \in \Gamma^+_{\epsilon,\mu}, \tag{2.13}$$

where $\epsilon > 0$, $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$,

$$\Gamma_{\epsilon,\mu} = \{\lambda \in \mathbb{C} : |\arg(\lambda)| = \mu, |\lambda| \geq \epsilon\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \mu, |\lambda| = \epsilon\},$$

and $\Gamma^-_{\epsilon,\mu}$ ($\Gamma^+_{\epsilon,\mu}$) is the area lying on the left (right) hand side of the contour $\Gamma_{\epsilon,\mu}$. Now, we suppose $r = \epsilon^{1/\alpha}$ and $\omega = \mu/\alpha$. Thus we have $r > 0$ and $\pi/2 < \omega < \pi$. By the transformation $\sigma = \zeta^{1/\alpha}$, we obtain, for $0 < \alpha < 2$ and an arbitrary complex number β,

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} \frac{e^\sigma \sigma^{\alpha-\beta}}{\sigma^\alpha - z} d\sigma, \quad z^{1/\alpha} \in \Gamma^-_{r,\omega} \tag{2.14}$$

and

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} + \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} \frac{e^\sigma \sigma^{\alpha-\beta}}{\sigma^\alpha - z} d\sigma, \quad z^{1/\alpha} \in \Gamma^+_{r,\omega}. \tag{2.15}$$

For $\beta = 1$, we set $E_{\alpha,\beta}(z) = E_\alpha(z)$, and, for $\alpha = \beta = 1$, we get that $E_{\alpha,\beta}(z)$ is nothing but exponential function e^z.

Now, we give the asymptotic formulas for the Mittag-Leffler function. For $0 < \alpha < 2$, β is an arbitrary complex number, and μ is an arbitrary number such that

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\},$$

then, for an arbitrary integer $p \geq 1$, the following expansions hold, those are

$$\begin{gathered} E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} - \sum_{n=1}^{p} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-1-p}), \\ |z| \to \infty, \ |\arg(z)| \leq \mu, \end{gathered} \tag{2.16}$$

and

$$E_{\alpha,\beta}(z) = -\sum_{n=1}^{p} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-1-p}), \quad |z| \to \infty, \ \mu \leq |\arg(z)| \leq \pi. \tag{2.17}$$

Next, we also provide the estimates of the behaviour of the Mittag-Leffler function in different parts of the complex plane, those are, there are real constants D_1, D_2, and D_3 such that

$$|E_{\alpha,\beta}(z)| \leq D_1(1+|z|)^{(1-\beta)/\alpha} e^{\mathrm{Re}(z^{1/\alpha})} + \frac{D_2}{1+|z|}, \quad |z| \geq 0, \ \arg(z)| \leq \mu. \tag{2.18}$$

and

$$|E_{\alpha,\beta}(z)| \leq \frac{D_3}{1+|z|}, \quad |z| \geq 0, \ \mu \leq |\arg(z)| \leq \pi. \tag{2.19}$$

We also have the Laplace transform of Mittag-Leffler function, that is

$$\mathcal{L}\left(t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)\right)(\xi) = \frac{\xi^{\alpha-\beta}}{\xi^\alpha - \lambda}, \quad \mathrm{Re}(\xi) > |\lambda|^{1/\alpha}.$$

Another Properties of Mittag-Leffler function which will be used frequently later are

$$D_t E_\alpha(\lambda t^\alpha) = \lambda t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha), \ t > 0, \tag{2.20}$$

$$D_t^\alpha E_\alpha(\lambda t^\alpha) = \lambda E_\alpha(\lambda t^\alpha), \ t \geq 0, \tag{2.21}$$

$$E_\alpha(\lambda t^\alpha) = J_t^{1-\alpha} t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha), \ t > 0. \tag{2.22}$$

The following well known Proposition show us the application of Mittag-Leffler function to fractional ordinary differential equations.

Proposition 2.1. *Let $\lambda \in \mathbb{C}$ and f be given complex functions defined in $(0,\infty)$. If $v : [0,\infty) \to \mathbb{C}$ is a continuous function solving the fractional ordinary differential equation*

$$\begin{aligned} D_t^\alpha v(t) &= \lambda v(t) + f(t), \quad t > 0, \\ v(0) &= v_0, \end{aligned} \tag{2.23}$$

then it is given uniquely by

$$v(t) = E_\alpha(\lambda t^\alpha)v_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) f(s)ds, \quad t > 0. \tag{2.24}$$

Proof. We use the Laplace tranform method to derive (2.24). Thus the laplace transform of (2.23) is

$$\xi^\alpha \mathcal{L}(v)(\xi) - \xi^{\alpha-1} v_0 = \lambda \mathcal{L}(v)(\xi) + \mathcal{L}(f)(\xi).$$

Then

$$\mathcal{L}(v)(\xi) = \frac{\xi^{\alpha-1}}{\xi^\alpha - \lambda} v_0 + \frac{\xi^{\alpha-\alpha}}{\xi^\alpha - \lambda} \mathcal{L}(f)(\xi), \quad \mathrm{Re}(\xi) > |\lambda|^{1/\alpha}.$$

By the invers of Laplace transform, it follows that

$$v(t) = E_\alpha(\lambda t^\alpha)v_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^\alpha) f(s)ds, \quad t > 0.$$

This representation of v is unique by the uniqueness property of the invers of the Laplace tranform. $\square$

For more details concerning the fractional integrals and derivatives and the Mittag-Lefller function, we refer to Kilbas et al. [3] or Podlubny [8].

3. Solution Operators for Fractional Evolution Equations

In this section, we construct solution operators for the fractional Cauchy problem (1.7) with A which is (almost) sectorial and derive their properties. Note that every (almost) sectorial operator is closed since its resolvent set is not empty. For related results in the case of A satisfying (1.5) and (1.6), one can refer to [10]. The authors in [10] needed a third object, namely the semigroup associated with A, to derive the properties of their solution operators. Here, we do not use it. In the case of $\alpha = 1$ or the evolution equation involving the first order time derivative, Periago et al. [7] studied what is called by the semigroup of growth order $1-\gamma$. Concerning this kind of semigroup, Prato [9] is the one who introduced it for nonnegative integer growth order. Then Okazawa [4] defined it for positive growth order.

If $Au, u, f \in L^1((0,\infty); H)$, by (2.10), we have the Laplace transform of the problem (1.7), that is

$$-\lambda^{\alpha-1}u_0 + \lambda^{\alpha}\mathcal{L}(u)(\lambda) = A\mathcal{L}(u)(\lambda) + \mathcal{L}(f)(\lambda).$$

Then we get

$$(\lambda^{\alpha} - A)\mathcal{L}(u)(\lambda) = \lambda^{\alpha-1}u_0 + \mathcal{L}(f)(\lambda), \tag{3.1}$$

and, by the invers of the Laplace transform, we may obtain

$$u(t) = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}\lambda^{\alpha-1}R(\lambda^{\alpha}; A)u_0 d\lambda + \int_0^t \left(\frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}R(\lambda^{\alpha}; A)d\lambda\right) f(t-\tau)d\tau, \tag{3.2}$$

where Γ is any vertical line $\mathrm{Re}(\lambda) = c$ such that c is greater than all real part of all singularities of the integrand of the integral (3.2). If both integrals of (3.2) exist, we have a solution to the problem (1.7). These motivate in defining solution operators for the problem (1.7).

3.1. Homogeneous Problem.

The first term of the right hand side of (3.2) motivates in defining a solution operator for the homogeneous case of the problem (1.7). Now, we consider the operator

$$\frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^{\alpha-1}R(\lambda^{\alpha}; A)d\lambda,$$

where $r > 0$, $\pi/2 < \omega < \theta$, and

$$\Gamma_{r,\omega} = \{\lambda \in \mathbb{C} : |\arg(\lambda)| = \omega, |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\arg(\lambda)| \leq \omega, |\lambda| = r\}$$

is oriented counterclockwise. Note that $\lambda \mapsto e^{\lambda t}\lambda^{\alpha-1}R(\lambda; A)$ is analytic on Σ_{θ}. Furthermore, for each $\lambda \in \Gamma_{r,\omega}$, it holds that

$$\|e^{\lambda t}\lambda^{\alpha-1}R(\lambda^{\alpha}; A)\| \leq M|\lambda|^{\alpha(1-\gamma)-1}e^{t|\lambda|\cos\arg(\lambda)}.$$

Therefore, for $t > 0$, we have

$$\left\|\int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^{\alpha-1}R(\lambda; A)d\lambda\right\| \leq 2M\int_r^{\infty} \rho^{\alpha(1-\gamma)-1}e^{t\rho\cos\omega}d\rho + Mr^{\alpha(1-\gamma)}\int_{-\omega}^{\omega} e^{tr\cos\varphi}d\varphi.$$

Thus we can define

$$S_{\alpha}(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^{\alpha-1}R(\lambda^{\alpha}; A)d\lambda, \quad t > 0. \tag{3.3}$$

By the Cauchy's theorem, the integral form (3.3) is independent of $r > 0$ and $\omega \in (\pi/2, \theta)$.

The properties of the family $\{S_\alpha(t)\}_{t>0}$ are given in the following theorem.

Theorem 3.1. *Let A be an (almost) sectorial operator and $S_\alpha(t)$ be an operator defined by* (3.3). *Then the following statements hold.*

(i) *$S_\alpha(t) \in B(H)$ and there exists $C_1 = C_1(\alpha,\gamma) > 0$ such that*

$$\|S_\alpha(t)\| \le C_1 t^{-\alpha(1-\gamma)}, \quad t > 0,$$

(ii) *$S_\alpha(t) \in B(H; D(A))$ for $t > 0$, and if $x \in D(A)$ then $AS_\alpha(t)x = S_\alpha(t)Ax$. Moreover, there exists $C_2 = C_2(\alpha,\gamma) > 0$ such that*

$$\|AS_\alpha(t)\| \le C_2 t^{-\alpha}(t^{-\alpha(1-\gamma)} + 1), \quad t > 0,$$

(iii) *The function $t \mapsto S_\alpha(t)$ belongs to $C^\infty((0,\infty); B(H))$ and it holds that*

$$S_\alpha^{(n)}(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{t\lambda}\lambda^{\alpha+n-1}R(\lambda^\alpha; A)d\lambda,\ n = 1,2,\dots$$

and there exist $M_n = M_n(\alpha,\gamma) > 0, n = 1,2,\dots$ such that

$$\|S_\alpha^{(n)}(t)\| \le M_n t^{-\alpha(1-\gamma)-n}, \quad t > 0.$$

Moreover, it has an analytic continuation $S_\alpha(z)$ to the sector $\Sigma_{\theta-\pi/2}$ and, for $z \in \Sigma_{\theta-\pi/2}$, $\eta \in (\pi/2,\theta)$, it holds that

$$S_\alpha(z) = \frac{1}{2\pi i}\int_{\Gamma_{r,\eta}} e^{\lambda z}\lambda^{\alpha-1}R(\lambda^\alpha; A)d\lambda,$$

(iv) *For $x \in H$,*

$$S_\alpha(t)S_\alpha(s)x = S_\alpha(s)S_\alpha(t)x, \quad s,t > 0.$$

Proof.

(i) We suppose $\cos\omega = -a$ for some $0 < a < 1$. Then, for $t > 0$,

$$\begin{aligned}
&\left\|\int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha; A)d\lambda\right\| = \left\|\int_{\Gamma_{t^{-1},\omega}} e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha; A)d\lambda\right\| \\
&\le 2M\int_{t^{-1}}^\infty e^{-ta\rho}\rho^{\alpha(1-\gamma)-1}d\rho + M\int_{-\omega}^{\omega} e^{\cos\varphi}t^{-\alpha(1-\gamma)}d\varphi \\
&= 2Mt^{-\alpha(1-\gamma)}a^{-\alpha(1-\gamma)}\int_a^\infty e^{-u}u^{\alpha(1-\gamma)-1}du \\
&\quad + Mt^{-\alpha(1-\gamma)}\int_{-\omega}^{\omega} e^{\cos\varphi}d\varphi.
\end{aligned}$$

It means that $S_\alpha(t) \in B(H)$ and there exists $C_1 = C_1(\alpha,\gamma) > 0$ such that

$$\|S_\alpha(t)\| \le C_1 t^{-\alpha(1-\gamma)}, \quad t > 0.$$

(ii) By the definiton of resolvent operator, for each $\lambda \in \Gamma_{r,\omega}$, $R(\lambda^\alpha; A) : D(R(\lambda^\alpha; A)) \subset H \to D(A)$ is bounded and $D(R(\lambda^\alpha; A))$ is dense in H. This and the closedness of A imply $R(\lambda^\alpha; A)x \in D(A)$ for each $x \in H$. Therefore $e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha; A)x \in D(A)$. Moreover, $\lambda \mapsto e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha; A)x$ is integrable along $\Gamma_{r,\omega}$.

Now, we consider the function $\lambda \mapsto e^{\lambda t}\lambda^{\alpha-1}AR(\lambda^\alpha;A)x$, $x \in H$. Since $AR(\lambda;A) = \lambda R(\lambda;A) - I$, we have

$$\int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^{\alpha-1}AR(\lambda^\alpha;A)xd\lambda = \int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^{2\alpha-1}R(\lambda^\alpha;A)xd\lambda - \int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^{\alpha-1}xd\lambda.$$

Then, for $t > 0$, we get

$$\begin{aligned}&\left\|\int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^{2\alpha-1}R(\lambda^\alpha;A)d\lambda\right\| = \left\|\int_{\Gamma_{t^{-1},\omega}} e^{\lambda t}\lambda^{2\alpha-1}R(\lambda^\alpha;A)d\lambda\right\|\\ &\leq 2Mt^{-\alpha(2-\gamma)}a^{-\alpha(2-\gamma)}\int_a^\infty e^{-u}u^{\alpha(2-\gamma)-1}du\\ &\quad + Mt^{-\alpha(2-\gamma)}\int_{-\omega}^{\omega} e^{\cos\varphi}d\varphi\end{aligned}$$

and

$$\frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^{\alpha-1}d\lambda = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}.$$

It means that $\lambda \mapsto e^{\lambda t}\lambda^{\alpha-1}AR(\lambda^\alpha;A)x$ is integrable along $\Gamma_{r,\omega}$ for each $x \in H$. Therefore, since A is closed, we find that, for each $x \in H$,

$$S_\alpha(t)x = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^{\alpha-1}R(\lambda^\alpha;A)xd\lambda \in D(A),$$
$$AS_\alpha(t)x = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^{\alpha-1}AR(\lambda^\alpha;A)xd\lambda.$$

Moreover, there exists $C_2 = C_2(\alpha,\gamma) > 0$ such that

$$\|AS_\alpha(t)\| \leq C_2 t^{-\alpha}(t^{-\alpha(1-\gamma)} + 1), \quad t > 0.$$

Thus we obtain $S_\alpha(t) \in B(H;D(A))$. Since $AR(\lambda;A) = R(\lambda;A)A$, we also have that, for each $x \in D(A)$, $AS_\alpha(t)x = S_\alpha(t)Ax$.

(iii) Now, observe that, for any $\lambda \in \mathbb{C}$ and $t, h > 0$ such that $t < t+h < 2t$ and $\mathrm{Re}(\lambda) \leq 0$, we have

$$|e^{(t+h)\lambda} - e^{t\lambda}| = |e^{t\lambda}(e^{h\lambda}-1)| = e^{t\mathrm{Re}(\lambda)}\left|\int_0^h \lambda e^{\lambda s}ds\right| \leq e^{t\mathrm{Re}(\lambda)}|\lambda|h.$$

If $\mathrm{Re}(\lambda) \geq 0$, we obtain

$$|e^{(t+h)\lambda} - e^{t\lambda}| \leq e^{2t\mathrm{Re}(\lambda)}|\lambda|h.$$

Similarly, for any $\lambda \in \mathbb{C}$ and $h < 0$ such that $0 < t/2 < t+h < t$ and $\mathrm{Re}(\lambda) \leq 0$, we find

$$|e^{(t+h)\lambda} - e^{t\lambda}| \leq e^{\frac{t}{2}\mathrm{Re}(\lambda)}|\lambda||h|.$$

If $\mathrm{Re}(\lambda) \geq 0$, we get

$$|e^{(t+h)\lambda} - e^{t\lambda}| \leq e^{t\mathrm{Re}(\lambda)}|\lambda||h|.$$

Thus, for $h > 0$ and $t < t + h < 2t$,

$$\left|\frac{e^{(t+h)\lambda} - e^{t\lambda}}{h}\right| \le f(t,\lambda) = \begin{cases} e^{t\mathrm{Re}(\lambda)}|\lambda|, & \mathrm{Re}(\lambda) \le 0, \\ e^{2t\mathrm{Re}(\lambda)}|\lambda|, & \mathrm{Re}(\lambda) \ge 0, \end{cases} \tag{3.4}$$

and, for $h < 0$ and $t/2 < t + h < t$,

$$\left|\frac{e^{(t+h)\lambda} - e^{t\lambda}}{h}\right| \le g(t,\lambda) = \begin{cases} e^{\frac{t}{2}\mathrm{Re}(\lambda)}|\lambda|, & \mathrm{Re}(\lambda) \le 0, \\ e^{t\mathrm{Re}(\lambda)}|\lambda|, & \mathrm{Re}(\lambda) \ge 0. \end{cases} \tag{3.5}$$

Therefore

$$\frac{S_\alpha(t+h) - S_\alpha(t)}{h} = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} \frac{e^{(t+h)\lambda} - e^{t\lambda}}{h}\lambda^{\alpha-1}R(\lambda^\alpha; A)d\lambda. \tag{3.6}$$

Next, note that $\lambda \mapsto f(t,\lambda)|\lambda|^{\alpha(1-\gamma)-1}$ and $\lambda \mapsto g(t,\lambda)|\lambda|^{\alpha(1-\gamma)-1}$ are integrable along $\Gamma_{r,\omega}$. Then, by taking a limit as $h \to 0$ and applying the Dominated Convergence Theorem to (3.6), we obtain

$$S'_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{t\lambda}\lambda^\alpha R(\lambda^\alpha; A)d\lambda, \quad t > 0.$$

It means $S_\alpha(t)$ is differentiable at $t \in (0,\infty)$. By using the similar procedure as above, we can deduce that $S'_\alpha(t)$ is also differentiable at $t \in (0,\infty)$. By induction, we can obtain that $t \mapsto S_\alpha(t)$ belongs to $C^\infty((0,\infty); B(H))$ and

$$S_\alpha^{(n)}(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{t\lambda}\lambda^{\alpha+n-1}R(\lambda^\alpha; A)d\lambda, \quad t > 0.$$

Moreover, for $t > 0$,

$$\begin{aligned}
&\left\|\int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^{\alpha+n-1}R(\lambda^\alpha; A)d\lambda\right\| = \left\|\int_{\Gamma_{t^{-1},\omega}} e^{\lambda t}\lambda^{\alpha+n-1}R(\lambda^\alpha; A)d\lambda\right\| \\
&\le 2M\int_{t^{-1}}^\infty e^{-ta\rho}\rho^{\alpha(1-\gamma)+n-1}d\rho + M\int_{-\omega}^{\omega} e^{\cos\varphi}t^{-\alpha(1-\gamma)-n}d\varphi \\
&= 2Mt^{-\alpha(1-\gamma)-n}a^{-\alpha(1-\gamma)-n}\int_a^\infty e^{-u}u^{\alpha(1-\gamma)+n-1}du \\
&\quad + Mt^{-\alpha(1-\gamma)-n}\int_{-\omega}^{\omega} e^{\cos\varphi}d\varphi.
\end{aligned}$$

Hence there exist $M_n = M_n(\alpha,\gamma) > 0$, $n = 1, 2, \ldots$ such that

$$\|S_\alpha^{(n)}(t)\| \le M_n t^{-\alpha(1-\gamma)-n}, \quad t > 0.$$

Now, let $0 < \delta < \theta - \pi/2$ and $\eta = \theta - \delta$. We suppose $z \in \Sigma_{\eta-\pi/2}$ and $\lambda = |\lambda|e^{\pm\eta i}$, $|\lambda| \ge r$. Then $z\lambda = |z||\lambda|e^{i(\arg(z)\pm\eta)}$ with $\pi/2 < \arg(z) + \eta <$

$3\pi/2$ and $-3\pi/2 < \arg(z) - \eta < -\pi/2$. This means that $\mathrm{Re}(z\lambda) < 0$. Then

$$\begin{aligned}\|S_\alpha(z)\| &= \left\| \frac{1}{2\pi i} \int_{\Gamma_{r,\eta}} e^{z\lambda}\lambda^{\alpha-1}R(\lambda^\alpha; A)d\lambda \right\| \\ &= \left\| \frac{1}{2\pi i} \int_{\Gamma_{|z|^{-1},\eta}} e^{z\lambda}\lambda^{\alpha-1}R(\lambda^\alpha; A)d\lambda \right\| \\ &\le \frac{|z|^{-\alpha(1-\gamma)}}{2\pi} \int_{\Gamma_{1,\eta}} e^{\mathrm{Re}(\rho)}|\rho|^{\alpha(1-\gamma)-1}|d\rho|.\end{aligned}$$

Thus, for $z \in \Sigma_{\eta-\pi/2}$, the function

$$z \mapsto S_\alpha(z) = \frac{1}{2\pi i} \int_{\Gamma_{r,\eta}} e^{z\lambda}\lambda^{\alpha-1}R(\lambda^\alpha; A)d\lambda.$$

is bounded. Furthermore, the function is analytic in the sector $\Sigma_{\eta-\pi/2}$. Also observe that the union of the sector $\Sigma_{\eta-\pi/2}$ is $\Sigma_{\theta-\pi/2}$. This completes the proof.

(iv) First, note that, for each $\mu \in \Gamma_{r,\omega}$, the function $\lambda \mapsto (\mu - \lambda)^{-1}$ is analytic in $\Sigma_\omega \setminus \{\lambda \in \mathbb{C} : 0 < |\lambda| \le r, -\omega < \arg(\lambda) < \omega\}$. Therefore, for each $\lambda \in \Sigma_\omega \setminus \{\lambda \in \mathbb{C} : 0 < |\lambda| \le r, -\omega < \arg(\lambda) < \omega\}$, we can define a bounded operator

$$(\lambda - A)^{-1} = \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} (\lambda - \mu)^{-1}R(\mu; A)d\mu. \tag{3.7}$$

Next, we select $r' > r^{1/\alpha}$ and $\pi/2 < \omega' < \theta$ such that $\alpha\omega' < \omega$. Thus, by (2.14) and (3.7), we get

$$\begin{aligned}S_\alpha(t) &= \frac{1}{2\pi i} \int_{\Gamma_{r',\omega'}} e^{t\lambda}\lambda^{\alpha-1}R(\lambda^\alpha; A)d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_{r',\omega'}} e^{t\lambda}\lambda^{\alpha-1} \left(\frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} (\lambda^\alpha - \mu)^{-1}R(\mu; A)d\mu \right) d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} \left(\frac{1}{2\pi i} \int_{\Gamma_{r't,\omega'}} e^\rho \rho^{\alpha-1}(\rho^\alpha - \mu t^\alpha)^{-1} d\rho \right) R(\mu; A)d\mu \\ &= \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} E_\alpha(\mu t^\alpha)R(\mu; A)d\mu.\end{aligned} \tag{3.8}$$

Now, we suppose $0 < r < r'$ and $\pi/2 < \omega' < \omega$. Consider that

$$\begin{aligned}S_\alpha(t)S_\alpha(s) &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_{r,\omega}} E_\alpha(\lambda t^\alpha)R(\lambda; A)d\lambda \int_{\Gamma_{r',\omega'}} E_\alpha(\mu s^\alpha)R(\mu; A)d\mu \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_{r,\omega}} \int_{\Gamma_{r',\omega'}} E_\alpha(\lambda t^\alpha)E_\alpha(\mu s^\alpha) \frac{R(\lambda; A) - R(\mu; A)}{\mu - \lambda} d\lambda d\mu \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_{r,\omega}} E_\alpha(\lambda t^\alpha)R(\lambda; A)d\lambda \int_{\Gamma_{r',\omega'}} \frac{E_\alpha(\mu s^\alpha)}{\mu - \lambda} d\mu \\ &\quad - \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_{r',\omega'}} E_\alpha(\mu s^\alpha)R(\mu; A)d\mu \int_{\Gamma_{r,\omega}} \frac{E_\alpha(\lambda t^\alpha)}{\mu - \lambda} d\lambda.\end{aligned} \tag{3.9}$$

By (2.19), the Mittag-Leffler function $E_{\alpha,\beta}(z)$ tends to 0 in the sector $\{z \in \mathbb{C} : \omega \leq \arg(z) \leq 2\pi - \omega\}$ as $|z| \to \infty$. Therefore, by the Cauchy's theorem, we can find that

$$\int_{\Gamma_{r,\omega}} \frac{E_\alpha(\lambda t^\alpha)}{\mu - \lambda} d\lambda = 0, \quad \mu \in \Gamma_{r',\omega'}.$$

By the similar argument, we can also obtain that

$$\int_{\Gamma_{r',\omega'}} \frac{E_\alpha(\mu s^\alpha)}{\mu - \lambda} d\mu = 2\pi i E_\alpha(\lambda s^\alpha), \quad \lambda \in \Gamma_{r,\omega}.$$

Then (3.9) is now reduced to

$$S_\alpha(t) S_\alpha(s) = \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} E_\alpha(\lambda t^\alpha) E_\alpha(\lambda s^\alpha) R(\lambda; A) d\lambda. \tag{3.10}$$

Hence $S_\alpha(t)S_\alpha(s) = S_\alpha(s)S_\alpha(t)$ for $s, t > 0$.

□

3.2. **Inhomogeneous Problem.** The second term of the right hand side of (3.2) motivates in defining another solution operator, besides the solution operator (3.3), for the inhomogeneous case of the problem (1.7). Thus our concern now is to consider the operator

$$\frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} e^{\lambda t} R(\lambda^\alpha; A) d\lambda.$$

Note that, for each $\lambda \in \Gamma_{r,\omega}$, we have $\|e^{\lambda t} R(\lambda^\alpha; A)\| \leq M|\lambda|^{-\alpha\gamma} e^{t|\lambda| \cos \arg(\lambda)}$. Then, for all $t > 0$, we get

$$\left\| \int_{\Gamma_{r,\omega}} e^{\lambda t} R(\lambda; A) d\lambda \right\| \leq 2M \int_r^\infty \rho^{-\alpha\gamma} e^{t\rho \cos \omega} d\rho + M r^{-\alpha\gamma+1} \int_{-\omega}^{\omega} e^{tr \cos \varphi} d\varphi.$$

Thus we can also define

$$P_\alpha(t) = \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} e^{\lambda t} R(\lambda^\alpha; A) d\lambda, \quad t > 0. \tag{3.11}$$

By Cauchy's Theorem again, the integral form of (3.11) does not depend on the choice of $r > 0$ and $\omega \in (\pi/2, \theta)$. The properties of the family $\{P_\alpha(t)\}_{t>0}$ are given in the following theorem.

Theorem 3.2. *Let A be an (almost) sectorial operator and $P_\alpha(t)$ be an operator defined by (3.11). Then the following statements hold.*

(i) *$P_\alpha(t) \in B(H)$ and there exists $L_1 = L_1(\alpha, \gamma) > 0$ such that*

$$\|P_\alpha(t)\| \leq L_1 t^{\alpha\gamma - 1}, \quad t > 0,$$

(ii) *$P_\alpha(t) \in B(H; D(A))$ for all $t > 0$, and if $x \in D(A)$ then $AP_\alpha(t)x = P_\alpha(t)Ax$. Moreover, there exists $L_2 = L_2(\alpha, \gamma) > 0$ such that*

$$\|AP_\alpha(t)\| \leq L_2 t^{\alpha(\gamma-1)-1}, \quad t > 0,$$

(iii) *The function $t \mapsto P_\alpha(t)$ belongs to $C^\infty((0, \infty); B(H))$ and it holds that*

$$P_\alpha^{(n)}(t) = \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} e^{t\lambda} \lambda^n R(\lambda^\alpha; A) d\lambda, \ n = 1, 2, \ldots$$

and there exist $K_n = K_n(\alpha, \gamma) > 0, n = 1, 2, \dots$ such that

$$\|P_\alpha^{(n)}(t)\| \le K_n t^{\alpha\gamma - n - 1}, \quad t > 0.$$

Moreover, it has an analytic continuation $P_\alpha(z)$ to the sector $\Sigma_{\theta-\pi/2}$ and, for $z \in \Sigma_{\theta-\pi/2}$, $\eta \in (\pi/2, \theta)$, it holds that

$$P_\alpha(z) = \frac{1}{2\pi i} \int_{\Gamma_{r,\eta}} e^{\lambda z} R(\lambda^\alpha; A) d\lambda,$$

(iv) *For $x \in H$,*

$$P_\alpha(s) P_\alpha(t) x = P_\alpha(t) P_\alpha(s) x, \quad s, t > 0.$$

Proof.

(i) We suppose again $\cos\omega = -a$ for some $0 < a < 1$. Then, for $t > 0$,

$$\begin{aligned}
&\left\| \int_{\Gamma_{r,\omega}} e^{\lambda t} R(\lambda^\alpha; A) d\lambda \right\| = \left\| \int_{\Gamma_{t^{-1},\omega}} e^{\lambda t} R(\lambda^\alpha; A) d\lambda \right\| \\
&\le 2M \int_{t^{-1}}^\infty e^{-ta\rho} \rho^{-\alpha\gamma} d\rho + M r^{-\alpha\gamma+1} \int_{-\omega}^{\omega} e^{tr\cos\varphi} d\varphi \\
&= 2M t^{\alpha\gamma-1} a^{\alpha\gamma-1} \int_a^\infty e^{-u} u^{-\alpha\gamma} du + M t^{\alpha\gamma-1} \int_{-\omega}^{\omega} e^{\cos\varphi} d\varphi.
\end{aligned}$$

Therefore we can conclude that $P_\alpha(t) \in B(H)$ and there exists $L_1 = L_1(\alpha, \gamma) > 0$ such that

$$\|P_\alpha(t)\| \le L_1 t^{\alpha\gamma-1}, \quad t > 0.$$

(ii) We have that, for each $x \in H$, $e^{\lambda t} R(\lambda^\alpha; A) x \in D(A)$ and $\lambda \mapsto e^{\lambda t} R(\lambda^\alpha; A) x$ is integrable along $\Gamma_{r,\omega}$. Next, we consider the function $\lambda \mapsto e^{\lambda t} A R(\lambda^\alpha; A) x$, $x \in H$. Since $AR(\lambda; A) = \lambda R(\lambda; A) - I$ and

$$\int_{\Gamma_{r,\omega}} e^{\lambda t} d\lambda = 0,$$

we get

$$\begin{aligned}
\int_{\Gamma_{r,\omega}} e^{\lambda t} A R(\lambda^\alpha; A) x d\lambda &= \int_{\Gamma_{r,\omega}} e^{\lambda t} \lambda^\alpha R(\lambda^\alpha; A) x d\lambda - \int_{\Gamma_{r,\omega}} e^{\lambda t} x d\lambda \\
&= \int_{\Gamma_{r,\omega}} e^{\lambda t} \lambda^\alpha R(\lambda^\alpha; A) x d\lambda.
\end{aligned}$$

Then, for $t > 0$, we obtain

$$\begin{aligned}
&\left\| \int_{\Gamma_{r,\omega}} e^{\lambda t} \lambda^\alpha R(\lambda^\alpha; A) d\lambda \right\| = \left\| \int_{\Gamma_{t^{-1},\omega}} e^{\lambda t} \lambda^\alpha R(\lambda^\alpha; A) d\lambda \right\| \\
&\le 2M t^{-\alpha(1-\gamma)-1} a^{-\alpha(1-\gamma)-1} \int_a^\infty e^{-u} u^{\alpha(1-\gamma)} du \\
&\quad + M t^{-\alpha(1-\gamma)-1} \int_{-\omega}^{\omega} e^{\cos\varphi} d\varphi.
\end{aligned}$$

It means that $\lambda \mapsto e^{\lambda t} AR(\lambda^\alpha; A)x$ is integrable along $\Gamma_{r,\omega}$. Therefore, since A is closed, we get that, for each $x \in H$,

$$P_\alpha(t)x = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda t} R(\lambda^\alpha; A)x d\lambda \in D(A),$$
$$AP_\alpha(t)x = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda t} AR(\lambda^\alpha; A)x d\lambda.$$

Moreover, there exists $L_2 = L_2(\alpha,\gamma) > 0$ such that

$$\|AP_\alpha(t)\| \le L_2 t^{-\alpha(1-\gamma)-1}, \quad t > 0.$$

Therefore $P_\alpha(t) \in B(H; D(A))$. Since $AR(\lambda; A) = R(\lambda; A)A$, we also get that, for each $x \in D(A)$, $AP_\alpha(t)x = P_\alpha(t)Ax$.

(iii) From the proof of theorem 3.1(iii), we have that, for $h > 0$ and $t < t+h < 2t$,

$$\left|\frac{e^{(t+h)\lambda} - e^{t\lambda}}{h}\right| \le f(t,\lambda) = \begin{cases} e^{t\mathrm{Re}\lambda}|\lambda|, & \mathrm{Re}\lambda \le 0,\\ e^{2t\mathrm{Re}\lambda}|\lambda|, & \mathrm{Re}\lambda \ge 0,\end{cases}$$

and, for $h < 0$ and $t/2 < t+h < t$,

$$\left|\frac{e^{(t+h)\lambda} - e^{t\lambda}}{h}\right| \le g(t,\lambda) = \begin{cases} e^{\frac{t}{2}\mathrm{Re}\lambda}|\lambda|, & \mathrm{Re}\lambda \le 0,\\ e^{t\mathrm{Re}\lambda}|\lambda|, & \mathrm{Re}\lambda \ge 0.\end{cases}$$

Now, we consider

$$\frac{P_\alpha(t+h) - P_\alpha(t)}{h} = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} \frac{e^{(t+h)\lambda} - e^{t\lambda}}{h} R(\lambda^\alpha; A)d\lambda. \tag{3.12}$$

Note that $\lambda \mapsto f(t,\lambda)|\lambda|^{-\alpha\gamma}$ and $\lambda \mapsto g(t,\lambda)|\lambda|^{-\alpha\gamma}$ are integrable along $\Gamma_{r,\omega}$ for $t > 0$. Then, by taking a limit as $h \to 0$ and applying the Dominated Convergence Theorem to (3.12), we obtain

$$P'_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{t\lambda}\lambda R(\lambda^\alpha; A)d\lambda, \quad t > 0.$$

It means $P_\alpha(t)$ is differentiable at $t \in (0,\infty)$. Similarly, we can deduce that $P'_\alpha(t)$ is also differentiable at $t \in (0,\infty)$. By induction, we can obtain that $t \mapsto P_\alpha(t)$ belongs to $C^\infty((0,\infty); B(H))$ and

$$P_\alpha^{(n)}(t) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{t\lambda}\lambda^n R(\lambda^\alpha; A)d\lambda, \quad t > 0.$$

Moreover, for $t > 0$,

$$\left\|\int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^n R(\lambda^\alpha; A)d\lambda\right\| = \left\|\int_{\Gamma_{t^{-1},\omega}} e^{\lambda t}\lambda^n R(\lambda^\alpha; A)d\lambda\right\|$$
$$\le 2M\int_{t^{-1}}^\infty e^{-ta\rho}\rho^{-\alpha\gamma+n}d\rho + M\int_{-\omega}^{\omega} e^{\cos\varphi} t^{\alpha\gamma-n-1}d\varphi$$
$$= 2Mt^{\alpha\gamma-n-1}a^{\alpha\gamma-n-1}\int_a^\infty e^{-u}u^{-\alpha\gamma+n}du + Mt^{\alpha\gamma-n-1}\int_{-\omega}^{\omega} e^{\cos\varphi}d\varphi.$$

Hence there exist $K_n = K_n(\alpha,\gamma) > 0$, $n = 1, 2, \ldots$ such that

$$\|P_\alpha^{(n)}(t)\| \le K_n t^{\alpha\gamma-n-1}, \quad t > 0.$$

Furthermore, it has an analytic continuation $P_\alpha(z)$ to the sector $\Sigma_{\theta-\pi/2}$, that is, for $z \in \Sigma_{\theta-\pi/2}$, $\eta \in (\pi/2, \theta)$, we can find that

$$P_\alpha(z) = \frac{1}{2\pi i}\int_{\Gamma_{r,\eta}} e^{\lambda z} R(\lambda^\alpha; A)d\lambda$$

(see the proof of Theorem 3.1(iii)).

(iv) As in the proof of Theorem 3.1(iv), we select $r' > r^{1/\alpha}$ and $\pi/2 < \omega' < \theta$ such that $\alpha\omega' < \omega$. Then we have, for $x \in H$,

$$\begin{aligned} P_\alpha(t)x &= \frac{1}{2\pi i}\int_{\Gamma_{r',\omega'}} e^{t\lambda} R(\lambda^\alpha; A)xd\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_{r',\omega'}} e^{t\lambda}\left(\frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} (\lambda^\alpha-\mu)^{-1}R(\mu; A)d\mu\right)xd\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}}\left(\frac{t^{\alpha-1}}{2\pi i}\int_{\Gamma_{r't,\omega'}} e^\rho \rho^{\alpha-\alpha}(\rho^\alpha-\mu t^\alpha)^{-1}d\rho\right)R(\mu; A)xd\mu \\ &= \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} t^{\alpha-1}E_{\alpha,\alpha}(\mu t^\alpha)R(\mu; A)xd\mu. \end{aligned} \tag{3.13}$$

By using the similar way as used in the proof of Theorem 3.1(iv), we can find that

$$P_\alpha(t)P_\alpha(s) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} (ts)^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)E_{\alpha,\alpha}(\lambda s^\alpha)R(\lambda; A)d\lambda \tag{3.14}$$

and hence $P_\alpha(t)P_\alpha(s) = P_\alpha(s)P_\alpha(t)$ for $s, t > 0$.

□

The following theorem states some identities concerning the operators $S_\alpha(t)$ and $P_\alpha(t)$ including our new semigroup-like property.

Theorem 3.3. *Let A be an (almost) sectorial operator, $S_\alpha(t)$ and $P_\alpha(t)$ be operators defined by* (3.3) *and* (3.11)*, respectively. Then the following statements hold.*

(i) *For $x \in H$ and $t > 0$,*

$$S_\alpha(t)x = J_t^{1-\alpha}P_\alpha(t)x, \quad D_tS_\alpha(t)x = AP_\alpha(t)x,$$

(ii) *For $x \in D(A)$ and $s, t > 0$,*

$$D_t^\alpha S_\alpha(t)x = AS_\alpha(t)x,$$
$$S_\alpha(t+s)x = S_\alpha(t)S_\alpha(s)x - A\int_0^t\int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)}P_\alpha(\tau)P_\alpha(r)xdrd\tau.$$

Proof.

(i) By using (2.22), (3.8), and (3.13), we get, for $x \in H$,

$$\begin{aligned} S_\alpha(t)x &= \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} E_\alpha(\lambda t^\alpha)R(\lambda;A)xd\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} J_t^{1-\alpha}\left(t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)\right)R(\lambda;A)xd\lambda \\ &= J_t^{1-\alpha}\left(\frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)R(\lambda;A)xd\lambda\right) \\ &= J_t^{1-\alpha}P_\alpha(t)x, \quad t>0. \end{aligned}$$

Next, by Theorem 3.1(iii), the identity $AR(\lambda;A) = \lambda R(\lambda;A) - I$, and the equation

$$\int_{\Gamma_{r,\omega}} e^{\lambda t}d\lambda = 0,$$

we find, for $x \in H$,

$$\begin{aligned} D_tS_\alpha(t)x &= \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda t}\lambda^\alpha R(\lambda^\alpha;A)xd\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda t}AR(\lambda^\alpha;A)xd\lambda + \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} e^{\lambda t}xd\lambda \\ &= AP_\alpha(t)x, \quad t>0. \end{aligned}$$

(ii) We prove the first identity of this part. Observe that, by Theorem 3.2(i), for $x \in H$ and $s,t>0$,

$$\begin{aligned} \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\|P_\alpha(\tau)x\|d\tau &\le L_1(\alpha,\gamma)\int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\tau^{\alpha\gamma-1}d\tau\|x\| \\ &= L_1(\alpha,\gamma)t^{-\alpha+\alpha\gamma}\int_0^1 \frac{(1-s)^{-\alpha}}{\Gamma(1-\alpha)}s^{\alpha\gamma-1}ds\|x\| \\ &= \frac{L_1(\alpha,\gamma)B(\alpha\gamma,1-\alpha)}{\Gamma(1-\alpha)}t^{-\alpha+\alpha\gamma}\|x\| \end{aligned}$$

and, by Theorem 3.2(i) and 3.2(ii), for $x \in D(A)$ and $s,t>0$,

$$\begin{aligned} \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\|AP_\alpha(\tau)x\|d\tau &= \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\|P_\alpha(\tau)Ax\|d\tau \\ &\le L_1(\alpha,\gamma)\int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}\tau^{\alpha\gamma-1}d\tau\|Ax\| \\ &= L_1(\alpha,\gamma)t^{-\alpha+\alpha\gamma}\int_0^1 \frac{(1-s)^{-\alpha}}{\Gamma(1-\alpha)}s^{\alpha\gamma-1}ds\|Ax\| \\ &= \frac{L_1(\alpha,\gamma)B(\alpha\gamma,1-\alpha)}{\Gamma(1-\alpha)}t^{-\alpha+\alpha\gamma}\|Ax\|, \end{aligned}$$

where

$$B(a,b) = \int_0^1 s^{a-1}(1-s)^{b-1}ds, \quad a,b>0,$$

is the Beta Function. Then, by the closedness of A, for $x \in D(A)$,

$$A\int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}P_\alpha(\tau)xd\tau = \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)}AP_\alpha(\tau)xd\tau.$$

Thus, by the identities in part (i), for $x \in D(A)$,

$$\begin{aligned} AS_\alpha(t)x &= AJ_t^{1-\alpha}P_\alpha(t)x \\ &= \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} AP_\alpha(\tau)xd\tau \\ &= \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} D_\tau S_\alpha(\tau)xd\tau \\ &= D_t^\alpha S_\alpha(t)x, \quad t>0. \end{aligned}$$

The first identity in this part is shown.

Now, we prove the second identity of this part, the semigroup-like property. The first step, we show that the complex function F defined by

$$F(t) = \int_0^s \frac{(t+s-r)^{-\alpha}}{\Gamma(1-\alpha)} D_r E_\alpha(\lambda r^\alpha)dr$$

is defined in $(0,\infty)$ for any $\lambda \in \mathbb{C}$ with $s>0$. Let $\pi\alpha/2 < \phi < \min\{\pi, \pi\alpha\}$. Note that, by (2.19) and (2.20), for $\lambda \in \mathbb{C}$ with $\phi \le |\arg(\lambda)| \le \pi$,

$$\begin{aligned} |F(t)| &\le \frac{|\lambda|}{\Gamma(1-\alpha)} \int_0^s (t+s-r)^{-\alpha} r^{\alpha-1} |E_{\alpha,\alpha}(\lambda r^\alpha)|dr \\ &\le \frac{|\lambda|D_3}{\Gamma(1-\alpha)} \int_0^s (t+s-r)^{-\alpha} r^{\alpha-1}(1+|\lambda|r^\alpha)^{-1}dr \\ &\le \frac{|\lambda|D_3}{\Gamma(1-\alpha)} \int_0^s (s-r)^{-\alpha} r^{\alpha-1}dr \\ &= \frac{|\lambda|D_3 B(\alpha,1-\alpha)}{\Gamma(1-\alpha)}, \quad t>0, \end{aligned}$$

and, by (2.18) and (2.20), for $\lambda \in \mathbb{C}$ with $|\arg(\lambda)| \le \phi$,

$$\begin{aligned} |F(t)| \le& \frac{|\lambda|D_2 B(\alpha,1-\alpha)}{\Gamma(1-\alpha)} \\ &+ \frac{|\lambda|D_1}{\Gamma(1-\alpha)} \int_0^s (t+s-r)^{-\alpha} r^{\alpha-1}(1+|\lambda|r^\alpha)^{\frac{1-\alpha}{\alpha}} e^{r\mathrm{Re}(\lambda^{1/\alpha})}dr \\ \le& \frac{|\lambda|D_2 B(\alpha,1-\alpha)}{\Gamma(1-\alpha)} \\ &+ \frac{|\lambda|(1+|\lambda|s^\alpha)^{\frac{1-\alpha}{\alpha}} D_1}{\Gamma(1-\alpha)} \int_0^s (s-r)^{-\alpha} r^{\alpha-1} e^{r\mathrm{Re}(\lambda^{1/\alpha})}dr \\ =& \frac{|\lambda|D_2 B(\alpha,1-\alpha)}{\Gamma(1-\alpha)} + \frac{|\lambda|(1+|\lambda|s^\alpha)^{\frac{1-\alpha}{\alpha}} D_1 D_4}{\Gamma(1-\alpha)}, \quad t>0, \end{aligned}$$

where

$$D_4 = \int_0^1 (1-\rho)^{-\alpha} \rho^{\alpha-1} e^{s\rho\mathrm{Re}(\lambda^{1/\alpha})}d\rho.$$

It means that F is defined in $(0,\infty)$ for any $\lambda \in \mathbb{C}$.

The next step, we prove that $E_\alpha(\lambda(t+s)^\alpha)$ solves the problem

$$\begin{aligned} D_t^\alpha v(t) &= \lambda v(t) - F(t), \quad t>0, \\ v(0) &= E_\alpha(\lambda s^\alpha). \end{aligned} \tag{3.15}$$

Consider that

$$\begin{aligned} D_t^\alpha E_\alpha(\lambda(t+s)^\alpha) &= \int_0^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} D_\tau E_\alpha(\lambda(\tau+s)^\alpha)d\tau \\ &= \int_s^{t+s} \frac{(t+s-r)^{-\alpha}}{\Gamma(1-\alpha)} D_r E_\alpha(\lambda r^\alpha)dr \\ &= \int_0^{t+s} \frac{(t+s-r)^{-\alpha}}{\Gamma(1-\alpha)} D_r E_\alpha(\lambda r^\alpha)dr \\ &\quad - \int_0^{s} \frac{(t+s-r)^{-\alpha}}{\Gamma(1-\alpha)} D_r E_\alpha(\lambda r^\alpha)dr \\ &= D_\tau^\alpha E_\alpha(\lambda\tau^\alpha)|_{\tau=t+s} - F(t), \quad t>0. \end{aligned} \tag{3.16}$$

By (2.21), we see that $E_\alpha(\lambda(t+s)^\alpha)$ solves the problem (3.15) and, by Proposition 2.1, it is represented uniquely by

$$E_\alpha(\lambda(t+s)^\alpha) = E_\alpha(\lambda t^\alpha)E_\alpha(\lambda s^\alpha) - \int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(\lambda\tau^\alpha)F(t-\tau)d\tau.$$

Furthermore, by (2.20), we obtain

$$\begin{aligned} E_\alpha(\lambda(t+s)^\alpha) = E_\alpha(\lambda t^\alpha)E_\alpha(\lambda s^\alpha) - \int_0^t\int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} \\ \cdot(\tau r)^{\alpha-1}\lambda E_{\alpha,\alpha}(\lambda\tau^\alpha)E_{\alpha,\alpha}(\lambda r^\alpha)drd\tau. \end{aligned} \tag{3.17}$$

The last step, by using (3.17), we show the semigroup-like property. By (3.14), the closedness of A, the identity $AR(\lambda;A) = \lambda R(\lambda;A) - I$, and the equation

$$\int_{\Gamma_{r,\omega}} E_{\alpha,\alpha}(\lambda\tau^\alpha)E_{\alpha,\alpha}(\lambda r^\alpha)d\lambda = 0,$$

we have, for each $x \in H$,

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} (\tau r)^{\alpha-1}E_{\alpha,\alpha}(\lambda\tau^\alpha)E_{\alpha,\alpha}(\lambda r^\alpha)\lambda R(\lambda;A)xd\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} (\tau r)^{\alpha-1}E_{\alpha,\alpha}(\lambda\tau^\alpha)E_{\alpha,\alpha}(\lambda r^\alpha)AR(\lambda;A)xd\lambda \\ &= AP_\alpha(\tau)P_\alpha(r)x, \quad \tau, r>0. \end{aligned} \tag{3.18}$$

Now, observe that, by Theorem 3.2(i), for $x \in H$ and $s,t>0$,

$$\begin{aligned} &\int_0^t\int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)}\|P_\alpha(\tau)P_\alpha(r)x\|\,drd\tau \\ &\le L_1(\alpha,\gamma)^2\int_0^t\int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)}\tau^{\alpha\gamma-1}r^{\alpha\gamma-1}drd\tau\|x\| \\ &\le \frac{L_1(\alpha,\gamma)^2B(\alpha\gamma,1-\alpha)B(\alpha\gamma,1-\alpha+\alpha\gamma)}{\Gamma(1-\alpha)}(t+s)^{-\alpha+2\alpha\gamma}\|x\| \end{aligned}$$

and, by Theorem 3.2(i) and 3.2(ii), for $x \in D(A)$ and $s, t > 0$,

$$\begin{aligned}
&\int_0^t \int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} \|AP_\alpha(\tau)P_\alpha(r)x\| \, dr d\tau \\
&= \int_0^t \int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} \|P_\alpha(\tau)P_\alpha(r)Ax\| \, dr d\tau \\
&\le \frac{L_1(\alpha,\gamma)^2 B(\alpha\gamma, 1-\alpha) B(\alpha\gamma, 1-\alpha+\alpha\gamma)}{\Gamma(1-\alpha)} (t+s)^{-\alpha+2\alpha\gamma} \|Ax\|.
\end{aligned}$$

Thus, by the closedness of A, for $x \in D(A)$ and $s, t > 0$,

$$\begin{aligned}
A \int_0^t \int_0^s & \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} P_\alpha(\tau)P_\alpha(r) x dr d\tau \\
&= \int_0^t \int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} AP_\alpha(\tau)P_\alpha(r) x dr d\tau.
\end{aligned} \tag{3.19}$$

Finally, by (3.10), (3.17), (3.18), and (3.19), we obtain, for $x \in D(A)$ and $s, t > 0$,

$$\begin{aligned}
S_\alpha(t+s)x &= S_\alpha(t)S_\alpha(s)x - \frac{1}{2\pi i} \int_{\Gamma_{r,\omega}} R(\lambda; A) \int_0^t \int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} \\
&\quad \cdot (\tau r)^{\alpha-1} \lambda E_{\alpha,\alpha}(\lambda \tau^\alpha) E_{\alpha,\alpha}(\lambda r^\alpha) x dr d\tau d\lambda \\
&= S_\alpha(t)S_\alpha(s)x - \int_0^t \int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} AP_\alpha(\tau)P_\alpha(r) x dr d\tau \\
&= S_\alpha(t)S_\alpha(s)x - A \int_0^t \int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} P_\alpha(\tau)P_\alpha(r) x dr d\tau.
\end{aligned}$$

□

Next theorem shows us the behavior of the operator $S_\alpha(t)$ at t close to 0^+.

Theorem 3.4. *Let A be an (almost) sectorial operator and $S_\alpha(t)$ be an operator defined by* (3.3). *Then the following statements hold.*

(i) *If $x \in D(A)$ then $\lim_{t\to 0^+} S_\alpha(t)x = x$.*

(ii) *For every $x \in D(A)$ and $t > 0$,*

$$\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau) x d\tau \in D(A),$$

$$\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} AS_\alpha(\tau) x d\tau = S_\alpha(t)x - x,$$

(iii) *If $x \in D(A^2)$ then*

$$\lim_{t\mapsto 0^+} \frac{S_\alpha(t)x - x}{t^\alpha} = \frac{1}{\Gamma(\alpha+1)} Ax.$$

Proof.

(i) First, we assume $x \in D(A)$. By using the identity

$$\lambda R(\lambda; A)x = x + AR(\lambda; A)x = x + R(\lambda; A)Ax,$$

we get, for $t > 0$,

$$\begin{aligned} S_\alpha(t)x &= \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} \frac{e^{\lambda t}}{\lambda}\lambda^\alpha R(\lambda^\alpha; A)x d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} \frac{e^{\lambda t}}{\lambda} x d\lambda + \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} \frac{e^{\lambda t}}{\lambda} R(\lambda^\alpha; A)Ax d\lambda. \end{aligned}$$

Since $e^{\lambda t}$ tends to 0 in the sector $\{z \in \mathbb{C} : \omega \le \arg(z) \le 2\pi - \omega\}$ as $|\lambda| \to \infty$, we have that

$$\int_{\Gamma_{r,\omega}} \frac{e^{\lambda t}}{\lambda} d\lambda = 2\pi i$$

by Cauchy's theorem. Then we obtain

$$\begin{aligned} \|S_\alpha(t)x - x\| &= \left\| \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} \frac{e^{\lambda t}}{\lambda} R(\lambda^\alpha; A)Ax d\lambda \right\| \\ &= \left\| \frac{1}{2\pi i}\int_{\Gamma_{t^{-1},\omega}} \frac{e^{\lambda t}}{\lambda} R(\lambda^\alpha; A)Ax d\lambda \right\| \\ &\le \frac{M}{2\pi}\|Ax\| t^{\alpha\gamma}\int_{\Gamma_{1,\omega}} |e^\mu||\mu|^{-\alpha\gamma-1}|d\mu|. \end{aligned}$$

Hence we conclude $\|S_\alpha(t)x - x\| \to 0$ as $t \to 0^+$.

Remark 3.1. *Now, we suppose $x \in \overline{D(A)}$. Then there exists a sequence $\{x_n\} \subset D(A)$ such that $x_n \to x$ in H. Consider that, for $t > 0$,*

$$\begin{aligned} \|S_\alpha(t)x - x\| &\le \|S_\alpha(t)(x - x_n)\| + \|S_\alpha(t)x_n - x_n\| + \|x_n - x\| \\ &\le (C_1 t^{-\alpha(1-\gamma)} + 1)\|x_n - x\| + \|S_\alpha(t)x_n - x_n\|. \end{aligned}$$

Thus if A is sectorial or $\gamma = 1$, we obtain $\lim_{t\to 0^+}\|S_\alpha(t)x - x\| = 0$. But, if A is almost sectorial or $0 < \gamma < 1$, we can not conclude the similar result.

Next, let $y = \lim_{t\to 0^+} S_\alpha(t)x$. Since $S_\alpha(t)x \in D(A)$, then $y \in \overline{D(A)}$. Observe that, since $R(\lambda; A)x \in D(A)$, we can obtain

$$R(\lambda; A)y = \lim_{t\to 0^+} R(\lambda; A)S_\alpha(t)x = \lim_{t\to 0^+} S_\alpha(t)R(\lambda; A)x = R(\lambda; A)x.$$

Therefore $x = y \in \overline{D(A)}$.

Thus, we can conclude that for A which is sectorial, $x \in \overline{D(A)}$ is the sufficient and necessary condition for $\lim_{t\to 0^+} S_\alpha(t)x = x$.

(ii) We take $\lambda \in \rho(A)$ and $x \in D(A)$. Then, by part (i), (2.7), and the first identity in Theorem 3.3(ii), we have

$$\begin{aligned}
&\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)x d\tau = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} (\lambda - A)R(\lambda; A)S_\alpha(\tau)x d\tau \\
&= \lambda \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} R(\lambda; A)S_\alpha(\tau)x d\tau - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} R(\lambda; A)AS_\alpha(\tau)x d\tau \\
&= \lambda R(\lambda; A) \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)x d\tau - R(\lambda; A) \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} D_\tau^\alpha S_\alpha(\tau)x d\tau \\
&= \lambda R(\lambda; A) \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)x d\tau - R(\lambda; A)S_\alpha(t)x + \lim_{\epsilon \to 0^+} S_\alpha(\epsilon)R(\lambda; A)x \\
&= R(\lambda; A) \left(\lambda \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)x d\tau - (S_\alpha(t)x - x) \right).
\end{aligned}$$

Since $R(\lambda; A)y \in D(A)$ for each $y \in H$, then

$$\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)x d\tau \in D(A)$$

and, furthermore,

$$\begin{aligned}
(\lambda - A) &\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)x d\tau \\
&= \lambda \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)x d\tau - (S_\alpha(t)x - x).
\end{aligned}$$

Therefore

$$A \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)x d\tau = S_\alpha(t)x - x.$$

Next, by Theorem 3.1(i) and 3.1(ii), for $x \in D(A)$ and $t > 0$,

$$\begin{aligned}
\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \|AS_\alpha(\tau)x\| d\tau &= \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \|S_\alpha(\tau)Ax\| d\tau \\
&\le C_1(\alpha, \gamma) \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \tau^{-\alpha(1-\gamma)} d\tau \|Ax\| \\
&= \frac{C_1(\alpha, \gamma)B(1 - \alpha(1-\gamma), \alpha)}{\Gamma(\alpha)} t^{\alpha\gamma} \|Ax\|.
\end{aligned}$$

Thus, by the closedness of A, we obtain

$$\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} AS_\alpha(\tau)x d\tau = S_\alpha(t)x - x.$$

(iii) Given $\varepsilon > 0$ and $y \in D(A)$. Then, by part (i), there exists $t_0 > 0$ such that, for $0 < t < t_0$, it holds that $\|S_\alpha(t)y - y\| < \varepsilon$. Thus we have

$$\begin{aligned}
&\left\| \frac{\Gamma(\alpha+1)}{t^\alpha} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau) y d\tau - y \right\| \\
&= \left\| \frac{\Gamma(\alpha+1)}{t^\alpha} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(S_\alpha(\tau)y - y\right) d\tau \right\| \\
&\leq \frac{\Gamma(\alpha+1)}{t^\alpha} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \|S_\alpha(\tau)y - y\| \, d\tau \\
&< \varepsilon \frac{\Gamma(\alpha+1)}{t^\alpha} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} d\tau = \varepsilon.
\end{aligned}$$

Therefore, by part (ii), we obtain, for $x \in D(A^2)$,

$$\lim_{t\to 0^+} \frac{S_\alpha(t)x - x}{t^\alpha} = \lim_{t \to 0^+} \frac{1}{t^\alpha} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau) A x d\tau = \frac{1}{\Gamma(\alpha+1)} Ax.$$

Remark 3.2. *Based on Remark 3.1, if $x \in D(A), Ax \in \overline{D(A)}$, we have the same conclusion when A is sectorial.*

□

The following proposition provides the representation formula for the resolvent operator $R(\lambda^\alpha; A)$ in term $S_\alpha(t)$.

Proposition 3.1. *Let $A : D(A) \subset H \to H$ be an (almost) sectorial operator. For every $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$ and $x \in D(A)$,*

$$\lambda^{\alpha-1} R(\lambda^\alpha; A)x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt.$$

Proof. We suppose $x \in D(A)$. By the Theorem 3.1(i), for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$, we have $t \mapsto e^{-\lambda t} S_\alpha(t)$ is integrable over $(0, \infty)$. By Theorem 3.4(ii), we have that

$$\int_0^\infty e^{-\lambda t} \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau) A x d\tau dt = \int_0^\infty e^{-\lambda t} \left(S_\alpha(t)x - x\right) dt.$$

Equivalently,

$$\lambda^{-\alpha} \mathcal{L}(S_\alpha(t))(\lambda) A x = \mathcal{L}(S_\alpha(t))(\lambda) x - \lambda^{-1} x.$$

It follows that

$$\mathcal{L}(S_\alpha(t))(\lambda)(\lambda^\alpha - A)x = \lambda^{\alpha-1} x.$$

Therefore we obtain

$$\lambda^{\alpha-1} R(\lambda^\alpha; A)x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt, \quad \mathrm{Re}(\lambda) > 0.$$

□

We also have the representation of the resolvent operator $R(\lambda^\alpha; A)$ in term $P_\alpha(t)$ as stated below.

Proposition 3.2. *Let $A : D(A) \subset H \to H$ be an (almost) sectorial operator. For every $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$ and $x \in D(A)$,*

$$R(\lambda^\alpha; A)x = \int_0^\infty e^{-\lambda t} P_\alpha(t) x dt.$$

Proof. Based on Proposition 3.1, for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$ and $x \in D(A)$,

$$R(\lambda^\alpha; A)x = \lambda^{1-\alpha} \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt.$$

By the first identity in Theorem 3.3(i), we get

$$\int_0^\infty e^{-\lambda t} S_\alpha(t) x dt = \int_0^\infty e^{-\lambda t} J_t^{1-\alpha} P_\alpha(t) x dt = \lambda^{\alpha-1} \int_0^\infty e^{-\lambda t} P_\alpha(t) x dt.$$

Therefore

$$R(\lambda^\alpha; A)x = \int_0^\infty e^{-\lambda t} P_\alpha(t) x dt, \quad \mathrm{Re}(\lambda) > 0.$$

□

Then we obtain a unique solution to the problem (1.7).

Theorem 3.5. *Let $u \in C^1((0,\infty); H) \cap L^1((0,\infty); H)$, $u(t) \in D(A)$ for $t \in [0,\infty)$, $Au \in L^1(0,\infty); H)$, $f \in L^1((0,\infty); D(A))$, and $Af \in L^1((0,\infty); H)$. If u is a solution to the problem* (1.7) *then*

$$u(t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s) f(s) ds, \quad t > 0. \tag{3.20}$$

Proof. We suppose u is a solution to the problem (1.7). Then, by the Laplace tranform, we get (3.1). Since A is an (almost) sectorial operator, we have that, for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$,

$$\mathcal{L}(u)(\lambda) = \lambda^{\alpha-1} R(\lambda^\alpha; A)u_0 + R(\lambda^\alpha; A)\mathcal{L}(f)(\lambda).$$

By Proposition 3.1 and 3.2, for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$,

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} u(t) dt &= \int_0^\infty e^{-\lambda t} S_\alpha(t) u_0 dt + \int_0^\infty e^{-\lambda t} P_\alpha(t) dt \int_0^\infty e^{-\lambda t} f(t) dt \\
&= \int_0^\infty e^{-\lambda t} S_\alpha(t) u_0 dt + \int_0^\infty e^{-\lambda t} P_\alpha(t) * f(t) dt \\
&= \int_0^\infty e^{-\lambda t} \left\{ S_\alpha(t) u_0 + \int_0^t P_\alpha(t-s) f(s) ds \right\}.
\end{aligned}$$

By the uniqueness of the Laplace transform, we obtain

$$u(t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s) f(s) ds, \quad t > 0.$$

□

Remark 3.3. *In particular, for $f = 0$, the function*

$$t \mapsto u(t) = S_\alpha(t)u_0, \quad t > 0,$$

is the unique solution to the homogeneous case of the problem (1.7).

4. Fractional Powers of (Almost) Sectorial Operators

We consider the fractional power of the operator A

$$A^{-\beta}x = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} \lambda^{-\beta}R(\lambda;A)xd\lambda,\ x\in H,\ \beta>1-\gamma,$$

and

$$A^{\beta}x = A(A^{\beta-1}x) = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}} \lambda^{\beta-1}R(\lambda;A)Axd\lambda,\ x\in D(A),\ 0<\beta<\gamma.$$

Both integrals above are independent of $r>0$ and $\omega\in(\pi/2,\theta)$. As for fractional powers of operators notion in more details, one can refer to [11]. Here, we state some results concerning some estimates involving A^{β} and the operators families $\{S_\alpha(t)\}_{t>0}$, $\{P_\alpha(t)\}_{t>0}$ generated by the (almost) sectorial operator A. These estimates are analogous to those as stated in Theorem 6.13 in [5] for analytic semigroups. We derive the estimates directly from the definition of A^{β}. Then we obtain our main results by employing these estimates.

Proposition 4.1. *For each $0<\beta<\gamma$, there exist positive constants $C_1'=C_1'(\alpha,\beta,\gamma)$ and $C_2'=C_2'(\alpha,\beta,\gamma)$ such that, for all $x\in H$,*

$$\|A^{\beta}S_\alpha(t)x\| \le C_1' t^{-\alpha}(t^{-\alpha(\beta-\gamma)}+1)\|x\|,\quad t>0, \tag{4.1}$$

$$\|A^{\beta}P_\alpha(t)x\| \le C_2' t^{-\alpha(\beta-\gamma)-1}\|x\|,\quad t>0. \tag{4.2}$$

Proof. We prove (4.1) first. Let $\pi/2<\omega'<\theta$ such that $\omega'<\omega/\alpha$ and $r'>r^{1/\alpha}$. Thus, for $x\in H$ and $t>0$, we have

$$\begin{aligned}
A^{\beta}S_\alpha(t)x &= \left(\frac{1}{2\pi i}\right)^2\int_{\Gamma_{r,\omega}}\lambda^{\beta-1}R(\lambda;A)d\lambda\int_{\Gamma_{r',\omega'}}e^{\mu t}\mu^{\alpha-1}AR(\mu^{\alpha};A)xd\mu\\
&= \left(\frac{1}{2\pi i}\right)^2\int_{\Gamma_{r,\omega}}\lambda^{\beta-1}R(\lambda;A)d\lambda\int_{\Gamma_{r',\omega'}}e^{\mu t}\mu^{2\alpha-1}R(\mu^{\alpha};A)xd\mu\\
&\quad -\left(\frac{1}{2\pi i}\right)^2\int_{\Gamma_{r,\omega}}\lambda^{\beta-1}R(\lambda;A)d\lambda\int_{\Gamma_{r',\omega'}}e^{\mu t}\mu^{\alpha-1}xd\mu\\
&= (I-II)x
\end{aligned}$$

where

$$\begin{aligned}
I &= \left(\frac{1}{2\pi i}\right)^2\int_{\Gamma_{r,\omega}}\int_{\Gamma_{r',\omega'}}\lambda^{\beta-1}e^{\mu t}\mu^{2\alpha-1}\frac{R(\lambda;A)-R(\mu^{\alpha};A)}{\mu^{\alpha}-\lambda}d\mu d\lambda\\
&= \left(\frac{1}{2\pi i}\right)^2\int_{\Gamma_{r,\omega}}\lambda^{\beta-1}R(\lambda;A)d\lambda\int_{\Gamma_{r',\omega'}}\frac{e^{\mu t}\mu^{2\alpha-1}}{\mu^{\alpha}-\lambda}d\mu\\
&\quad -\left(\frac{1}{2\pi i}\right)^2\int_{\Gamma_{r',\omega'}}e^{\mu t}\mu^{2\alpha-1}R(\mu^{\alpha};A)d\mu\int_{\Gamma_{r,\omega}}\frac{\lambda^{\beta-1}}{\mu^{\alpha}-\lambda}d\lambda\\
&= III-IV.
\end{aligned}$$

Note that

$$II = \frac{1}{2\pi i}\int_{\Gamma_{r,\omega}}\lambda^{\beta-1}R(\lambda;A)d\lambda\cdot\frac{t^{-\alpha}}{\Gamma(1-\alpha)},$$

$$
\begin{aligned}
III &= t^{-\alpha}\left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_{r,\omega}} \lambda^{\beta-1} R(\lambda;A)d\lambda \int_{\Gamma_{r't,\omega'}} \frac{e^{\rho}\rho^{\alpha-(1-\alpha)}}{\rho^{\alpha}-\lambda t^{\alpha}} d\rho \\
&= \frac{t^{-\alpha}}{2\pi i}\int_{\Gamma_{r,\omega}} \lambda^{\beta-1}E_{\alpha,1-\alpha}(\lambda t^{\alpha})R(\lambda;A)d\lambda \\
&= \frac{t^{-\alpha\beta-\alpha}}{2\pi i}\int_{\Gamma_{rt^{\alpha},\omega}} \eta^{\beta-1}E_{\alpha,1-\alpha}(\eta)R(\eta t^{-\alpha};A)d\eta,
\end{aligned}
$$

and

$$
IV = t^{-\alpha\beta-\alpha}\left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_{r't,\omega'}} e^{\rho}\rho^{2\alpha-1}R(\rho^{\alpha}t^{-\alpha};A)d\rho \int_{\Gamma_{rt^{\alpha},\omega}} \frac{\eta^{\beta-1}}{\rho^{\alpha}-\eta}d\eta.
$$

Next, by (2.18) and (2.19), for $\eta \in \Gamma_1 = \{\eta \in \Gamma_{rt^{\alpha},\omega} : \alpha\omega' < |\arg(\eta)| < 2\pi - \alpha\omega'\}$,

$$
|E_{\alpha,1-\alpha}(\eta)| \le \frac{D_3}{1+|\eta|},
$$

and, for $\eta \in \Gamma_2 = \{\eta \in \Gamma_{rt^{\alpha},\omega} : |\arg(\eta)| \le \alpha\omega'\}$,

$$
|E_{\alpha,1-\alpha}(\eta)| \le D_1(1+|\eta|)e^{\mathrm{Re}(\eta^{1/\alpha})} + \frac{D_2}{1+|\eta|}.
$$

Since the integrals involved do not depend on the choice of $r>0$ and $\pi/2<\omega<\theta$, then, by taking $r=t^{-\alpha}$, we get

$$
\begin{aligned}
&\left\|\int_{\Gamma_{rt^{\alpha},\omega}} \eta^{\beta-1}E_{\alpha,1-\alpha}(\eta)R(\eta t^{-\alpha};A)d\eta\right\| \le MD_3 t^{\alpha\gamma}\int_{\Gamma_1}\frac{|\eta|^{\beta-\gamma-1}}{1+|\eta|}|d\eta| \\
&+ Mt^{\alpha\gamma}\int_{\Gamma_2}\left(D_1(1+|\eta|)e^{\mathrm{Re}(\eta^{1/\alpha})}+\frac{D_2}{1+|\eta|}\right)|d\eta| < \infty.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\left\|\int_{\Gamma_{r't,\omega'}} e^{\rho}\rho^{2\alpha-1}R(\rho^{\alpha}t^{-\alpha};A)d\rho\right\| &\le Mt^{\alpha\gamma}\int_{\Gamma_{1,\omega'}} e^{\mathrm{Re}(\rho)}|\rho|^{2\alpha-\alpha\gamma-1}|d\rho| < \infty, \\
\left|\int_{\Gamma_{rt^{\alpha},\omega}} \frac{\eta^{\beta-1}}{\rho^{\alpha}-\eta}d\eta\right| &\le \int_{\Gamma_{1,\omega}} \frac{|\eta|^{\beta-1}}{|\rho^{\alpha}-\eta|}|d\eta| < \infty,
\end{aligned}
$$

and, clearly,

$$
\left\|\int_{\Gamma_{r,\omega}} \lambda^{\beta-1}R(\lambda;A)d\lambda\right\| < \infty.
$$

Hence there exists $C_1' = C_1'(\alpha,\beta,\gamma) > 0$ such that

$$
\|A^{\beta}S_{\alpha}(t)x\| \le C_1' t^{-\alpha}(t^{-\alpha(\beta-\gamma)}+1)\|x\|,\ x\in H,\ t>0.
$$

Next, we prove (4.2). Since

$$
\int_{\Gamma_{r',\omega'}} e^{\mu t}d\mu = 0
$$

and $AR(\mu;A) = \mu R(\mu;A) - I$, we find

$$\begin{aligned}
A^\beta P_\alpha(t)x &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_{r,\omega}} \lambda^{\beta-1} R(\lambda;A)d\lambda \int_{\Gamma_{r',\omega'}} e^{\mu t} AR(\mu^\alpha;A)xd\mu \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_{r,\omega}} \int_{\Gamma_{r',\omega'}} \lambda^{\beta-1} e^{\mu t}\mu^\alpha \frac{R(\lambda;A)-R(\mu^\alpha;A)}{\mu^\alpha-\lambda} d\mu d\lambda \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_{r,\omega}} \lambda^{\beta-1} R(\lambda;A)d\lambda \int_{\Gamma_{r',\omega'}} \frac{e^{\mu t}\mu^\alpha}{\mu^\alpha-\lambda} xd\mu \\
&\quad - \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_{r',\omega'}} e^{\mu t}\mu^\alpha R(\mu^\alpha;A)d\mu \int_{\Gamma_{r,\omega}} \frac{\lambda^{\beta-1}}{\mu^\alpha-\lambda} xd\lambda \\
&= (V - VI)x.
\end{aligned}$$

Note that

$$\begin{aligned}
V &= t^{-1}\left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_{r,\omega}} \lambda^{\beta-1} R(\lambda;A)d\lambda \int_{\Gamma_{r't,\omega'}} \frac{e^\rho \rho^\alpha}{\rho^\alpha - \lambda t^\alpha} d\rho \\
&= \frac{t^{-1}}{2\pi i} \int_{\Gamma_{r,\omega}} \lambda^{\beta-1} E_{\alpha,0}(\lambda t^\alpha) R(\lambda;A)d\lambda \\
&= \frac{t^{-\alpha\beta-1}}{2\pi i} \int_{\Gamma_{rt^\alpha,\omega}} \eta^{\beta-1} E_{\alpha,0}(\eta) R(\eta t^{-\alpha};A)d\eta
\end{aligned}$$

and

$$VI = t^{-\alpha\beta-1}\left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_{r't,\omega'}} e^\rho \rho^\alpha R(\rho^\alpha t^{-\alpha};A)d\rho \int_{\Gamma_{rt^\alpha,\omega}} \frac{\eta^{\beta-1}}{\rho^\alpha-\eta} d\eta.$$

Observe that, by (2.18) and (2.19) again, for $\eta \in \Gamma_1 = \{\eta \in \Gamma_{rt^\alpha,\omega} : \alpha\omega' < |\arg(\eta)| < 2\pi - \alpha\omega'\}$,

$$|E_{\alpha,0}(\eta)| \le \frac{D_3}{1+|\eta|},$$

and, for $\eta \in \Gamma_2 = \{\eta \in \Gamma_{rt^\alpha,\omega} : |\arg(\eta)| \le \alpha\omega'\}$,

$$|E_{\alpha,0}(\eta)| \le D_1(1+|\eta|)^{1/\alpha} e^{\mathrm{Re}(\eta^{1/\alpha})} + \frac{D_2}{1+|\eta|}.$$

By taking $r = t^{-\alpha}$ again, we have

$$\begin{aligned}
&\left\| \int_{\Gamma_{rt^\alpha,\omega}} \eta^{\beta-1} E_{\alpha,1-\alpha}(\eta) R(\eta t^{-\alpha};A)d\eta \right\| \le MD_3 t^{\alpha\gamma} \int_{\Gamma_1} \frac{|\eta|^{\beta-\gamma-1}}{1+|\eta|} |d\eta| \\
&+ Mt^{\alpha\gamma} \int_{\Gamma_2} \left(D_1(1+|\eta|)^{1/\alpha} e^{\mathrm{Re}(\eta^{1/\alpha})} + \frac{D_2}{1+|\eta|} \right) |d\eta| < \infty,
\end{aligned}$$

and

$$\left\| \int_{\Gamma_{r't,\omega'}} e^\rho \rho^\alpha R(\rho^\alpha t^{-\alpha};A)d\rho \right\| \le Mt^{\alpha\gamma} \int_{\Gamma_{1,\omega'}} e^{\mathrm{Re}(\rho)} |\rho|^{\alpha-\alpha\gamma} |d\rho| < \infty.$$

Then there exists $C_2' = C_2'(\alpha,\beta,\gamma) > 0$ such that

$$\|A^\beta P_\alpha(t)x\| \le C_2' t^{-\alpha(\beta-\gamma)-1} \|x\|, \; x \in H, \; t > 0.$$

□

Next, we observe that $-1 < -\alpha(\beta-\gamma) - 1 < -\alpha - \alpha(\beta-\gamma) < 0$. Now, let $\xi_\zeta = \alpha(\zeta - \gamma) + 1$, for $0 < \zeta < \gamma$. Then we get $0 < \xi_\beta - \alpha - \alpha(\beta-\gamma) = 1 - \alpha < 1$. Note also that $-1 < \xi_\beta - \alpha < 1$. It means that $\xi_\beta - \alpha$ may be negative. However, by assuming $\beta > (1 - 1/\alpha + \gamma)^+$, we find $\xi_\beta - \alpha > 0$, where $x^+ = \max\{0, x\}$, $x \in \mathbb{R}$. Thus we obtain

$$t^{\xi_\beta}\|A^\beta S_\alpha(t)x\| \le C_1' t^{\xi_\beta - \alpha}(t^{-\alpha(\beta-\gamma)} + 1)\|x\|, \quad t > 0, \tag{4.3}$$

$$t^{\xi_\beta}\|A^\beta P_\alpha(t)x\| \le C_2'\|x\|, \quad t > 0. \tag{4.4}$$

Consequently, we have the following.

Corollary 4.1. *For each $\beta > (1 - 1/\alpha + \gamma)^+$ and $x \in H$,*

$$t^{\xi_\beta}\|A^\beta S_\alpha(t)x\| \le 2C_1'\|x\|, \quad 0 < t \le 1, \tag{4.5}$$

$$t^{\xi_\beta}\|A^\beta P_\alpha(t)x\| \le C_2'\|x\|, \quad t > 0, \tag{4.6}$$

and

$$t^{\xi_\beta}\|A^\beta S_\alpha(t)x\| \to 0, \quad \text{as } t \to 0^+. \tag{4.7}$$

Remark 4.1. *If $\beta = 1 - 1/\alpha + \gamma > 0$, implying $\xi_\beta - \alpha = 0$, the estimates* (4.5) *and* (4.6) *also hold for all $x \in H$. As for the limit* (4.7)*, it remains valid for all $x \in H$ if A is sectorial ($\gamma = 1$). In the case of A which is almost sectorial ($0 < \gamma < 1$), it is valid only for $x \in D(A)$.*

Now, observe that, for $x \in D(A^\beta)$ with $1 - \gamma < \beta < \gamma$ and $1/2 < \gamma \le 1$, we have $AS_\alpha(t)x = A^{1-\beta}S_\alpha(t)A^\beta x$, $AP_\alpha(t)x = A^{1-\beta}P_\alpha(t)A^\beta x$, and $AP_\alpha(s)P_\alpha(t)x = P_\alpha(s)A^{1-\beta}P_\alpha(t)A^\beta x$, for $s, t > 0$. Then, by using the same method as used in the proof of Theorem 3.3(ii), we obtain a theorem that is similar to Theorem 3.3(ii).

Theorem 4.1. *Let $1 - \gamma < \beta < \gamma$ with $1/2 < \gamma \le 1$. Then, for every $x \in D(A^\beta)$ and $s, t > 0$,*

$$D_t^\alpha S_\alpha(t)x = AS_\alpha(t)x,$$
$$S_\alpha(t+s)x = S_\alpha(t)S_\alpha(s)x - A\int_0^t\int_0^s \frac{(t+s-\tau-r)^{-\alpha}}{\Gamma(1-\alpha)} P_\alpha(\tau)P_\alpha(r)x\,dr\,d\tau.$$

Next, we get the following proposition.

Proposition 4.2. *Let $1 - \gamma < \beta < \gamma$ and $1/2 < \gamma \le 1$. Then there exists $C_3' = C_3'(\alpha, \beta, \gamma) > 0$ such that, for all $x \in D(A^\beta)$,*

$$\|S_\alpha(t)x - x\| \le C_3' t^{-\alpha(-\beta-\gamma+1)}\|A^\beta x\|, \quad t > 0. \tag{4.8}$$

Proof. Let $1 - \gamma < \beta < \gamma$, $1/2 < \gamma \le 1$, and $x \in D(A^\beta)$. By using (4.1), the first identity in Theorem 4.1, and following the way used in proving Theorem 3.4(ii), we also have that

$$A\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)x\,d\tau = S_\alpha(t)x - x. \tag{4.9}$$

Now, observe that, by (2.2) and the first identity in Theorem 3.3(i),

$$\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)x\,d\tau = J_t^\alpha J_t^{1-\alpha} P_\alpha(t)x = \int_0^t P_\alpha(\tau)x\,d\tau. \tag{4.10}$$

Next, by (4.2), we find

$$\begin{aligned}\int_0^t \|AP_\alpha(s)x\|ds &= \int_0^t \|A^{1-\beta}P_\alpha(s)A^\beta x\|ds \\ &\le C_2'(\alpha, 1-\beta, \gamma)\int_0^t s^{-\alpha(1-\beta-\gamma)-1}ds\|A^\beta x\| \\ &= C_3' t^{-\alpha(1-\beta-\gamma)}\|A^\beta x\|\end{aligned} \tag{4.11}$$

where $C_3' = C_2'(\alpha, 1-\beta, \gamma)$. Then, by the closedness of A,

$$A\int_0^t P_\alpha(\tau)xd\tau = \int_0^t AP_\alpha(\tau)xd\tau. \tag{4.12}$$

Thus, by (4.9), (4.10), (4.12), and (4.11),

$$\|S_\alpha(t)x - x\| \le \int_0^t \|AP_\alpha(s)x\|ds \le C_3' t^{-\alpha(1-\beta-\gamma)}\|A^\beta x\|, \quad t > 0.$$

□

Furthermore, by using (4.8) and the same method as used in the proof of Theorem 3.4(ii) and 3.4(iii), we have a theorem that is similar to Theorem 3.4.

Theorem 4.2. *Let $1-\gamma < \beta < \gamma$ with $1/2 < \gamma \le 1$. Then*

(i) *If $x \in D(A^\beta)$ then $\lim_{t\to 0^+} S_\alpha(t)x = x$,*

(ii) *For every $x \in D(A^\beta)$ and $t > 0$,*

$$\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} S_\alpha(\tau)xd\tau \in D(A),$$

$$\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} AS_\alpha(\tau)xd\tau = S_\alpha(t)x - x,$$

(iii) *If $x \in D(A)$ and $Ax \in D(A^\beta)$ then*

$$\lim_{t\mapsto 0^+} \frac{S_\alpha(t)x - x}{t^\alpha} = \frac{1}{\Gamma(\alpha+1)}Ax.$$

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